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UACE MATHEMATICS PAPER 1 2016 guide

SECTION A (40 marks)

Answer all questions in this section

1. Without using mathematical tables or calculators, find the value of
$$\frac{(\sqrt{5}+2)^2 - (\sqrt{5}-2)^2}{8\sqrt{5}}$$
2. Find the angle between the lines 2x –y = 3 and 11x + 2y = 13
3. Evaluate $\int_{\frac{1}{2}}^{1} 10x\sqrt{(1-x^2)}\,dx$
4. Solve the equation $\frac{dy}{dx} = 1 + y^2$ given that when y = 1 when x = 0.
5. Given that $2x^2 + 7x - 4$, $x^2 + 3x - 4$ and $7x^2 + ax - 8$ have a common factor, find the
   (a) Factors of $2x^2 + 7x - 4$ and $x^2 + 3x - 4$
   (b) Value of a in $7x^2 + ax - 8$.
6. Solve the equation sin2θ + cos2θcos4θ = cos4θcos6θ for $0 \le \theta \le \frac{\pi}{4}$.
7. Using small changes; show $(244)^{\frac{1}{5}} = 3\frac{1}{405}$.
8. Three points A(2, -1, 0), B(-2, 5, -4) and C are on a straight line such that 3AB = 2AC. Find the coordinates of C.

SECTION B (60MARKS)

Answer any five questions from this section. All questions carry equal marks

9. (a) If $z_1 = \frac{2i}{1+3i}$ and $z_2 = \frac{3+2i}{5}$, find $|z_1 - z_2|$
   (b) Given the complex number z = x + iy
      (i) Find $\frac{z+i}{z+2}$
      (ii) Show that the locus of $\frac{z+i}{z+2}$ is a straight line when its imaginary part is zero. State the gradient of the line.
10. (a) solve the equation $\cos 2x = 4\cos^2 x - 2\sin^2 x$ for $0 \le \theta \le 180^0$
    (b) Show that if sin(x + α) = psin(x – α) then tan x = $\left(\frac{p+1}{p-1}\right)$tanα. Hence solve the equation sin(x + α) = psin(x – α) for p = 2 and α= 20^0.
11. Given that $x = \frac{t^2}{1+t^3}$ and $y = \frac{t^3}{1+t^3}$, find $\frac{d^2y}{dx^2}$.
12. (a) Line A is the intersection of two planes whose equations are 3x –y + z = 2 and x + 5y +2z = 6. Find the equation of the line.

(b) Given that line B is perpendicular to the plane 3x –y + z = 2and passes through the point C(1, 1, 0), find the

(i) Cartesian equation of line B

(ii) angle between line B and line A in (a) above

13. (a) Find $\int \frac{1+\sqrt{x}}{2\sqrt{x}}$dx

(b) The gradient of the tangent at any point on a curve is x - $\frac{2y}{x}$. The curve passes through the point (2, 4), find the equation of the curve.

14. (a) The point P($at_1^2, 2at_1$) and Q($at_2^2, 2at_2$) are on parabola $y^2 = 4ax$. OP is perpendicular to OQ, where O is the origin. Show that $t_1t_2 + 4 = 0$.

(b) The normal to the rectangular hyperbola xy = 8 at point (4, 2) meets the asymptotes at M and N. find the length MN.

15. (a) Prove by induction

1.3 + 2.4 + … + n(n + 2) = $\frac{1}{6}n(n+1)(2n+7)$ for all values of n.

(b) A man deposits Shs. 150,000 at the beginning of every year in a micro finance bank with the understanding that at the end of the seven years he is paid back his money with 5% per annum compound interest. How much does he receive?

16. (a) If $x^2 + 3y^2 = k$, where k is constant, find $\frac{dy}{dx}$at the point (1, 2).

(c) A rectangular field of area 7200m2 is to be fenced using a wire mesh. On one side of the field. Is a straight river. This side is not to be fenced. Find the dimension of the field hat will minimize the amount of wire mesh to be used.

End

Solution

SECTION A (40 marks)

Answer all questions in this section

1. Without using mathematical tables or calculators, find the value of

$\frac{(\sqrt{5}+2)^2-(\sqrt{5}-2)^2}{8\sqrt{5}} = \frac{[(\sqrt{5}+2)+(\sqrt{5}-2)][(\sqrt{5}+2)-(\sqrt{5}-2)]}{8\sqrt{5}} = \frac{(2\sqrt{5})(4)}{8\sqrt{5}} = \frac{8\sqrt{5}}{8\sqrt{5}} = 1$

Or

$\frac{(\sqrt{5}+2)^2-(\sqrt{5}-2)^2}{8\sqrt{5}} = \frac{(5+4\sqrt{5}+4)-(5-4\sqrt{5}+4)}{8\sqrt{2}} = \frac{5+4\sqrt{5}+4-5+4\sqrt{5}-4}{8\sqrt{5}} = \frac{8\sqrt{5}}{8\sqrt{5}} = 1$

2. Find the angle between the lines 2x –y = 3 and 11x +2 y = 13

$\tan\theta = \frac{m_1-m_2}{1+m_1m_2}$

For 11x + y = 13; 2y = -11x + 13; y = $\frac{-11}{2}$x + $\frac{13}{2}$ => $m_1 = \frac{-11}{2}$ = -5.5

For 2x –y = 3; y = 2x + 3; => $m_2 = 2$

$\tan\theta = \frac{-5.5-2}{1+(-5.5 \times 2)} = \frac{-7.5}{-10}$

$\theta = \tan^{-1}\left(\frac{7.5}{10}\right) = 36.87^0$

Alternatively

$n_1 = \begin{pmatrix}2\\-1\end{pmatrix}, n_2 = \begin{pmatrix}11\\2\end{pmatrix}$

$n_1 n_2 = \cos\theta |n_1||n_2|$

$n_1 n_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} . \begin{pmatrix} 11 \\ 2 \end{pmatrix}$ = 20

$n_1 = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

$n_2 = \sqrt{11^2 + 2^2} = \sqrt{125}$

$20 = \cos\theta(\sqrt{5})(\sqrt{125})$

cosθ = $\frac{20}{\sqrt{125}}$

θ = $\cos^{-1}$ 0.8 = 36 .87^0

3. Evaluate $\int_{\frac{1}{2}}^{1} 10x\sqrt{(1 - x^2)}\, dx$

$\int_{\frac{1}{2}}^{1} 10x\sqrt{(1 - x^2)}\, dx$

Let u = 1- x^2; du = -2xdx

$-\frac{1}{2} du = xdx$

x	u
$\frac{1}{2}$	$\frac{3}{4}$
1	0

⇨ $\int_{\frac{1}{2}}^{1} 10x\sqrt{(1 - x^2)}\, dx = \int_{\frac{1}{2}}^{1} 10\sqrt{(1 - x^2)} . xdx$

$= \int_{\frac{3}{4}}^{0} 10. u^{\frac{1}{2}}. -\frac{1}{2} du$

$= -5\int_{\frac{3}{4}}^{0} u^{\frac{1}{2}}\, du$

$= -5\left[\frac{2}{3} u^{\frac{3}{2}}\right]_{\frac{3}{4}}^{0}$

$= -\frac{10}{3}\left[0 - \left(\frac{3}{4}\right)^{\frac{3}{2}}\right]$

= 2.165

Or

By using limits of x, we drop out limits when integrating and bring them in after u has been substituted for x

⇨ $\int 10x\sqrt{(1 - x^2)}\, dx = \int 10\sqrt{(1 - x^2)} . xdx$

$= \int 10. u^{\frac{1}{2}}. -\frac{1}{2} du$

$= -5\int u^{\frac{1}{2}}\, du$

$= -5\left(\frac{2}{3} u^{\frac{3}{2}}\right) + c$

$= -\frac{10}{3}(1 - x^2)^{\frac{3}{2}} + c$

Now bringing in limits

$\int_{\frac{1}{2}}^{1} 10x\sqrt{(1 - x^2)}\, dx = -\frac{10}{3}\left[(1 - x^2)^{\frac{3}{2}}\right]_{\frac{1}{2}}^{1}$

$= -\frac{10}{3}\left[0 - \left(\frac{3}{4}\right)^{\frac{3}{2}}\right]$

$= -\frac{10}{3} \cdot - \left(\frac{3}{4}\right)^{\frac{3}{2}}$

= 2.165

Alternatively

Let u = $\sqrt{1 - x^2}$

$u^2 = 1 - x^2$

2udu = -2xdx

-udu = xdx

x	u
1	0
$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

$\int_{\frac{1}{2}}^{1} 10x\sqrt{(1 - x^2)}\, dx = \int_{\frac{\sqrt{3}}{2}}^{0} -10u.udu = -10\int_{\frac{\sqrt{3}}{2}}^{0} u^2 du$

$= -10\left[\frac{u^3}{3}\right]_{\frac{\sqrt{3}}{2}}^{0}$

$= \frac{-10}{3}\left[0 - \left(\frac{\sqrt{3}}{2}\right)^3\right]$

= 2.165

4. Solve the equation $\frac{dy}{dx} = 1 + y^2$ given that when y = 1 when x = 0.

$\frac{dy}{dx} = 1 + y^2$

$\frac{dy}{1+y^2} = dx$

$\int \frac{dy}{1+y^2} = \int dx$

$\tan^{-1}(y) = x + C$

Substituting y = 1 when x = 0

$\tan^{-1}(1) = C$

C = $\frac{\pi}{4}$

⇨ $\tan^{-1}(y) = x + \frac{\pi}{4}$

y = $\tan\left(x + \frac{\pi}{4}\right)$

5. Given that $2x^2 + 7x - 4$, $x^2 + 3x - 4$ and $7x^2 + ax - 8$ have a common factor, find the
   (a) Factors of $2x^2 + 7x - 4$ and $x^2 + 3x - 4$

$2x^2 + 7x - 4 = 2x^2 + 8x - x - 4$

$= 2x(x + 4) - 1(x + 4)$

$= (2x - 1)(x + 4)$

Hence the factors of $2x^2 + 7x - 4$ are $(2x - 1)$ and $(x+4)$

$x^2 + 3x - 4 = x(x + 4) - 1(x + 4)$

$= (x - 1)(x + 4)$

Hence the factors $x^2 + 3x - 4$ are $(x - 1)$ and $(x + 4)$

∴ the common factor of $2x^2 + 7x - 4$ and $x^2 + 3x - 4$ is $(x + 4)$

(b) Value of a in $7x^2 + ax - 8$.

Since $(x + 4)$ is the common factor of $2x^2 + 7x - 4$ and $x^2 + 3x - 4$; it implies that it a factor of $7x^2 + ax - 8$

Substituting for $x = -4$ in the equation $7x^2 + ax - 8$

$7(-4)^2 - 4a - 8 = 0$

$7 \times 16 - 4a - 8$

$112 - 8 - 4a = 0$

$a = 26$

6. Solve the equation $\sin 2\theta + \cos 2\theta \cos 4\theta = \cos 4\theta \cos 6\theta$ for $0 \leq \theta \leq \frac{\pi}{4}$.

$\sin 2\theta + \cos 2\theta \cos 4\theta = \cos 4\theta \cos 6\theta$

$\sin 2\theta = \cos 4\theta \cos 6\theta - \cos 2\theta \cos 4\theta$

$= \cos 4\theta(\cos 6\theta - \cos 2\theta)$

But $\cos P - \cos Q = -2\sin\frac{(P+Q)}{2}\sin\frac{(P-Q)}{2}$

⇨ $\sin 2\theta = -2\cos 4\theta\left[\sin\frac{(6\theta+2\theta)}{2}\ \sin\frac{(6\theta-2\theta)}{2}\right]$

$\sin 2\theta = -2\cos 4\theta \sin 4\theta \sin 2\theta$

$\sin 2\theta + 2\cos 4\theta \sin 4\theta \sin 2\theta = 0$

$\sin 2\theta(1 + 2\cos 4\theta \sin 4\theta) = 0$

$\sin 2\theta(1 + \sin 8\theta) = 0$

either

$\sin 2\theta = 0$

$2\theta = \sin^{-1}(0) = 0, \pi, 2\pi$

$\theta = 0, \frac{\pi}{2}, \pi$

or

$(1 + \sin 8\theta) = 0$

$\sin 8\theta = -1$

$8\theta = \sin^{-1}(-1) = 270^0$ or $\frac{3\pi}{2}$

$\theta = \frac{3\pi}{16}$

Hence $\theta = 0, \frac{3\pi}{16}$

7. Using small changes; show $(244)^{\frac{1}{5}} = 3\frac{1}{405}$.

Let $y = x^{\frac{1}{5}}$

$\frac{dy}{dx} = \frac{1}{5}x^{-\frac{4}{5}}$

$= \frac{1}{5x^{\frac{4}{5}}}$

$= \frac{1}{5\left(x^{\frac{1}{5}}\right)^4}$

$\delta y = \frac{1}{5\left(x^{\frac{1}{5}}\right)^4}\delta x$

Taking x = 243 and $\delta x = 1$

$$\delta y = \frac{1}{5\left(243^{\frac{1}{5}}\right)^4} . 1 = \frac{1}{405}$$

$$(x + \delta x) = y + \delta y\text{b}$$

$$= \sqrt[5]{243} + \frac{1}{405}$$

$$= 3 + \frac{1}{405}$$

$$= 3\frac{1}{405}$$

8. Three points A(2, -1, 0), B(-2, 5, -4) and C are on a straight line such that 3AB = 2AC. Find the coordinates of C.

3(AB) = 2AC

$$\frac{3}{2} AB = AC$$

$$\frac{3}{2}(OB - OA\,) = OC - OA$$

$$\frac{3}{2}\left[\begin{pmatrix}-2\\5\\-4\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}\right] = \begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}$$

$$\frac{3}{2}\begin{pmatrix}-4\\6\\-4\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}$$

$$\begin{pmatrix}-6\\9\\-6\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}$$

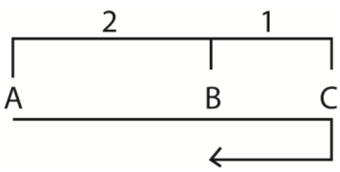
$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-6\\9\\-6\end{pmatrix} + \begin{pmatrix}2\\-1\\0\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-4\\8\\-6\end{pmatrix}$$

Hence coordinates of C are (-4, 8, -6)

Alternatively

Using ratio theorem



C divides externally in the ratio 3: -1

$$OC = \frac{3(OB) - 1(OA)}{3 + (-1)}$$

$$OC = \frac{1}{2}\left\{3\begin{pmatrix}-2\\5\\-4\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}\right\}$$

$$= \frac{1}{2}\begin{pmatrix}-8\\16\\12\end{pmatrix} = \begin{pmatrix}-4\\8\\-6\end{pmatrix}$$

Hence C(-4, 8, -6)

Alternatively

B divides AC internally in ration of 2:1

$$\begin{pmatrix}-2\\5\\-4\end{pmatrix} = \frac{1}{3}\left\{2\begin{pmatrix}x\\y\\z\end{pmatrix} + \begin{pmatrix}2\\-1\\0\end{pmatrix}\right\}$$

$$3\begin{pmatrix}-2\\5\\-4\end{pmatrix} = 2\begin{pmatrix}x\\y\\z\end{pmatrix} + \begin{pmatrix}2\\-1\\0\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{1}{2}\left\{\begin{pmatrix}-6\\15\\12\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}\right\}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-8\\16\\-12\end{pmatrix} = \begin{pmatrix}-4\\8\\-6\end{pmatrix}$$

Hence C(-4, 8, -6)

SECTION B (60MARKS)

Answer any five questions from this section. All questions carry equal marks

9. (a) If $z_1 = \frac{2i}{1+3i}$ and $z_2 = \frac{3+2i}{5}$, find $|z_1 - z_2|$

$z_1 - z_2 = \frac{2i}{1+3i} - \frac{3+2i}{5}$,

$= \frac{10i-(1+3i)(3+2i)}{5(1+3i)} = \frac{10i-[3+2i+9i-6]}{5(1+3i)}$

$= \frac{10i-11i+3}{5(1+3i)} = \frac{3-i}{5(1+3i)}$

$= \frac{(3-i)(3-i)}{5(1+3i)(3-i)} = \frac{3-9i-i-3}{5(1+9)} = \frac{-i}{5}$

$|z_1 - z_2| = \sqrt{0^2 - \left(-\frac{1}{5}\right)^2} = \frac{1}{5}$

Alterative 2

$z_1 = \frac{2i}{1+3i} = \frac{2i\,(1-3i)}{(1+3i)(1-3i)} = \frac{2i+6}{1+9} = \frac{2i+6}{10} = \frac{3+2i}{5}$

$z_1 - z_2 = \frac{3+2i}{5} - \frac{3+2i}{5} = \frac{-i}{5}$

$|z_1 - z_2| = \sqrt{0^2 - \left(-\frac{1}{5}\right)^2} = \frac{1}{5}$

(c) Given the complex number z = x + iy

(i) Find $\frac{z+i}{z+2}$

$\frac{z+i}{z+2} = \frac{x+i(1+y)}{(x+2)+iy}$

$= \frac{[x+i(1+y)][(x+2)-iy]}{[(x+2)+iy][(x+2)-iy]}$

$= \frac{x[(x+2)-iy]-i(1+y)[(x+2)-iy]}{(x+2)^2+y^2}$

$= \frac{x^2+2x-ixy+i(x+2+xy+2y)+y+y^2}{(x+2)^2+y^2}$

$= \frac{x^2+2x+y^2+y+i(2+x+2y)}{(x+2)^2+y^2}$

(ii) Show that the locus of $\frac{z+i}{z+2}$ is a straight line when its imaginary part is zero. State the gradient of the line.

If imaginary part is zero

$(2 + x + 2y) = 0$

$2y = -x - 2$

y = $-\frac{1}{2}x + 1$

comparing with y = mx + c

the gradient = $-\frac{1}{2}$

10. (a) solve the equation cos2x = $4\cos^2 x - 2\sin^2 x$ for $0 \leq \theta \leq 180^0$

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$\cos^2 x - \sin^2 x = 4\cos^2 x - 2\sin^2 x$

$3\cos^2 x - \sin^2 x = 0$

$4\cos^2 x - 1 = 0$

$(2\cos x + 1)(2\cos x - 1) = 0$

Either

$2\cos x + 1 = 0$

$\cos x = -\frac{1}{2}$

$x = \cos^{-1}(-\frac{1}{2}) = 120^0$

Or

$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$x = \cos^{-1}(\frac{1}{2}) = 60^0$

$\therefore x(60^0, 120^0)$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$= \frac{4}{2}(1 + \cos 2x) - \frac{2}{2}(1 - \cos 2x)$

$= 2 + 2\cos 2x - 1 + \cos 2x$

$2\cos 2x + 1 = 0$

$\cos 2x = -\frac{1}{2}$

$2x = \cos_{-1}(-\frac{1}{2}) = 120^0, 240^0$

$x = 60^0, 120^0$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$\cos^2 x - \sin^2 x = 4\cos^2 x - 2\sin^2 x$

$3\cos^2 x - \sin^2 x = 0$

$\text{Sin}^2 x = 3\cos^2 x$

$\tan^2 x = 3$

$\tan x = \pm\sqrt{3}$

Either

$\tan x = \sqrt{3}$

$x = \tan^{-1}\sqrt{3} = 60^0$

Or

tanx = $-\sqrt{3}$

x = $\tan^{-1} -\sqrt{3}$ = 120^0

Hence x = 60^0, 120^0

Alternatively

cos2x = 4cos^{2}x -2sin^{2}x

1-2sin^{2}x = 4(1-sin^{2}x)-2sin^{2}x

1 = 4 – 4sin^{2}x

4sin^{2}x = 3

sin^{2}x = $\frac{3}{4}$

sinx = $\pm\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

x = 60^0, 120^0

Alternatively

cos2x = 4cos^{2}x -2sin^{2}x

1-2sin^{2}x = 4cos^{2}x -2sin^{2}x

4cos^{2}x = 1

Cosx = $\pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

x = 60^0, 120^0

(b) Show that if sin(x + α) = psin(x – α) then tan x = $\left(\frac{p+1}{p-1}\right)$tanα. Hence solve the equation sin(x + α) = psin(x – α) for p = 2 and α= 20^0.

sinxcosα + coxsinα = p(sinxcosα - coxsinα)

cosxsinα (p + 1) = sinxcosα (p - 1)

cosxsinα$\left(\frac{p+1}{P-1}\right)$ = sinxcosα

$\frac{cosxsin\alpha}{\text{sinxcos}\alpha}\left(\frac{p+1}{p-1}\right) = \frac{sinxcos\alpha}{\text{sinxcos}\alpha}$

$tanx = \left(\frac{p+1}{P-1}\right)\tan\alpha$

For sin(x + 20^0) = 2sin(x – 20^0)

tanx = $\frac{2+1}{2-1}tan20^0$ = 3tan20^0

x = $\tan^{-1}(3tan20^0)$ = 47.52^0

11. Given that x = $\frac{t^2}{1+t^3}$ and y =$\frac{t^3}{1+t^3}$, find$\frac{d^2y}{dx^2}$.

$$x = \frac{t^2}{1+t^3}$$

$$\frac{dx}{dt} = \frac{2t(1+t^3)-3t^4}{(1+t^3)^2} = \frac{2t+2t^4-3t^4}{(1+t^3)^2} = \frac{2t-t^4}{(1+t^3)^2} = \frac{t(2-t^3)}{(1+t^3)^2}$$

$$y = \frac{t^3}{1+t^3}$$

$$\frac{dy}{dt} = \frac{3t^2(1+t^3)-t^3(3t^2)}{(1+t^3)^2} = \frac{3t^2+3t^5-3t^5}{(1+t^3)^2} = \frac{3t^2}{(1+t^3)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= \frac{3t^2}{(1+t^3)^2}\cdot\frac{(1+t^3)^2}{t(2-t^3)} = \frac{3t^2}{t(2-t^3)} = \frac{3t}{2-t^3}$$

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dx}{dt}$

$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{3(2-t^3)-3t(-3t^3)}{(2-t^3)^2} = \frac{6-3t^3+9t^3}{(2-t^3)^2} = \frac{6+6t^3}{(2-t^3)^2} = \frac{6(1+t^3)}{(2-t^3)^2}$

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dx}{dt}$

$= \frac{6(1+t^3)}{(2-t^3)^2} . \frac{(1+t^3)^2}{t(2-t^3)} = \frac{6(1+t^3)^3}{t(2-t^3)^3} = \frac{6}{t}\left(\frac{1+t^3}{2-t^3}\right)^3$

12. (a) Line A is the intersection of two planes whose equations are
3x –y + z = 2 and x + 5y +2z = 6. Find the equation of the line.

3x –y + z = 2 ………………….. (i)

x + 5y +2z = 6 ………………… (ii)

5eqn. (i) + eqn. (ii)

15x -5y +5z = 10

\+ x + 5y + 2z = 6

16x + 7z = 16

Let x = λ

16λ + 7z = 16

z = $\frac{1}{7}(16 - 16\lambda)$

Substituting for x and z in equation (i)

3λ – y + $\frac{1}{7}(16 - 16\lambda)$ = 2

21λ – 7y + 16 - 16λ = 14

y = $\frac{1}{7}(2 + 5\lambda)$

let λ = 1 + 7μ

=> x = 1 + 7μ

y = $\frac{1}{7}(2 + 5(1 + 7\mu))$ = $\frac{1}{7}(2 + 5 + 35\mu)$ = 1 + 5μ

z = $\frac{1}{7}(16 - 16(1 + 7\mu))$= $\frac{1}{7}(16 - 16 - 16\ x7\mu)$)= -16μ

$\underline{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}7\\5\\-16\end{pmatrix}$

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}7\\5\\-16\end{pmatrix}$

$\frac{x-1}{7} = \frac{y-1}{5} = \frac{-z}{16}$

(b) Given that line B is perpendicular to the plane 3x –y + z = 2and passes through the point C(1, 1, 0), find the

(i) Cartesian equation of line B

Normal to the plane b = 3i –j + k

r = a + λb

$= \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\1\end{pmatrix}$

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\1\end{pmatrix}$

$\frac{x-1}{3} = \frac{y-1}{-1} = \frac{z}{1}$

(ii) angle between line B and line A in (a) above

Let $b_1 = 7i + 5j - 16k$ and $b_2 = 3i - j + k$ and θ = angle between line A and line B

$b_1.b_2 = |b_1||b_2|cos\theta$

$b_1.b_2 = (7i + 5j - 16k).(3i - j + k)$

$= 21 - 5 - 16 = 0$

$|b_1||b_2|cos\theta = 0$

$\cos\theta = 0$

$\theta = \cos^{-1} 0 = 90^0$

13. (a) Find $\int \frac{1+\sqrt{x}}{2\sqrt{x}}dx$

Let $u = \sqrt{x}$

$u^2 = x$

$2udu = dx$

$=> \int \frac{1+\sqrt{x}}{2\sqrt{x}}dx = \int \frac{1+u}{2u}, 2udu = \int(i+u)du = u + \frac{1}{2}u^2 + c = \sqrt{x} + \frac{x}{2} + c$

Alternatively

Let $\sqrt{x} = tanu$

$\frac{1}{2\sqrt{x}} = sec^2udu$

$dx = 2\sqrt{x}sec^2udu$

$\int \frac{1+\sqrt{x}}{2\sqrt{x}}dx = \int \left(\frac{1+\tan u}{2\sqrt{x}}\right).2\sqrt{x}sec^2udu$

$= \int(1 + tanu)sec^2udu$

$= \int sec^2udu + \int tanusec^2udu$

$= tanu + \frac{1}{2}tan^2u + c$

$= \sqrt{x} + \frac{x}{2} + c$

(b) The gradient of the tangent at any point on a curve is $x - \frac{2y}{x}$. The curve passes through the point (2, 4), find the equation of the curve.

$\frac{dy}{dx} = x - \frac{2y}{x}$

$\frac{dy}{dx} + \frac{2y}{x} = x$

$the\ integrating\ factor\ \lambda = e^{\int\frac{2}{x}dx} = e^{2Inx} = x^2$

Multiplying through by λ

$x^2\frac{dy}{dx} + 2xy = x^3$

$main\ function = x^2y$

⇨ $\frac{d}{dx}(x^2y) = x^3$

$\int \frac{d}{dx}(x^2y)dx = \int x^3dx$

$x^2y = \frac{1}{4}x^4 + c$

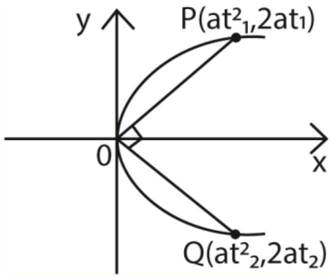
At (2,4)

4 x 4 = 4 + c => c = 12

$x^2y = \frac{1}{4}x^4 + 12$ or $y = \frac{1}{4}x^2 + \frac{12}{x^2}$ or $4x^2y = x^4 + 48$

14. (a) The point P($at_1^2, 2at_1$) and Q($at_2^2, 2at_2$) are on parabola $y^2 = 4ax$. OP is perpendicular to OQ, where O is the origin. Show that $t_1t_2 + 4 = 0$.

OP.OQ = 0



Gradient of OP, = $m_1 = \frac{2at_1}{at_1^2} = \frac{2}{t_1}$

Gradient of OQ $= m_2 = \frac{2at_2}{at_2^2} = \frac{2}{t_2}$

But $m_1m_2 = -1$

$\frac{2}{t_1} \cdot \frac{2}{t_2} = -1$

$t_1t_2 + 4 = 0$

Alternatively

OP.OQ = 0

$\begin{pmatrix} at_1^2 \\ at_1 \end{pmatrix} \begin{pmatrix} at_2^2 \\ at_2 \end{pmatrix} = 0$

$at_1^2 . at_2^2 + 2at_1 . 2at_2 = 0$

$aa^2t_1t_2(t_1t_2 + 4) = 0$

⇨ $t_1t_2 + 4 = 0$

Alternatively

$\overline{PQ}^2 = \overline{OP}^2 + \overline{OQ}^2$

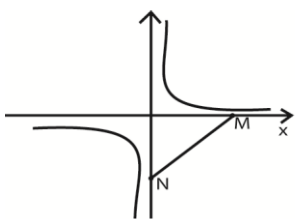
$a^2(t_2^2 - t_1^2) + 4a^2(t_2 - t_1)^2 = a^2t_1^4 + 4a^2t_1^2 + a^2t_2^4 + 4a^2t_2^2$

$\cancel{a^2t_2^4} - 2a^2t_1^2t_2^2 + \cancel{a^2t_1^4} + \cancel{4a^2t_2^2} - 8a^2t_1t_2 + \cancel{4a^2t_1^2} = \cancel{a^2t_1^4} + \cancel{4a^2t_1^2} + \cancel{a^2t_2^4} + \cancel{4a^2t_2^2}$

$-2a^2t_1^2t_2^2 - 8a^2t_1t_2 = 0$

$t_1t_2 + 4 = 0$

(b) The normal to the rectangular hyperbola xy = 8 at point (4, 2) meets the asymptotes at M and N. find the length MN.



The equation of the normal to a rectangular hyperbola $xy = c^2$ at a point (ct, $\frac{c}{t}$) is given by

$t^3x = ty + c(t^4 - 1)$

Comparing $xy = c^2$ with xy = 8

⇨ $c^2 = 8$; $c = 2\sqrt{2}$

Also comparing point $(ct, \ ^c/_t)$ with (4, 2)

⇨ ct = 4

$(2\sqrt{2})t = 4$

t $= \frac{4}{2\sqrt{2}} = \sqrt{2}$

Find the equation of the normal by substituting for c and t.

$(\sqrt{2})^3 = (\sqrt{2})y + 2\sqrt{2}\left[(\sqrt{2})^4 - 1\right]$

$(\sqrt{2})^2 = y + 2\left[(\sqrt{2})^4 - 1\right]$

$2x = y + 6$

y = 2x -6

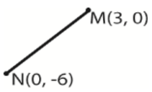
The normal drawn from the curve meets the asymptotes at the x-axis (M) and y-axis N as shown above

At point, y = 0

⇨ 2x = 6; x= 3, M(3, 0)

At point, x = 0

⇨ y =-6; N(0, -6)



$\overline{NM} = \sqrt{(3-0)^2 + (0-6)^2}$ = $3\sqrt{5}$ = 6.708units

Alternatively

y + $x\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{y}{x}$

At (4, 2)

$\frac{dy}{dx} - \frac{2}{4} = -\frac{1}{2}$

Hence gradient of normal at (4,2) is 2

Finding thee equation of the normal

$\frac{y-2}{x-2} = 2$

y = 2x- 6

Along y-axis at N, x = 0 => y = -6, N(0, -6)

Along y-axis at M, y = 0 => x = 3, M(3, 0)

$\overline{NM} = \sqrt{(3-0)^2 + (0-6)^2}$ = $3\sqrt{5}$ = 6.708units

15. (a) Prove by induction

1.3 + 2.4 + … + n(n + 2) = $\frac{1}{6}n(n+1)(2n+7)$ for all values of n.

Suppose n = 1

L.H.S = 1 x 3 = 3

R.H.S = $\frac{1}{6}x1(1+1)(2+7)=3$

L.H.S = R.H.S, hence the series holds for n =1

Suppose n = 2

L.H.S= 1 x 3 + 2 x 4 = 11

R.H.S = $\frac{1}{6}\ x\ 2(2+1)(4+7)=11$

L.H.S = R.H.S, hence the series holds for n =2

Suppose n = k

1.3 + 2.4 + … + k(k + 2) = $\frac{1}{6}k(k+1)(2k+7)$

For n = k + 1

1.3 + 2.4 + … + k(k + 2), (k + 1)(k + 3) = $\frac{1}{6}k(k+1)(2k+7)$+(k +1)(k + 3)

=(k + 1)$\left[\frac{1}{6}k(2k+7)+(k+3\right]$

= $\frac{1}{6}(k+1)(2k^2+13k+18)$

= $\frac{1}{6}(k+1)(2k^2+4k+9k+18)$

= $\frac{1}{6}(k+1(k+2)(2k+9)$

= $\frac{1}{6}(k+1)(k+2)[2(k+1)+7]$

Which is equal to R.H.S when n = k + 1

It holds for n = 1, 2, 3 …, hence it holds for all integral values of n.

(b) A man deposits Shs. 150,000 at the beginning of every year in a micro finance bank with the understanding that at the end of the seven years he is paid back his money with 5% per annum compound interest. How much does he receive?

Using amount, A = P$\left(1+\frac{r}{100}\right)^n$

=150000$\left(1\ +\ \frac{5}{100}\right)^7$=211,065.06

Alternatively

1st year

P = 150,000

He is paid back principal plus interest; P$\left(1\ +\ \frac{5}{100}\right)$ = 150,000$\left(1\ +\ \frac{5}{100}\right)$=157,500

2nd year

P = 157, 500

He is paid back principal plus interest; P$\left(1\ +\ \frac{5}{100}\right)$ = 157500$\left(1\ +\ \frac{5}{100}\right)$=165375

3rd year

P = 165375

Interest = $\frac{5}{10}\ x\ 165375=8268.75$

He is paid back principal plus interest; P$\left(1\ +\ \frac{5}{100}\right)$ = 165375 $\left(1\ +\ \frac{5}{100}\right)$=173643.75

4th year

P=173643.75

He is paid back principal plus interest; P$\left(1\ +\ \frac{5}{100}\right)$ = 173643.75$\left(1\ +\ \frac{5}{100}\right)$=182325.94

5th year

P =182325.94

His paid back principal plus interest; P$\left(1\ +\ \frac{5}{100}\right)$ = 182325.94$\left(1\ +\ \frac{5}{100}\right)$=191442.23

6^{th} year

P = 191442.23

He is paid back principal plus interest; $P\left(1 + \frac{5}{100}\right)$ = 191442.23$\left(1 + \frac{5}{100}\right)$=201014.35

7^{th} year

P = 201014.35

He is paid back principal plus interest; $P\left(1 + \frac{5}{100}\right)$ = 201014.35$\left(1 + \frac{5}{100}\right)$=211,065.06

∴ by the 7^{th} year he has accumulated shs. **211,065.06**

16. (a) If $x^2 + 3y^2 = k$, where k is constant, find $\frac{dy}{dx}$at the point (1, 2).

$x^2 + 3y^2 = k$

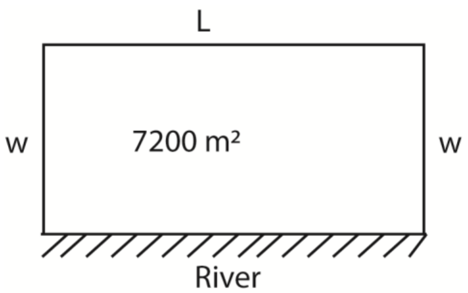
$2x + 6y\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-2x}{6y}$$

Substituting for (x, y) = (1, 2)

$\frac{dy}{dx} = \frac{-2}{6(2)} = \frac{-2}{12} = \frac{-1}{6}$

(b) A rectangular field of area 7200m2 is to be fenced using a wire mesh. On one side of the field. Is a straight river. This side is not to be fenced. Find the dimension of the field hat will minimize the amount of wire mesh to be used.



Area = 7200

Lw = 7200

$L = \frac{7200}{w}$ ……………………………….……. (i)

Perimeter, P = 2w + L ………………… (ii)

Substituting eqn. (i) into eqn. (ii)

$P = 2w + \frac{7200}{w}$

$\frac{dp}{dw} = 2 - \frac{7200}{w^2}$

Minimum perimeter occurs when $\frac{dp}{dw} = 0$

⇨ $2 - \frac{7200}{w^2} = 0$

$2w^2 = 7200$

$w^2 = 3600$

$w = \pm 60$ or w = 60

from eqn. (ii)

L= $\frac{7200}{60} = 120m$

Hence the dimensions are 60m x 120m

Thank you

Dr. Bbosa Science