Variable acceleration

This occurs when the rate of change of velocity is not constant.

Differential calculus

Let r = displacement, v = velocity and a = acceleration and are functions of time, t.

velocity, v = $\frac{dr}{dt}$ while acceleration, a = $\frac{dv}{dt} = \frac{d^2r}{dt^2}$

Differentiation; displacement => velocity => acceleration

Example 1

A particle moves along a straight line such that after t seconds, its displacement from a fixed point is r metres where r = $8t^2$i – t^4j. Find

(a) velocity after t seconds (b) velocity after 1s (c) speed after 1 s

Solution

$v_{(t=t)} = \frac{dr}{dt} = \frac{d(8t^2i - t^4j)}{dt}$

= $(16ti – 4t^3j)ms^{-1}$

(b) when t = 1

$v_{(t=1)}$= $(16i -4j)ms^{-1}$

(c) speed = $|v_{(t=1)}| = \sqrt{16^2 - (-4)^2}$

=$16.49ms^{-2}$

Example 2

A particle move along a straight line such that after t seconds its displacement from a fixed point is s in metres where s = 2sinti + 3costj. Find

(a) acceleration after t seconds (b) acceleration after $\frac{\pi}{2}s$ (c) magnitude of acceleration after $\frac{\pi}{2}s$

(a) $v_{(t=t)} = \frac{dr}{dt} = \frac{d(2\text{sinti} + 3\text{costj})}{dt}$

= $(2costi – 3sintj)ms^{-1}$

$a_{(t=t)} = \frac{dv}{dt} = \frac{d(2\text{costi} - 3\text{sintj})}{dt}$

=$(-2sinti – 3costj)ms^{-2}$

(b) when t = $\frac{\pi}{2}$s

$a_{(t=\frac{\pi}{2})}$ = $(-2\sin\left(\frac{\pi}{2}\right)i – 3\cos\left(\frac{\pi}{2}\right)j) = -2i\ ms^{-2}$

(c) $\left|a_{(t=\frac{\pi}{2})}\right| = \sqrt{(-2)^2 - 0^2}$ = $2ms^{-2}$

Example 3

A particle moves along a straight line such that after t seconds its displacement from a fixed point is s where $s = \begin{pmatrix} \sin 2t \\ t+1 \\ cost + \sin t \end{pmatrix}$. Find

(a) velocity when t = $\frac{\pi}{2}s$ (b) speed when t= $\frac{\pi}{2}s$ (c) acceleration when t =$\frac{\pi}{2}s$

Solution

$v_{(t=t)} = \frac{dr}{dt} = \frac{d[\sin 2ti + (t+1)j + (cost + sint)k]}{dt}$

=2cos2ti + j + (sint –cost)k

When t = $\frac{\pi}{2}s$

$v_{(t=\frac{\pi}{2})} = 2\cos\left(\frac{2\pi}{2}\right)i + j + (\sin\frac{\pi}{2} - \cos\frac{\pi}{2})k$

= (-2i + j –k)ms^{-1}

(b) $\left|v_{(t=\frac{\pi}{2})}\right| = \sqrt{(-2)^2 + 1^2 + (-1)^2}$

= 2.45ms^{-1}

(c) $a_{(t=t)} = \frac{dv}{dt} = \frac{d[2cos2ti + j + (sint - cost)k]}{dt}$

= -4sin2ti + (cost + sint)k

$a_{(t=\frac{\pi}{2})} = -4sin2\left(\frac{\pi}{2}\right)i + [\cos\frac{\pi}{2} + \sin\frac{\pi}{2}]k$

a = kms^{-2}

Example 4

A particle moves in x-y plane such that its position at any time t is given by r= $(3t^2\text{-}1)$i + $(4t^3 + t -1)$j.
Find (a) speed after time t = 2 (b) magnitude of acceleration after t = 2s

$v_{(t=t)} = \frac{dr}{dt} = \frac{d[(3t^2-1)i + (4t^3+t-1)]}{dt}$

= 6ti + $(12t^2 + 1)$j ms^{-1}

when t = 2s

$v_{(t=2)} = [6 \; x \; 2i + (12 \; x \; 2^2 + 1)j]ms^{-1}$

= [12i + 49j] ms^{-1}

speed = $\left|v_{(t=2)}\right| = \sqrt{12^2 + 49^2}$ = 50.45ms^{-1}

(b) $a_{(t=t)} = \frac{dv}{dt} = \frac{d[6ti + (12t^2+1)j]}{dt}$

= (6i + 24tj)ms^{-2}

when t = 2s

$a_{(t=2)}$= 6i =+ (24 x 2)j = (6i + 48i)ms^{-2}

$\left|a_{(t=2)}\right|$= $\sqrt{6^2 + 48^2}$ = 48ms^{-2}

Revision exercise 1

1. The position vector of a particle at any time (t) is given by r(t) = [$(t^2 + 4t)$I + $(3t - t^3)$j]m. Find the speed of the particle at t =3seconds. [26ms^{-1}]
2. The displacement of a particles after t seconds is given by r = t^3i + 9tj. Find the speed when t = 2s. [15ms^{-1}]
3. The displacement of a particle after t seconds is given by s = $2\sqrt{3}$ sinti + 8costj. find the speed when t = $\frac{\pi}{6}s$. [5ms^{-1}]
4. The displacement of a particle after t seconds is given by r = $8t^3$i + $2t^2$j. Find
   (i) acceleration when t = 1s [(6i+ 4j)ms^{-2}]
   (ii) magnitude of the acceleration when t = 1s [7.21ms^{-2}]

5. The velocity of a particle after t seconds is given by v = sin2ti – costj. Find
   (i) acceleration when t = $\frac{\pi}{6}s$ [(-2i + j)ms^{-2}]
   (ii) magnitude of acceleration when t = $\frac{\pi}{6}s$ [2.24ms^{-2}]
6. The velocity of a particle after t seconds is given by v = 2t^{2}i + 6j. find the magnitude of acceleration when t = 3. [12ms^{-2}]
7. A particle of mass 6kg moves such that its displacement s = $\begin{pmatrix} t^2 - 5 \\ t^2 - 3t + 2 \end{pmatrix}$m. Find the
   (a) velocity after time t [2ti + (2t – 3)j ms^{-1}]
   (b) speed of the particle at t = 2s [15ms^{-1}]
   (c) acceleration and hence determine the force acting on the particle [2i + 2j)ms^{-2}, (12i + 12j)N]
8. A particle of mass 2kg moves such that its displacement s = $\begin{pmatrix} t^2 - 4t - 5 \\ t^2 - 4t + 3 \end{pmatrix}$m. Find
   (a) speed of the particle at t= 2s [0ms^{-1}]
   (b) force acting on the particle [(4i + 4j)N]
9. A particle of mass 4kg moves such that its displacement s = (t^3 –t^2 – 4t + 3)I + (t^3 -2t^2 + 3t -7)j. Find the
   (i) the speed of the particle at t= 4s [50.21ms^{-1}]
   (ii) magnitude of the force acting on the particle when t = $\frac{2}{3}s$ [8N]
10. A particle of mass 0.5kgmoves such that its displacement r =$\begin{pmatrix} 4sin2t \\ 2cost - 1 \end{pmatrix}$m. Find the
    (a) velocity of the particle after t = $\frac{\pi}{6}s$ [(4i – j)ms^{-1}]
    (b) force acting on the particle at any time t [(-8sin2ti –costj)N]
11. A particle of mass 2kg moves such that its displacement r = (2 – cos3t)I + (6sin2t)j. Find the
    (a) velocity of the particle at t = $\frac{\pi}{6}s$[(3i + 6j)ms^{-1}]
    (b) force acting on the particle at t = πs[-18N]
12. A particle moves such that its displacement s = (2sint +sin2t)I + (4cost + cos2t)j. Find the
    (a) velocity of the particle at t = $\frac{\pi}{3}s$ [-3$\sqrt{3}jms^{-1}$
    (b) acceleration of the particle at t= $\frac{\pi}{2}s$ [(-2i + 4j)ms^{-2}]

Integral calculus

If r, v or a are function of time t;

velocity, v = $\int adt + c$ and displacement, r = $\int vdt + c$

integration; acceleration => velocity => displacement

Example 5

The velocity of the particle v = 3t^{2}i + 10tj. Given that the displacement is 4i- 4j at t = 0. Find the distance of the body from the origin when t = 2s.

r = $\int vdt + c$

$r_{(t=t)} = \int(3t^2i + 10tj)dt + C$

$r_{(t=t)} = t^3i + 5t^2j + C$

$At\ t = 0, r = 4i - 4j$

4i -4j = 0^{3}i + 5x 0^{2}j +C

c = 4i -4j

$r_{(t=t)} = [(t^3 + 4)i + (5t^2 - 4)j]$

when t = 2s

$r_{(t=2)} = [(2^3 + 4)i + (5x2^2 - 4)j]$ = (12i + 16j)m

$|r_{(t=2)}| = \sqrt{12^2 + 16^2}$ = 20m

Example 6

A particle is accelerated from rest at the origin with an acceleration of (2t + 4)ms^{-2}. Find

(a) velocity attained after t = 2s (b) distance travelled at t = 1s

Solution

v = $\int adt + c$

v = $\int (2t + 4)dt + c$

$v_{(t=t)} = 2t^2 + 4t + c$

At t = 0, v = 0

0 = $2(0)^2 + 4(0) + c$

c =0

$v_{(t=t)} = 2t^2 + 4t$

$v_{(t=2)} = 2x2^2 + 4x2$ = 12ms^{-1}

r = $\int vdt + c$

$r_{(t=t)} = \int (2t^2 + 4t\ )dt + c$

$r_{(t=t)} = \frac{t^3}{3} + 2t^2 + c$

At t = 0, r = 0

$0 = \frac{0^3}{3} + 2x\ 0^2 + c$

c = 0

$r_{(t=t)} = \frac{t^3}{3} + 2t^2$

$r_{(t=1)} = \frac{1^3}{3} + 2x1^2$ = 2.33m

Example 7

A particle is accelerated from rest at the origin with acceleration of (4t + 2)I – 3j. Find

(i) velocity attained after t = 3s (ii) speed after 3s

Solution

v = $\int adt + c$

v = $\int \{(4t + 2)i - 3j\}dt + c$

$v_{(t=t)} = (2t^2 + 2t)i - 3tj + c$

t = 0, v = 0

0 =$(2\ x\ 0^2 + 2\ x\ 0)i - 3\ x\ 0j + c$

c = 0

$v_{(t=t)} = (2t^2 + 2t)i - 3tj$

$v_{(t=3)} = (2\ x\ 3^2 + 2\ x\ 3)i - 3x\ 3j$

=24i -9j

Speed = $|v_{(t=3)}| = \sqrt{24^2 + (-9)^2}$ = 25.63ms^{-1}

Example 8

A particle starts from origin (0, 0). Its acceleration in ms^{-2} at time t seconds is given by a = 6ti -4j.

Find its speed after t =2s

v = $\int adt + c$

v = $\int (6ti - 4j)dt + c$

$v_{(t=t)}$ = $3t^2$i -4tj + c

At t = 0, v= 0

0 = 3 x 0^2i -4 x 0j + c

c = 0

$v_{(t=t)}$ =$3t^2$i -4tj

$v_{(t=2)}$ =3 x 2^2i -4 x 2tj

=12i- 8j

$v_{(t=2)} = \sqrt{12^2 + (-8)^2} = 14.42ms^{-2}$

Example 9

An object of mass 5kg is initially at rest at a point whose position vector is -2i+ j. If it is acted on by a force F = 2i + 3j -4k. Find

(i) acceleration

F = ma

$a_{(t=t)} = \frac{1}{5}(2\text{i} + 3\text{j} - 4\text{k})ms^{-2}$

(ii) speed after t= 3s

v = $\int adt + c$

$v_{(t=t)} = \frac{1}{5}\int(2\text{i} + 3\text{j} - 4\text{k}) + c$

$= \frac{1}{5}(2\text{ti} + 3\text{tj} - 4\text{tk}) + c$

At t = 0, v = 0

0 = $\frac{1}{5}(2 \text{ x } 0\text{i} + 3 \text{ x } 0\text{j} - 4 \text{ x } 0\text{k}) + c$

c =0

$v_{(t=t)} = \frac{1}{5}(2\text{ti} + 3\text{tj} - 4\text{tk})$

$v_{(t=3)} = \frac{1}{5}(2 \text{ x } 3\text{i} + 3 \text{ x } 3\text{j} - 4 \text{ x } 3\text{k})$

$= \frac{1}{5}(6\text{i} + 9\text{j} - 12\text{tk})$

$|v_{(t=3)}| = \frac{1}{5}\sqrt{6^2 + 9^2 + (-12)^2}$ = 3.23ms^{-1}

(iii) its distance from the origin after 3 seconds

r = $\int vdt + c$

$r_{(t=t)} = \frac{1}{5}\int(2\text{ti} + 3\text{tj} - 4\text{tk}) + c$

$r_{(t=t)} = \frac{1}{5}(\text{t}^2\text{i} + 1.5\text{t}^2\text{j} - 2\text{t}^2\text{k}) + c$

At t = 0, r = -2i+ j

-2i+ j = $\frac{1}{5}(0^2\text{i} + 1.5 \text{ x } 0^2\text{j} - 2 \text{ x } 0^2\text{k}) + c$

c = -2i+ j

$r_{(t=t)} = \frac{1}{5}\{(\text{t}^2 - 10)\text{i} + (1.5\text{t}^2 + 5)\text{j} - 2\text{t}^2\text{k}\}$

$r_{(t=3)} = \frac{1}{5}\left((3^2 - 10)\text{i} + (1.5 \text{ x } 3^2 + 5)\text{j} - 18\text{k}\right)$

$= \frac{1}{5}(-\text{i} + 18.5\text{j} - 2 \text{ x } 3^2\text{k})$

$|r_{(t=3)}| = \sqrt{(-1)^2 + (18.5)^2 + 18^2}$=5.166m

Example 10

A particle starts from rest at point (2, 0, 0) and moves such that its acceleration in ms-2 at time t seconds is given by a = [16cos4ti + 8sin2tj + (sint -2sin2t)k]ms^{-2}. Find the

(a) speed when t = $\frac{\pi}{4}$.

a = [16cos4ti + 8sin2tj + (sint – 2sin2t)k]ms^{-2}

v = $\int adt$

= $\int[16\text{cos4ti} + 8\text{sin2tj} + (\text{sint} - 2\text{sin2t})\text{k}]\text{dt}$

= [4sin4ti – 4cos2tj + (-cos t + cos2t)k]+ c

At t = 0

0 = [4sin0i – 4cos0j + (-cos 0 + cos0)k]+ c

0 = -4j + c

c = 4j

⇨ v = [4sin4ti + (-4cos2t + 4)j + (-cos t + cos2t)k

At t = $\frac{\pi}{4}$

⇨ v = [4sinπi + (-4cos$\frac{\pi}{2}$ + 4)j + (-cos $\frac{\pi}{4}$ + cos$\frac{\pi}{2}$)k

= 4j – cos $\frac{\pi}{4}$ k

$|v| = \sqrt{4^2 + \left(-\cos\frac{\pi}{4}\right)^2} = \sqrt{16 + \frac{2}{4}} = \sqrt{16.5}$ = 4.062ms^{-1}

(b) distance from the origin when t = $\frac{\pi}{4}$

s = $\int vdt$

=$\int[4\text{sin4ti} + (-4\text{cos2t} + 4)\text{j} + (-\cos\text{t} + \text{cos2t})\text{k}$

=-cos4ti +(-2sin2t + 4t)j + (-sint + $\frac{1}{2}sin2t)k$ + c

At t = 0, s = 2i

By substitution

2i = -cos0 i +(-2sin0 + 4(0))j + (-sin0 + $\frac{1}{2}sin2\ (0))k$ + c

2i = -I + c

c = 3i

⇨ s = (-cos4t + 3)i +(-2sin2t + 4t)j + (-sint + $\frac{1}{2}sin2t)k$

At t = $\frac{\pi}{4}$

⇨ s = (-cosπ + 3)i +(-2sin$\frac{\pi}{2}$t + π)j + (-sin$\frac{\pi}{4}$ + $\frac{1}{2}sin\ \frac{\pi}{2})k$

= 4i + (π -2)j + $\left(-\frac{\sqrt{2}}{2}+\ \frac{1}{2}\right)k$

= 4i + 1.416j – 0.207k

$|s|$=$\sqrt{4^2+\ (1.416)^2+\ (-0.207)^2}$

= 4.24828

= 4.248 (3D)

Vector approach of finding work and power

Work done by a variable force is a dot product

W= F.d or $W_{(t=t)} = \int Fvdt$

Since power, P = F.v

Example 11

A particle of mass 4kg starts frm rest at the origin. It is acted on by a force F = (2ti + 3t^{2}j + 5k)N

Find the work done by the force after 3 second

Solution

F = ma

$a_{(t=t)} = \frac{1}{4}(2\text{ti}\ +\ 3\text{t}^2\text{j}\ +\ 5\text{k})$

v = $\int adt$ + c

$v_{(t=t)} = \frac{1}{4}\int(2\text{ti}\ +\ 3\text{t}^2\text{j}\ +\ 5\text{k})\,dt + c$

$v_{(t=t)} = \frac{1}{4}(\text{t}^2\text{i}\ +\ \text{t}^3\text{j}\ +\ 5\text{tk}) + c$

At t = 0, v = 0

0 = $\frac{1}{4}(0^2\text{i}\ +\ 0^3\text{j}\ +\ 5\text{ x }0\text{k}) + c$

c = 0

$v_{(t=t)} = \frac{1}{4}(\text{t}^2\text{i}\ +\ \text{t}^3\text{j}\ +\ 5\text{tk})$

$v_{(t=3)} = \frac{1}{4}(3^2\text{i}\ +\ 3^3\text{j}\ +\ 5\text{x }3\text{k})$

$v_{(t=t)} = \frac{1}{4}(9\text{i}\ +\ 27\text{j}\ +\ 15\text{k})$

$\left|v_{(t=t)}\right| = \frac{1}{4}\sqrt{9^2+27^2+15^2}$

= $\frac{1}{4}\sqrt{1{,}025}ms^{-1}$

$W_{(t=0\ and\ t=3)}$ =$\frac{1}{2}m({v_{t=3}}^2 - {v_{t=0}}^2)$

= $\frac{1}{2}$ x 4 $\left(\frac{1}{16}\ x\ 1{,}025\ - 0\right)$ = 129.375J

Alternatively

$P_{(t=t)} = F.v$

$P_{(t=t)} = \begin{pmatrix} 2t \\ 3t^2 \\ t \end{pmatrix} . \frac{1}{4}\begin{pmatrix} t^2 \\ t^3 \\ 5t \end{pmatrix} = \frac{1}{4}(2t^3 + 3t^5 + 5t^2)$

$W_{(t=0 \ and \ t=3)} = \int_0^3 F.vdt$

$= \frac{1}{4}\int_0^3 (2t^3 + 3t^5 + 5t^2)\, dt$

$= \frac{1}{4}\left[\frac{1}{2}t^4 + \frac{1}{2}t^6 + \frac{25}{2}t^2\right]_0^3$

$= \frac{1}{4}\left\{\left(\frac{1}{2}x\ 3^4 + \frac{1}{2}\ x\ 3^6 + \frac{25}{2}x\ 3^2\right) - 0\right\}$

= 129.375J

Example 12

A particle of mass 3kg moves along a straight line such that after t seconds its position vector is s metres where s = $\begin{pmatrix} 25t + 4t^2 \\ 50 + 2t^2 \\ 15 + 3t^2 \end{pmatrix}$. Find

(a) Magnitude of force

$v_{(t=t)} = \frac{ds}{dt} = \frac{d}{dt}\begin{pmatrix} 25t + 4t^2 \\ 50 + 2t^2 \\ 15 + 3t^2 \end{pmatrix} = \begin{pmatrix} 25 + 8t \\ 4t \\ 6t \end{pmatrix} ms^{-1}$

$a_{(t=t)} = \frac{dv}{dt} = \frac{d}{dt}\begin{pmatrix} 25 + 8t \\ 4t \\ 6t \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} ms^{-2}$

$F = 3\begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 24 \\ 12 \\ 18 \end{pmatrix}$

$|F| = \sqrt{12^2 + 12^2 + 18^2}$

= 32.31N

(b) Power when t = 2s

$P_{(t=t)} = F.v$

$v_{t=2} = \begin{pmatrix} 25 + 8\ x\ 2 \\ 4\ x\ 2 \\ 6\ x\ 2 \end{pmatrix} = \begin{pmatrix} 41 \\ 8 \\ 12 \end{pmatrix} ms^{-1}$

$P_{(t=2)} = \begin{pmatrix} 24 \\ 12 \\ 18 \end{pmatrix} . \begin{pmatrix} 41 \\ 8 \\ 12 \end{pmatrix} = 1{,}296ms^2$

(c) Work done on the particle between t = 1s and t=2s

$W_{(t=1 \ and \ t=2)} = \frac{1}{2}m({v_{t=2}}^2 - {v_{t=1}}^2)$

$v_{(t=1)} = \begin{pmatrix} 25 + 8\ x1 \\ 4\ x\ 1 \\ 6\ x\ 1 \end{pmatrix} = \begin{pmatrix} 33 \\ 4 \\ 6 \end{pmatrix} ms^{-1}$

$|v_{(t=1)}| = \sqrt{33^2 + 4^2 + 6^2} = \sqrt{1141}$ms^{-1}

$|v_{(t=1)}| = \sqrt{41^2 + 8^2 + 12^2} = \sqrt{1889}$ms^{-1}

$W_{(t=1 \ and \ t=2)} = \frac{1}{2}m({v_{t=2}}^2 - {v_{t=1}}^2)$

$= \frac{1}{2}$ x 3 $(1889 - 1141)$ = 1122J

Alternatively

$P_{(t=1)} = F.v$

$= \begin{pmatrix} 24 \\ 12 \\ 18 \end{pmatrix} . \begin{pmatrix} 25 + 8t \\ 4t \\ 6t \end{pmatrix} = (600 + 348t)$

$W_{(t=1 \ and \ t=2)} = \int_1^2 F.vdt$

$= \int_1^2 (600 + 348t)dt$

$[600t + 174t^2]_1^2$

$= (12000 + 696) - (6000 + 174)$

= 1122J

Example 13

A particle of mass 10kg starts from rest at a point A with position vector (4i + 3j + 2k)m. It is acted on by a constant force, F = (8i + 4j + 6k)N causing it to accelerate to B after 4s. Find the

(a) Magnitude of acceleration

F= ma

$a_{(t=t)} = \frac{1}{10}((8i + 4j + 6k))ms^{-2}$

$|a_{(t=t)}| = \frac{1}{10}\sqrt{8^2 + 4^2 + 6^2}$

= $1.077ms^{-2}$

(b) velocity at any time t

$v_{(t=t)} = \int adt + c$

$= \frac{1}{10}\int(8i + 4j + 6k)\,dt$

$v_{(t=t)} = \frac{1}{10}(8ti + 4tj + 6tk) + c$

At t = 0; v=0

$0 = \frac{1}{10}(8\,x\,0i + 4\,x\,0j + 6\,x\,0k) + c$

c = 0

$v_{(t=t)} = \frac{1}{10}(8ti + 4tj + 6tk)$

(c) position vector point B

$r_{(t=t)} = \int vdt + c$

$r_{(t=t)} = \frac{1}{10}\int(8ti + 4tj + 6tk)\,dt + c$

$r_{(t=t)} = \frac{1}{10}(4t^2i + 2t^2j + 3t^2k) + c$

At t = 0, OA = (4i + 3j + 2k)

(2i + 3j + 2k) $= \frac{1}{10}(4\,x\,0^2i + 2x\,0^2j + 3\,x\,0^2k) + c$

c = (4i + 3j + 2k)

$r_{(t=t)} = \frac{1}{10}(4t^2i + 2t^2j + 3t^2k)$ +(4i + 3j + 2k)

$r_{(t=t)} = \frac{1}{10}\{(4t^2 + 40)i + (2t^2 + 30)j + (3t^2 + 2)k\}$

(d) the displacement vector AB

$$OB_{(t=4)} = \frac{1}{10}\begin{pmatrix}4\,x\,4^2 + 40\\ 2\,x\,4^2 + 30\\ 3\,x\,4^2 + 20\end{pmatrix} = \begin{pmatrix}10.4\\ 3.2\\ 6.8\end{pmatrix}m$$

$$\overline{AB} = OB - OA = \begin{pmatrix}10.4\\ 3.2\\ 6.8\end{pmatrix} - \begin{pmatrix}4\\ 3\\ 2\end{pmatrix} = \begin{pmatrix}6.4\\ 3.2\\ 4.8\end{pmatrix}m$$

(e) work done by the force F after 4s.

$$W_{(t=4)} = F.r = \begin{pmatrix}8\\ 4\\ 6\end{pmatrix}.\begin{pmatrix}6.4\\ 3.2\\ 4.8\end{pmatrix} = 92.8J$$

Alternative 1

$$v_{(t=4)} = \frac{1}{10}\begin{pmatrix}8\,x4\\ 4\,x\,4\\ 6\,x\,4\end{pmatrix} = \begin{pmatrix}3.2\\ 1.6\\ 2.4\end{pmatrix}ms^{-1}$$

$v_{(t=4)} = \sqrt{3.2^2 + 1.6^2 + 2.4^2} = \sqrt{18.56}ms^{-1}$

$W_{(t=0\ and\ t=4)} = \frac{1}{2}m\left({v_{(t=4)}}^2 - {v_{(t=0)}}^2\right)$

$= \frac{1}{2}x\ 10(18.56 - 0)$ =92.8J

Alternative 2

$P_{(t=t)} = F.v$

$$P_{(t=t)} = \begin{pmatrix}8\\ 4\\ 6\end{pmatrix}.\frac{1}{10}\begin{pmatrix}8t\\ 4t\\ 6t\end{pmatrix} = 11.6t$$

$W_{(t=0\ and\ t=4)} = \int F.vdt = \int_0^4 11.6tdt$

$=[5.8t^2]_0^4$

= [5.8 x 4^2 – 0]

= 92.8J

Example 14

A particle move along a curve such that after t seconds its position vector is r where r = $\begin{pmatrix} t+1 \\ \frac{10}{3}t^3 - 6 \\ 4 - \frac{3}{2}t^2 \end{pmatrix}$.

The particle is acted on by a force F = (2ti + t^2j -2tk)N. Find

(a) Power at any time t.

$$v_{(t=t)} = \frac{dr}{dt} = \frac{d}{dt}\begin{pmatrix} t+1 \\ \frac{10}{3}t^3 - 6 \\ 4 - \frac{3}{2}t^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 10t^2 \\ -3t \end{pmatrix} ms^{-1}$$

$$P_{(t=t)} = F.v = \begin{pmatrix} 2t \\ t^2 \\ -2t \end{pmatrix}.\begin{pmatrix} 1 \\ 10t^2 \\ -3t \end{pmatrix} = 2t + 10t^4 + 6t^2$$

(b) work done by the force in the interval t = 1s and t = 3s

$$W_{(t=1\ and\ t=3)} = \int_1^3 F.v dt = \int_1^3 2t + 10t^4 + 6t^2 dt$$

$$= [t^2 + 2t^3 + 2t^5]_1^3$$

$W = (9 + 54 + 486) - (1 + 2 + 2)$ = 544J

Example 15

A particle of mass 2kg has a displacement vector s = $\begin{pmatrix} 2t^2 \\ 4t \\ 8t^3 \end{pmatrix}$m and a force F= $\begin{pmatrix} 3t \\ 4t^3 \\ 5t \end{pmatrix}N$. Find

(a) work done at t = 2s

W = F.v

$$W_{(t=t)} = \begin{pmatrix} 3t \\ 4t^3 \\ 5t \end{pmatrix}.\begin{pmatrix} 2t^2 \\ 4t \\ 8t^3 \end{pmatrix} = (6t^3 + 56t^4)J$$

$W_{(t=2)} = (6\ x\ 2^3 + 56\ x\ 2^4)$ = 944J

(b) Power when t = 4s

$$v_{(t=t)} = \frac{ds}{dt} = \frac{d}{dt}\begin{pmatrix} 2t^2 \\ 4t \\ 8t^3 \end{pmatrix} = \begin{pmatrix} 4t \\ 4 \\ 24t^2 \end{pmatrix} ms^{-1}$$

$$P_{(t=t)} = F.v = \begin{pmatrix} 3t \\ 4t^3 \\ 5t \end{pmatrix}.\begin{pmatrix} 4t \\ 4 \\ 24t^2 \end{pmatrix} = (12t^2 + 136t^3)W$$

$P_{(t=4)} = (12\ x\ 4^2 + 136\ x\ 4^3)W$ = 8,896W

Example 16

1. A force F = (2t i + j – 3t k)N acts on a particle of mass 2kg. The particle is initially at a point (0,0,0) and moving with a velocity (i + 2j-k)ms^{-1}. Determine the:

   (a) Magnitude of the acceleration of the particle after 2 seconds (04marks)

   F = (2t i + j – 3t k) = $\begin{pmatrix} 2t \\ 1 \\ -3t \end{pmatrix}$N

$a = \frac{F}{m} = \frac{1}{2}\begin{pmatrix} 2t \\ 1 \\ -3t \end{pmatrix} = \begin{pmatrix} t \\ 0.5 \\ -1.5t \end{pmatrix} ms^{-1}$

At t = 2s

$\underline{a}$=2i + 0.5j – 3k

$|\underline{a}| = \sqrt{2^2 + 0.5^2 + (-3)^2}$ = 3.64ms^{-2}

(b) Velocity of the particle after 2seconds (04marks)

$\underline{v} = \int \underline{a}dt = \int \begin{pmatrix} t \\ 0.5 \\ -1.5t \end{pmatrix} dt = \begin{pmatrix} 0.5t^2 \\ 0.5t \\ -7.5t^2 \end{pmatrix} + C$

At t = 0 initial velocity = (i + 2j –k)

$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + C$ => $C = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$\therefore \underline{v} = \begin{pmatrix} 0.5t^2 + 1 \\ 0.5t + 2 \\ -7.5t^2 - 1 \end{pmatrix}$

At t = 2s

$\underline{v} = \begin{pmatrix} 0.5(2)^2 + 1 \\ 0.5(2) + 2 \\ -7.5(2)^2 - 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix} ms^{-1}$

(c) Displacement of the particle after 2 seconds (04marks)

$\underline{r} = \int \underline{v}dt \int \begin{pmatrix} 0.5t^2 + 1 \\ 0.5t + 2 \\ -7.5t^2 - 1 \end{pmatrix} dt = \begin{pmatrix} \frac{t^3}{6} + t \\ \frac{t^2}{4} + 2t \\ -t - \frac{t^3}{4} \end{pmatrix} + C$

At t = 0; $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + C$

$C = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\underline{r} = \begin{pmatrix} t^3 + t \\ \frac{t^2}{4} + 2t \\ -t - \frac{t^3}{4} \end{pmatrix}$

At t = 2s

$\underline{r} = \begin{pmatrix} \frac{2^3}{6} + 2 \\ \frac{2^2}{4} + 2x2 \\ -2 - \frac{2^3}{4} \end{pmatrix} = \begin{pmatrix} \frac{10}{3} \\ 5 \\ -4 \end{pmatrix} m$

Example 17

1. A particle of mass 4kg starts from rest at point (2i- 3j + k)m. it moves with acceleration a= (4i + 2j -3k)ms^{-2} when a constant force F acts on it.
   Find the:
   (a) Force F (02marks)
   F = ma
   $= 4\begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 16 \\ 8 \\ -12 \end{pmatrix}$N or F = (16i + 8j -12k)N
   (b) Velocity at any time t. (04marks)
   $v = \int adt$
   v = at + c
   $= \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} t + c$
   At time t = 0, v = u = c = $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
   Substituting for c
   $v = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} t = \begin{pmatrix} 4t \\ 2t \\ -3t \end{pmatrix} ms^{-1}$
   or v = (4t i + 2t j -3t k)ms^{-1}
   (c) Work done by the force F after 6 seconds (06marks)
   Work done = force (F) x distance ($\underline{r}$)

$\underline{r} = \int v dt = \int \begin{pmatrix} 4t \\ 2t \\ -3t \end{pmatrix} dt = \begin{pmatrix} 2t^2 \\ t^2 \\ -1.5t^2 \end{pmatrix} + c$

$At\ t = 0, \underline{r} = c = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$Hence\ \underline{r} = \begin{pmatrix} 2t^2 \\ t^2 \\ -1.5t^2 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

After time t= 6seconds

$\underline{r} = \begin{pmatrix} 2(6)^2 \\ (6)^2 \\ -1.5(6)^2 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 74 \\ 33 \\ -53 \end{pmatrix}$

Work done = force (F) x distance ($\underline{r}$)

$= \begin{pmatrix} 16 \\ 8 \\ -12 \end{pmatrix} x \begin{pmatrix} 74 \\ 33 \\ -53 \end{pmatrix}$

= 1184 + 264 + 636 = 2,084J

Revision exercise

1. A particle of mass 3kg is acted on by a force F = (24t3i + (36t -16)j + 12k)N at time t. at time t = 0, the particle is at the pint with position vector (3, -1, 4) and moving with velocity (16i + 15j -8k)ms^{-1}. Determine the
   (i) acceleration of the particle at time t= 2s. [28.4253ms^{-2}]
   (ii) speed of the particle at t= 2s [42.9534ms^{-1}]
   (iii) distance of the particle from the origin at t = 2s [56.5155m]
2. A particle of mass 4kg is acted on by a force F = (6i -36t^{2}j + 54tk)N at time t. At time t = 0, the particle is the point with position vector (i -5j –k) and it velocity is (3i + 3j)ms^{-1}. Determine the
   (a) position vector of the particle a time t = 1s. $\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$m.
   (b) Distance of the particle from the origin a time t = 1s [6.1644m]

3. The acceleration f a particle is 6ti + 2j. Given that the velocity is (4i –j)ms^{-1} and displacement is (2i + 3j)m when t = 1s. Find the displacement when t =3s [(30i +5j)m]
4. If the velocity of a particle is 4cos2ti + 2sin2tj, given the displacement is 6i – 2j when t = $\frac{\pi}{4}s$. Find the distance of the body from the origin when t = πs. [5m]
5. If the acceleration of a particle is 9sin3ti + 2costj and the body id initially at rest. Find its velocity when t = $\frac{\pi}{6}s$. [(3i +j)ms^{-1}]
6. If the acceleration of a particle is 6sin6ti + 9cos3tj, given that the velocity is (i + 3j)ms^{-1} and displacement is (5i +2j)m when t = $\frac{\pi}{6}s$. Find the displacement when t = $\frac{\pi}{6}s$. [(5i +3j)m]
7. If the acceleration of the particle is 6ti + 6j -2k, given that the velocity is (3i + 6j -3k)ms^{-1} and displacement is (2i + 5j -2k)m when t =1s. Find
   (a) velocity when t= 2s [(12i + 12tj -5k)ms^{-1}]
   (b) displacement when t = 3s [(28i + 29j – 12k)m
8. If the acceleration f a particle is 2i +6j + 12t^{2}k, given that the velocity is (3i + k)ms^{-1} and displacement is (-i + k)m when t = 1s. Find the
   (i) velocity when t = 2s [(-i + 3j + 5k)
   (ii) displacement when t = 2s [(-3i + 8j + 19k)m
9. If the acceleration of a particle is (6ti – 2k)ms^{-2}, given that the velocity is (i + 12j -4k)ms^{-1} and the displacement is (3i + 6j)m when t = 2s, Find the
   (i) velocity when t = 4s [37i + 12j -8k)ms^{-1}
   (ii) displacement when t = 3s [11i + 18j -5k)m

10. The velocity of a particle v = $4t^3$i + 6tj – $3t^2$k. Given that the displacement is 14i + 6j-3k at t = 1s. Find the
    (i) acceleration when t = 5s [(108i + 6j -18k)ms^{-2}
    (ii) displacement when t = 0 [(13i + 3j – 2k)m]
11. The velocity of a particle v = ($3t^2$i – 10t)i + 2j – 6tk. Given that the displacement is -9i + 3j – 13k at t = 2s. Find the
    (i) acceleration when t = 5s [(20i – 6k)ms^{-2}]
    (ii) displacement when t = 3s [15i + 5j -28k)m]
12. The particle starts from rest at the origin moving with velocity of $v = \begin{pmatrix} 2cos2t + 11 \\ 3sin3t \\ 4 \end{pmatrix}$. Find the
    (i) speed when t = $\frac{\pi}{6}s$. [13ms^{-1}]
    (ii) displacement when t = $\frac{\pi}{2}s$. [(1.5πi + j -2πk)m]
    (iii) acceleration when t = πs [9jms^{-2}]

Thank you
Dr. Bbosa Science