Centre of gravity

This is the point where the resultant force due to attraction acts

General formula for the centre of gravity

Consider a system of particles of weight, w_1, w_2 …. w_n located at points with coordinates (x_1, y_1), (x_2, y_2) ….. (x_n, y_n) in the x-y plane

The resultant of weight $w_1 + w_2 + …. + w_n$ have a C.O.G at point $(\bar{x}, \bar{y})$
Taking moments along the y-axis

$(w_1 + w_2 + …. + w_n)\bar{x} = w_1x_1 + w_2x_2 + …. + w_nx_n$

$$\bar{x} = \frac{\sum w_1 x_i}{\sum w_i}$$

Similarly taking moments along the x-axis

$$\bar{y} = \frac{\sum w_1 x_i}{\sum w_i}$$

Alternatively:

$$\binom{\bar{x}}{\bar{y}} = \frac{\left[w_1\binom{x_1}{y_1} + w_2\binom{x_2}{y_2} + \cdots + w_n\binom{x_n}{y_n}\right]}{(w_1 + w_2 + \cdots + w_n)}$$

Or

$$\binom{\bar{x}}{\bar{y}} = \frac{\left[m_1 g\binom{x_1}{y_1} + m_2 g\binom{x_2}{y_2} + \cdots + m_n g\binom{x_n}{y_n}\right]}{(m_1 g + m_2 g + \cdots + w_n g)}$$

Example 1

Find the position of the centre of gravity of three particles of masses 1kg, 5kg, and 2g which lie on the y-axis at points (0, 2), (0, 4) and (0,5) respectively

$$\binom{\bar{x}}{\bar{y}} = \frac{\left[m_1 g\binom{x_1}{y_1} + m_2 g\binom{x_2}{y_2} + \cdots + m_n g\binom{x_n}{y_n}\right]}{(m_1 g + m_2 g + \cdots + w_n g)}$$

$$\binom{\bar{x}}{\bar{y}} = \frac{\left[1g\binom{0}{2} + 5g\binom{0}{4} + 2g\binom{0}{5}\right]}{(1g + 5g + 2g)}$$

$$\binom{\bar{x}}{\bar{y}} = \frac{\binom{0}{32}}{8} = \binom{0}{4}$$

Example 2

Find the coordinates of the centre of gravity of four particles of masses 5kg, 2kg, 2kg and 3kg which are situated at (3, 1), (4, 3), (5, 2) and (-3, 1) respectively

$$\binom{\bar{x}}{\bar{y}} = \frac{\left[m_{1g}\binom{x_1}{y_1}+m_{2g}\binom{x_2}{y_2}+\cdots+m_n g\binom{x_n}{y_n}\right]}{(m_1 g+ m_2 g+\cdots+w_n g)}$$

$$\binom{\bar{x}}{\bar{y}} = \frac{\left[5g\binom{3}{1}+2g\binom{4}{3}+2g\binom{5}{2}+3g\binom{-3}{1}\right]}{(5g+ 2g+2g+3g)}$$

$$\binom{\bar{x}}{\bar{y}} = \frac{\binom{24}{18}}{12} = \binom{2}{1.5}$$

Example 3

Three particles of masses 2kg, 1kg and 3kg are situated at (4, 3), (1, 0) and (a, b) respectively. If the centre of gravity of the system lies at (0,2), find the values of a and b.

$$\binom{\bar{x}}{\bar{y}} = \frac{\left[m_{1g}\binom{x_1}{y_1}+m_{2g}\binom{x_2}{y_2}+\cdots+m_n g\binom{x_n}{y_n}\right]}{(m_1 g+ m_2 g+\cdots+w_n g)}$$

$$\binom{0}{2} = \frac{\left[2g\binom{4}{3}+1g\binom{1}{0}+3g\binom{a}{b}\right]}{(2g+ 1g+3g)}$$

$$\binom{0}{2} = \frac{\binom{9+3a}{6+3b}}{6}$$

$0 = \frac{9+3a}{6}$, a = -3

$2 = \frac{6+3b}{6}$,

b = 2

Example 4

Find the coordinates of the centre of gravity of four particles of weight, 6N, 5N, 4N and 7N which are situated at (i +2j), (2i), (3j) and (4i + 2j) respectively.

$$\binom{\bar{x}}{\bar{y}} = \frac{\left[w_1\binom{x_1}{y_1}+w_2\binom{x_2}{y_2}+\cdots+w_n\binom{x_n}{y_n}\right]}{(w_1+ w_2+\cdots+w_n)}$$

$$\binom{\bar{x}}{\bar{y}} = \frac{\left[6\binom{1}{2}+5\binom{2}{0}+4\binom{0}{3}+7\binom{4}{2}\right]}{(6+5+4+7)} = \frac{\binom{44}{38}}{22}$$

$(\bar{x},\ \bar{y})$ = (2, 1.73)

Example 5

The rectangle EFGH has EF= 3m and EH = 2m, particles of masses 20g, 30g, 60g and 10g are placed at the mid-point of the sides EF, FG, GH and EH respectively. Find the distance of the centre of gravity of the system from each of the line EF and EH

$$\binom{\bar{x}}{\bar{y}} = \frac{\left[0.02\binom{1.5}{0}+0.03\binom{3}{1}+0.06\binom{1.5}{2}+0.01\binom{0}{1}\right]}{(0.02+0.03+0.06+0.01)} = \frac{\binom{0.21}{0.16}}{0.12}$$

$(\bar{x}, \bar{y})$ = (1.75, 1.33)

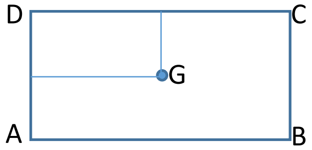
Revision exercise 1

1. Find the coordinates of the centre of gravity of four particles of masses 60g, 30g, 70g and 40g which are situated at (4i + 3j), (6i + 5j), (-6i + 5j) and (-5i – 2j) respectively. [-1, 3]
2. The rectangle ABCD has AB = 4cm and AD = 2cm, particles og masses 3kg, 5kg, 1kg and 7kg are placed at point points A, B, C and D respectively. Find the distance of the centre of gravity of the system from each line AB and AD. [1cm, 1.5cm]
3. Find the position of the centre of gravity of four particles of masses 5kg, 6kg, 2kg and 2kg which are situated at (5i – 7j), (-3i + 2j), (3i – 5j) and (i -6j) respectively. [(i - 3j)]
4. Particles of weight 1N, 2N, 3N and 4N are situated at (6i), (i – 5j), (3i + 2j) and (ai + bi) respectively. If the centre of gravity of the system lies at the points with position vector (2.5i – 2j), find the values of a and b. [2, -4]

5. Find the position of the centre of gravity of four particles of weight 2N, 1N, 5N and 2N which are situated at (4, -5), (1, 2), (3, -6) and (0, 3) respectively. [2.4, -3.2]
6. Particles of mass 1kg, 2kg and mkg are situated at (5, 2), (1, 5) and (1, -2) respectively. If the centre of gravity of this system lies at (2, $\bar{y}$), find the value of m and $\bar{y}$. [m = 1kg, $\bar{y} = 2.5$]
7. Particles of masses 2kg, 1kg and 3kg are situated on y-axis at (0, 7), (0, 4), and (0, -2) respectively. Where must a 6kg mass be placed to ensure that the centre of gravity of this system lies at the origin. [(0, -2)]
8. Particles of weight 5N, 4N and 3N are situated at (-5, 0), (4, 0.5), (-4, -3) respectively. Where must a 7N particles be placed to ensure that the centre of gravity of this system lies at the origin (3,1)

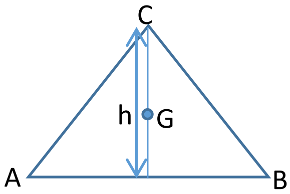
Centre of gravity of lamina

1. C.O. G of a rectangle



G = $\left(\frac{AB}{2}, \frac{AD}{2}\right)$

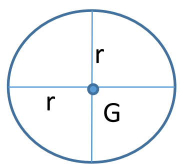
2. C.O. G of an isosceles triangle



G = $\left(\frac{AB}{2}, \frac{h}{3}\right)$

C.O.G lies along the line of symmetry a distance $\frac{h}{3}$

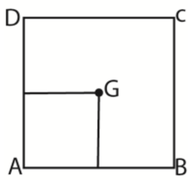
3. C.O. G of a circle



C.O.G lies along the centre

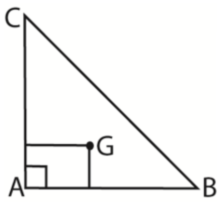
G = (r, r)

4. C.O.G of a square



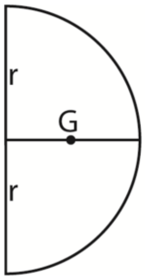
G = $\left(\frac{AB}{2}, \frac{AD}{2}\right)$

5. C.O.G right angled triangle



G = $\left(\frac{AB}{3}, \frac{AC}{3}\right)$

6. C.O.G of a semi-circle



7. C.O.G of **a sector of a circle** subtending an angle 2α at the centre line along the line of symmetry at a distance $\frac{2rsin\alpha}{3\alpha}$ from the centre.

8. C.O.G of a circular arc subtending an angle 2α at the centre line along the line of symmetry at a distance $\frac{rsin\alpha}{\alpha}$ from the centre.

C.O.G a semi-circular arc of radius r is at a distance $\frac{2r}{\pi}$ from the centre.

Example 6

Find the position of the centre of gravity of a uniform lamina in form of a triangle whose co-ordinates are

(i) (0, 0), (2, 6) and (4, 0)

$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{3}(0 + 2 + 4,\ 0 + 6 + 0)$ = [(2,2)]

(ii) (0, 3), (3, 0) and (6,3) [3,2]

(iii) (0, 0), (0,6) and (6,0) [2, 2]

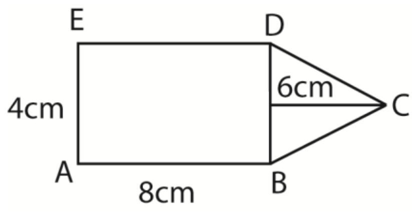
(iv) (0, 0), (0, 6) and (3,0) [1,2]

(v) (1, 0), (5, 0) and (0, 6) [2,2]

(vi) (3, 0), (6,0) and (0,6) [3, 2]

Example 7

The figure below shows a lamina formed by joining together a rectangular solid and triangular solid. Find the C.O.G of the composite lamina from AE and AB.



Lamina	Area (cm^2)	Weight	Distance of C.O.G from	
			AE	AB
ABDE	32	32w	4	2
BCD	12	12w	10	2
Composite	44	4w	$\bar{x}$	$\bar{y}$

C.O.G from AE

44w$\bar{x}$ = 32w x 4 + 12w x 10

$\bar{x}$ =5.64cm

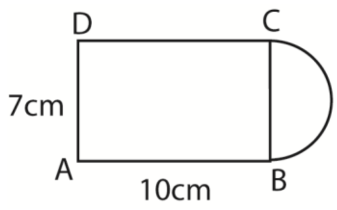
C.O.G from AB

44w$\bar{y}$ = 32w x 2 + 12w x 2

$\bar{y}$ =2cm

Example 8

The figure below shows a lamina formed b joining together a rectangular solid to a semi-circular solid. Find the C.O.G of the composite lamina



Lamina	Area (cm^2)	Weight	Distance of C.O.G from	
			AE	AB
ABCD	70	70w	5	3.5
Semi-circle	19.24	19.24w	10+ $\frac{4 \, x \, 3.5}{3\pi}$ =11.49	3.5
Composite	89.24	89.24w	$\bar{x}$	$\bar{y}$

C.O.G from AD

89.24w$\bar{x}$ = 19.24w x 11.49 + 70w x 5

$\bar{x}$ =6.4cm

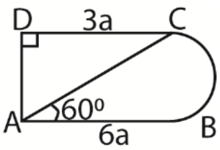
C.O.G from AB

89.24w$\bar{y}$ = 19.24w x 3.5 + 70w x 3.5

$\bar{y}$ =3.5cm

Example 9

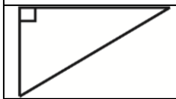
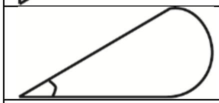
The figure below shows a uniform lamina consisting of a sector ABC of a circle centre A and of radius 6a and a triangle ADC where angle ADC = 90^0 and CAB = 60^0.



Show that the distance of the centre of gravity of the composite body from Ad is $\frac{27a\sqrt{3}}{4\pi+3\sqrt{3}}$

Solution

AD = $\sqrt{(6a)^2-(3a)^2} = 3a\sqrt{3}$; let w = weght per unit area

Portion	Area	Weight	C.O.G from AD
	$\frac{1}{2}x3a\ x3a\sqrt{3}$ $=4.5\sqrt{3a^2}$	$4.5\sqrt{3a^2}$w	$\frac{3a}{3}=a$
	$\frac{60^0}{360^0}\pi(6a)^2$ $=6\pi a^2$	$6\pi a^2 w$	$\frac{2\ x6a\ x\sin 30^0}{3\ x\frac{30^0\pi}{180^0}}\cos 30^0=\frac{6a\sqrt{3}}{\pi}$
Composite	$(4.5\sqrt{3a^2}+6\pi a^2)$	$(4.5\sqrt{3a^2}+6\pi a^2)$w	$\bar{x}$

$\bar{x}\left(\frac{9}{2}\sqrt{3}+6\pi\right)a^2w = 6\pi a^2w\ x\ \frac{6a\sqrt{3}}{\pi}+\frac{9}{2}\sqrt{3a^2}w\ x\ a$

$\bar{x}(9\sqrt{3}+12\pi) = 81a\sqrt{3}$

$\bar{x} = \frac{81a\sqrt{3}}{12\pi+9\sqrt{3}} = \frac{27a\sqrt{3}}{4\pi+3\sqrt{3}}$

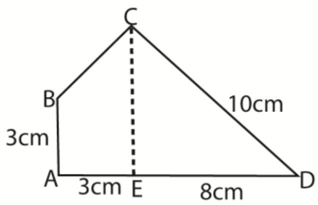
Example 10

Find the position of the centre of gravity of a uniform lamina in form of a triangle whose coordinates are (2, 2), (4, 6) and (0, 3)respectively

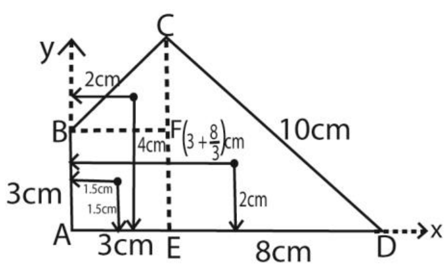
$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{3}(2+4+0, 2+6+3)$ = $\left(2,\ \frac{11}{3}\right)$

Example 11

Find the coordinates of the centre of the mass of the lamina shown below. Take A as the origin and AD, AB as the x – and y – axes respectively



Solution



Let m = mass per cm^2

Let $(\bar{x}, \bar{y})$= coordinates of C.O.G of composite

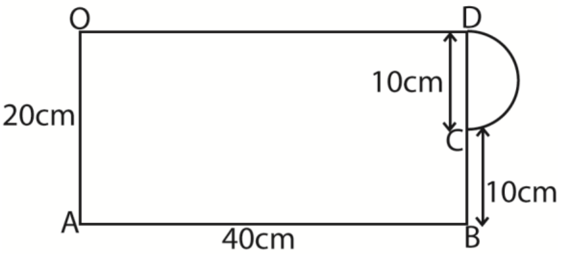
Portion	Area (cm^2)	Mass	C.O.G from AB (cm)	C.O.G from AD (cm)
ABFE	9	9m	1.5	1.5
BFC	4.5	4.5m	2	4
EDC	24	24m	$\left(3+\frac{8}{3}\right)=\frac{17}{3}$	2
ADCB	37.5	37.5m	$\bar{x}$	$\bar{y}$

$$\binom{\bar{x}}{\bar{y}} 37.5m = 9m\binom{1.5}{1.5} + 4.5m\binom{2}{4} + 24m\binom{\frac{17}{3}}{2}$$

$\bar{x} = 4.227cm$ and $\bar{y} = 2.12cm$

Example 12

The figure below is a lamina formed by welding together a rectangular metal sheet and a semi-circular meal sheet.



Find the position of the centre of gravity from the side OA and AB

Let w = weight per cm^2

Lamina	Weight	Distance of C.O.G from	
		OA	AB
ABDO	800w	20	10
Semi-circle	39.27w	$20 + \frac{3 \times 5}{8}$ =21.875	15
Composite	839.27w	$\bar{x}$	$\bar{y}$

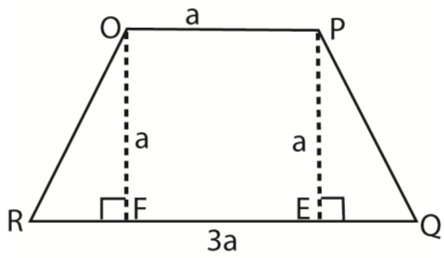
839.27w$\bar{x}$ = 800 w x 20 + 39.27w x 21.875; $\bar{x}$ = 20.088cm

839.27w$\bar{y}$ = 800 w x 10 + 39.27w x 15; $\bar{y}$ = 10.234cm

∴ Position of C.O.G from OA and AB is 20.088 cm and 10 .234cm respectively

Example 13

The figure OPQR below shows a metal sheet of uniform material cut in a shape of a trapezium; OP= a, RQ = 3a and the vertical height of P from RQ = a.



Calculate the centre of gravity of the mass from R and RQ

Let w = weight per cm^2

Lamina	Weight	Distance of C.O.G from	
		R	RQ
ROF	$\frac{1}{2}a^2$w	$\frac{2a}{3}$	$\frac{a}{3}$
OPFE	a^2w	$\frac{2a}{3}+\frac{a}{2}=\frac{7a}{6}$	$\frac{a}{2}$
PEQ	$\frac{1}{2}a^2$w	$\frac{7a}{6}+\frac{a}{3}=\frac{9a}{6}$	$\frac{a}{3}$
Composite	$2a^2$w	$\bar{x}$	$\bar{y}$

$2a^2w\bar{x} = \frac{1}{2}a^2$ w x $\frac{2a}{3}$ + a^2w x $\frac{7a}{6} + \frac{1}{2}a^2w\ x\ \frac{9a}{6}$ = $\left(\frac{2a^3}{6}+\frac{7a^3}{6}+\frac{9a^3}{6}\right)w = \frac{18a^3}{6}w;$

$\bar{x} = \frac{18a^3}{12a^2} = \frac{3a}{2}$

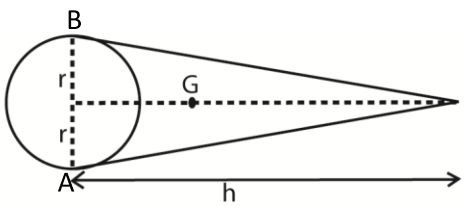
$2a^2w\bar{y} = \frac{1}{2}a^2$ w x $\frac{a}{3}$ + a^2w x $\frac{a}{2} + \frac{1}{2}a^2w\ x\ \frac{a}{3}$ = $\left(\frac{a^3}{6}+\frac{a^3}{2}+\frac{a^3}{6}\right)w = \frac{5a^3}{6}w$

$\bar{x} = \frac{5a^3}{12a^2} = \frac{5a}{12}$

∴ Centre of gravity is $\left(\frac{3a}{2},\frac{5a}{12}\right)$

Centre of gravity of solids

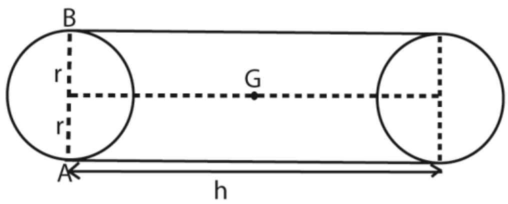
1. C.O.G of a solid cone, Pyramid and tetrahedron



C.O.G lies along the line of symmetry at a distance $\frac{h}{4}$ from AB

G = $\left(\frac{h}{4},r\right)$

2. C.O.G of solid cylinder



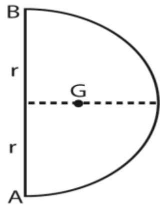
C.O.G lies along the line of symmetry at a distance $\frac{h}{2}$ from AB

G = $\left(\frac{h}{2},r\right)$

3. C.O.G of solid hemisphere

C.O.G lies along the line of symmetry at a distance $\frac{3r}{8}$ from AB

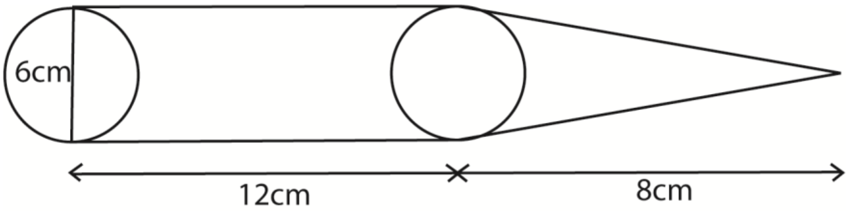
G = $\left(\frac{3r}{8},r\right)$



4. C.O.G of *a hollow(thin) hemisphere* of radius r is at a distance $\frac{h}{3}$ from the base
5. C.O.G of a hollow (thin) cone, pyramid of tetrahedron of height h is at a distance $\frac{h}{3}$ from the base

Example 14

The figure below shows a lamina formed by joining together a cylindrical solid to a conical solid. Find the centre of gravity of the composite from AE and AB



Solution

Let w = weight per volume

Solid	Volume (cm^3)	Weight	Distance of C.O.G from	
			AE	AB
Cylinder	339.29	339.29w	6	3
Cone	75.4	75.4w	12+$\frac{8}{4}$ = 14	3
Composite	414.69	414.69w	$\bar{x}$	$\bar{y}$

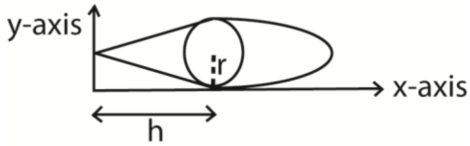
C.O.G from AE = 414.69w$\bar{x}$ = 339.29 x 6 + 75.4 x 14
$\bar{x}$ = 7.45

C.O.G from AB = 414.69w$\bar{y}$ = 339.29 x 3 + 75.4 x 3

$\bar{y}$ =3cm

Example 15

A body consists of a solid hemisphere of radius r joined to a solid right angled circular cone of base radius r and perpendicular height h. The pane surface of the cone and hemisphere coincide and both solids are made of the same uniform material. Show that the C.O.G of the body lies on the axis of symmetry at a distance $\frac{3r^2-h^2}{4(h+2r)}$ from the base of the cone.



Let w = weight per volume

Solid	Weight	C.O.G from y-axis
Cone	$\frac{1}{3}\pi r^2 hw$	$\frac{3h}{4}$
Hemisphere	$\frac{2}{3}\pi r^3 w$	$h+\frac{3r}{8}$
Composite	$\frac{1}{3}\pi r^2(h+2r)w$	$\bar{x}$

$\frac{1}{3}\pi r^2(h+2r)w\bar{x}$

$$=\frac{1}{3}\pi r^2 hw\ x\ \frac{3h}{4}+\frac{2}{3}\pi r^3 w\ x\left(h+\frac{3r}{8}\right)$$

(h+ 2r) $\bar{x} = \frac{3h^2}{4}+2r\left(h+\frac{3r}{8}\right)$

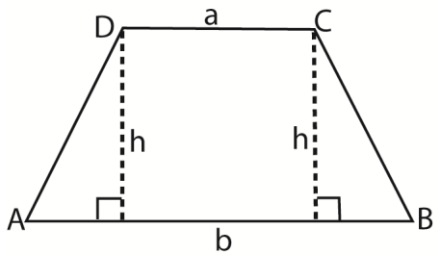
$\bar{x} = \frac{3h^2+8hr+3r^2}{4(h+2r)}$

C.O.G from the base = $\frac{3h^2+8hr+3r^2}{4(h+2r)}-h$

$$=\frac{3r^2-h^2}{4(h+2r)}$$

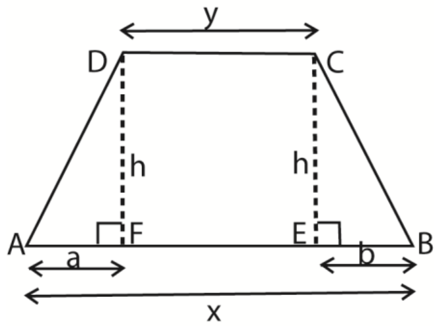
Revision exercise 2

1. ABCD is a trapezium in which AB and CD are parallel and length a and b respectively



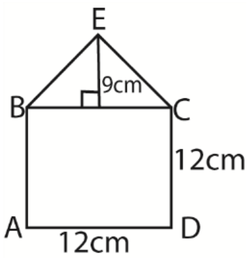
Prove that the distance of the centre of gravity of the centre of mass from AB is $\frac{1}{3}h\left(\frac{a+2b}{a+b}\right)$, where h is the distance between AB and CD

2. The figure ABCD below shows a metal sheet of uniform material cut in a shape of a trapezium: AB = x, CD = y, AF = a, EB = b and h is the vertical distance between AB and CD



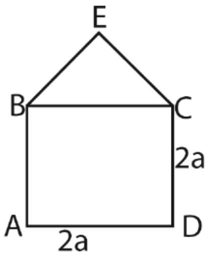
Prove that the distance of the centre of gravity of the centre of mass from AB is $\frac{1}{3}h\left(\frac{3y+a+b}{x+y}\right)$.

3. The figure below shows a sheet of metal in form of a square ABCD of side 12cm and an isosceles triangle BCE of height 9cm



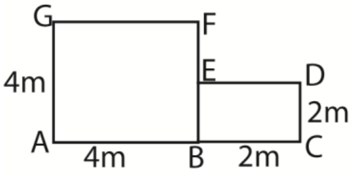
Find the distance of centre of gravity from line AB and AD [6cm, 8.45cm]

4. The figure below shows a sheet of a metal in form of a square ABCD of side 2a and an equilateral triangle BCE.



Find the distance of centre of gravity from line AB and AD [a, 1.896a]

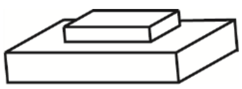
5. The diagram shows two uniform squares ABFG and BCDE joined together. The mass per unit area of BCDE is twice that ABFG.



Find the distance of the centre of gravity of the composite from AB and AG. $\left[1\frac{2}{3}m, 3m\right]$

6. Two uniform square lamina, each of side 3m are joined together to form a rectangular lamina 6m x 3m. The weight per unit area of one square is twice the weight per unit mass of the other. Find the distance of the centre of gravity from the edge of the square [0.5m]

7. Two solid cubes one of side 4cm and the other of side 2cm are made of the same uniform material. The smaller cube is glued centrally to the one of the faces of the larger cube as shown below



Find how far the centre of gravity of the composite body is from the common surface of the cube $\left[1\frac{2}{3}cm\right]$

8. A uniform semi-circular lamina of radius 6cm is joined to another uniform semi-circular lamina of radius 3cm. The centre of the straight edges of each lamina coincides, but the laminas do not overlap. If the two laminas are made of the same material, find the centre of gravity of the composite lamina formed. [On the axis of symmetry, $\frac{28}{5\pi}$cm into the larger semicircle from the common diameter]

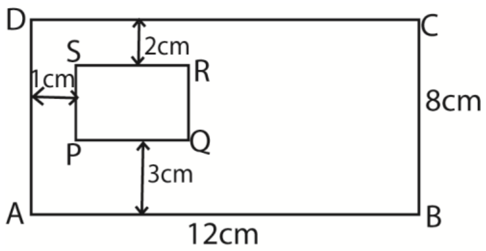
9. A uniform semi-circular lamina of radius 6cm is joined to another uniform semi-circular lamina of radius 3cm. The centre of the straight edges of each lamina coincides, but the laminas do not overlap. If the smaller semi-circle is made of a material having mass per unit area equal to twice that of the larger semi-circular lamina, find the position of the centre of gravity of the composite lamina formed. [On the axis of symmetry, $\frac{4}{\pi}$cm into the larger semicircle from the common diameter]

10. A solid right circular cylinder has a base of radius 3cm and height of 6cm. The cylinder's circular top form the base of a solid right circular cone of base radius 3cm and perpendicular height 4cm.The cylinder and the cone are made of the same uniform material. Find the position of the centre of gravity of the composite. [On the axis of symmetry, $3\frac{8}{11}$cm above the base of the cylinder]
11. A body consists of a solid hemisphere of radius 4cm joined to a solid circular cone of radius 4cm and perpendicular height 12cm. The plane surface of the cane and the hemisphere coincides and both solids are made of the same uniform material. Find the centre of gravity of the composite [On the axis of symmetry, 10.8cm from the tip of the cone]

Centre of gravity of a remainder.

Example 16

Find the centre of gravity of the remainder of the rectangle ABCD if a square PQRS is removed as shown below



Let w = weight per cm^2

Portions	Area (cm^2)	weight	Distance of C.O.G from	
			AD	AB
ABCD	96	96w	6	4
PQRS	9	9w	2.5	4.5
remainder	87	87w	$\bar{x}$	$\bar{y}$

C.O.G from AD

87w$\bar{x}$ = 96w x 6 – 9w x 2.5

$\bar{x}$ =6.362cm

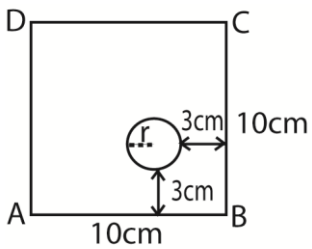
C.O.G from AB

87w$\bar{y}$ = 96w x 4 – 9w x 4.6

$\bar{y}$ = 3.95cm

Example 17

Find the centre of gravity of the remainder of the square ABCD offside 10cm if a circle of radius r = 3cm is removed as shown below



Let w = weight per cm^2

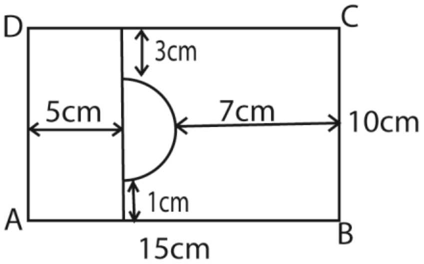
Portions	Area (cm^2)	weight	Distance of C.O.G from	
			AD	AB
ABCD	100	100w	5	5
PQRS	28.27	28.27w	4	6
remainder	71.73	71.73w	$\bar{x}$	$\bar{y}$

C.O.G from AD: 71.73w$\bar{x}$ = 100w x 5 – 28.27w x 5; $\bar{x}$= 5.39cm

C.O.G from AB: 71.73w$\bar{y}$ = 100w x 5 – 28.27w x 6; $\bar{x}$= 4.61cm

Example 18

A rectangle ABCD is of AB = 15cm and AD = 10cm. If a semi-circle is removed as shown below. Find the centre of gravity of the remainder solid



Let w = weight per cm^2

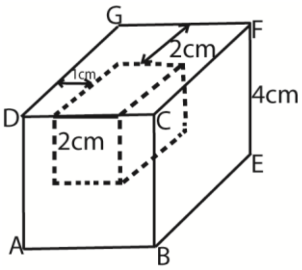
Portions	Area (cm^2)	weight	Distance of C.O.G from	
			AD	AB
ABCD	150	150w	7.5	5
Semi-circle	14.14	14.14w	$5+\frac{4 \times 3}{3\pi}$ =6.27	4
remainder	135.86	135.86w	$\bar{x}$	$\bar{y}$

C.O.G from AD: 135.86w$\bar{x}$ = 150w x 7.5 – 14.14w x 6.27; $\bar{x}$= 7.63cm

C.O.G from AB: 135.86w$\bar{y}$ = 150w x 5 – 14.14w x 4; $\bar{x}$= 5.1cm

Example 19

A solid cube of 4cm is made from a uniform material. From this a smaller cube is removed as shown



Find the centre of gravity of the remaining body

Let w = weight per cm^3

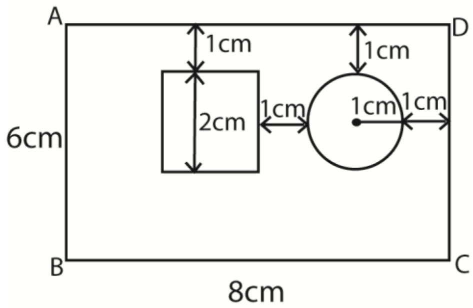
Portions	volume (cm^3)	weight	Distance of C.O.G from	
			AD	AB
Big cube	64	64w	2	2
Small cube	8	8w	3	3
remainder	56	56w	$\bar{x}$	$\bar{y}$

C.O.G from AD: 56w$\bar{x}$ = 64w x 2 – 8w x 2; $\bar{x}$= 2cm

C.O.G from AB: 56w$\bar{y}$ = 64w x 2 – 8w x 3; $\bar{y}$= 1.86cm

Example 20

ABCD is a uniform rectangular sheet of cardboard of length 8cm and width 6cm. a square and a circular hole are cut off from the cardboard as shown below. Calculate the position of the centre of gravity of the remaining sheet



Let w = weight per cm^3

Portions	volume (cm^3)	weight	Distance of C.O.G from	
			AB	BC
ABCD	48	64w	4	3
square	4	4w	3	4
Circle	3.14	3.142w	6	4
remainder	40.86	40.86w	$\bar{x}$	$\bar{y}$

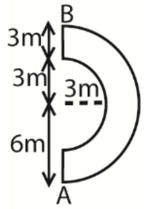
C.O.G from AB: 40.86w$\bar{x}$ = 48w x 4 – (4w x 3+ 3.14w x 6); $\bar{x}$= 3.944cm

C.O.G from BC: 40.86w$\bar{y}$ = 48w x 3 – (4w x 4+ 3.14w x 4); $\bar{y}$= 2.825cm

Revision exercise 3

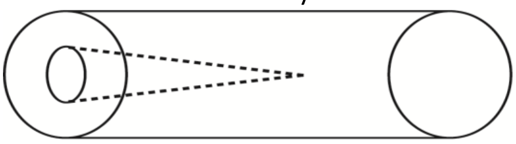
1. A circular lamina made of uniform material has its centre at the origin and radius 6cm. Two smaller circles are cut from this circle, one of radius 1cm and centre (-1, -3) and another of radius 3cm and centre (1, 2). Find the coordinates of the centre of gravity of the remaining shape. $\left[\frac{-4}{3}, \frac{-15}{26}\right]$

2. The diagram shows a uniform semi-circular lamina of radius 6m with a semi-circular portion of radius 3m missing.



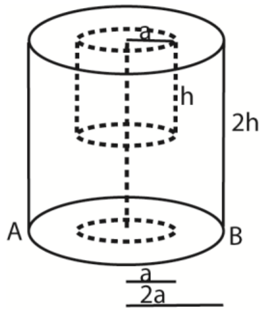
Find the distance of the centre of gravity of the remaining shape from AB as shown above $\left[\frac{28}{\pi}\right]$

3. A conical hole is made in one end of a right circular cylinder, The axis of symmetry of the cone is the same as that of the cylinder



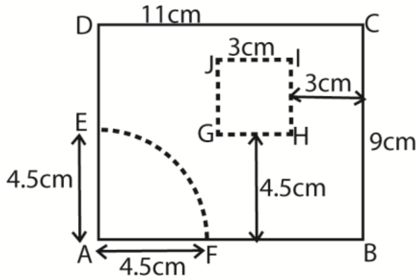
The cylinder is of radius 2cm and length 6cm. The conical hole penetrates 4cm into the cylinder and the circular hole at the end of the cylinder is of radius 1.5cm. Find the position of the centre of gravity of the remaining body. [on axis of symmetry, $2\frac{5}{17}$cm from undrilled end]

4. The diagram below shows a uniform cylinder of radius 2a and height 2h with a cylindrical hole of radius a and height h drilled centrally at one plane end.



Show that the centre of gravity of the remaining solid is $\frac{13}{14}h$ from base AB

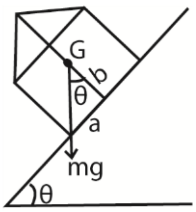
5. The figure below shows a uniform rectangular lamina ABCD with a square GHIJ of side 3cm and a quarter circular section AFE of radius 4.5cm cut off



Find the coordinates of the centre of gravity from sides AB and AD taken as the x and y axes respectively [6.149cm, 4.874cm]

Toppling

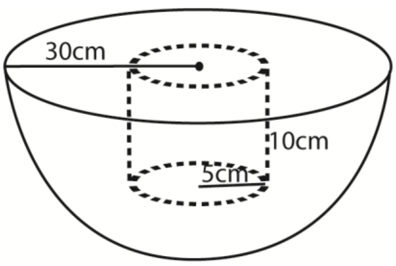
Consider a body which is resting on a slope which is rough to prevent slipping. When the angle of the slope is such that the weight acts through X the will be on the point of toppling.



Where, θ = $\tan^{-1}\left(\frac{a}{b}\right)$

Example 21

The figure shows a uniform hemispherical solid of radius 30cm with cylindrical hole of radius 5cm and height 10cm centrally drilled in it.



(a) Find the distance of centre of gravity from the flat surface

(b) If the figure is placed on an inclined plane with the flat surface in contact with the plane, calculate the angle that should be inclined, before toppling occurs assuming that sliding does not occur.

Solution

Let w = weight per cm^3

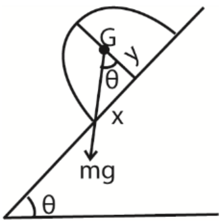
Portion	Volume (cm^3)	Weight	C.O.G from	
			Flat face (cm)	y-axis (cm)
Hemisphere	$\frac{2}{3}\pi r^3$ =5648.668	5648.668w	$\frac{3 \, x \, 30}{8}$ = 11.25	30
Cylinder	$\pi r^2 h$ = 785.398	785.398w	5	30
remainder	4863.270	4863.270w	$\bar{y}$	$\bar{x}$

C.O.G from flat face

4863.270w$\bar{y}$ = 5648.668w x 11.25 - 785.398w x 5; $\bar{y}$ = 12.26

C.O.G from y-axis

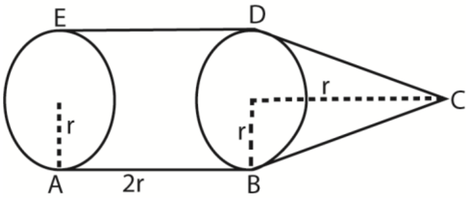
4863.270w$\bar{x}$ = 5648.668w x 30 - 785.398w x 30; $\bar{x}$ = 30



$\theta = \tan^{-1}\left(\frac{30}{12.26}\right) = 67.77^0$

Example 22

A body consists of a uniform solid cylinder of mass 6m, base radius r and height 2r, attached to a plane face of a uniform solid cone of mass 4m base radius r and height r.



(a) Find the position of the centre of gravity of the body
(b) The body is now placed with its plane face AE in contact with a horizontal table. The surface of the table is rough enough to prevent slipping as the table is slowly tilted. Find the angle through which the table has to be tilted when the body is on the point of toppling.

Let w = weight per cm^3

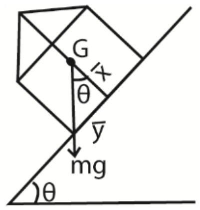
Portion	Weight	C.O.G from	
		AE(cm)	AB (cm)
Cylinder	6mg	r	r
cone	4mg	2r + $\frac{r}{4} = \frac{9r}{4}$	r
composite	10mg	$\bar{x}$	$\bar{y}$

C.O.G from AE

10mg$\bar{x}$ = 4mg x $\frac{9r}{4}$ + 6mg x r; $\bar{x}$ = 1.5r

C.O.G from AB

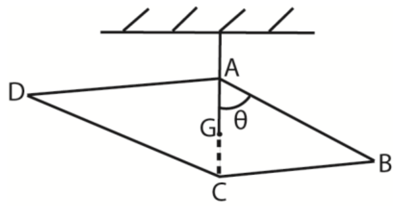
10mg$\bar{y}$ = 4mg x r + 6mg x r; $\bar{x}$ = r



$\theta = \tan^{-1}\left(\frac{r}{1.5r}\right) = 33.69^0$

Equilibrium of suspended lamina

Consider a lamina freely suspended from point A. The centre of gravity passes through the point of suspension

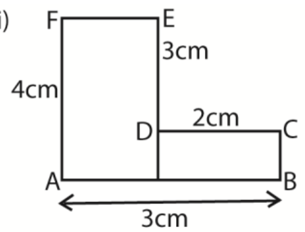


$\theta = \tan^{-1}\left(\frac{BC}{AB}\right)$

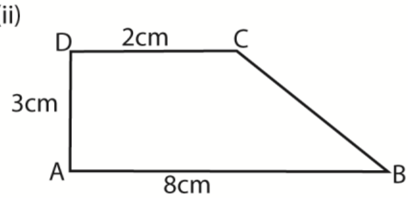
Example 23

The uniform laminas below freely suspended from point A. In each case find the angle that line AB makes with the vertical.

(i)

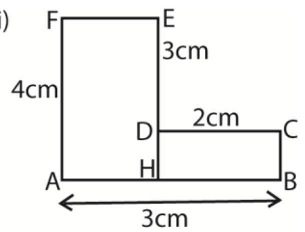


(ii)



(i) Solution

(i)



Let w= weight per cm^2

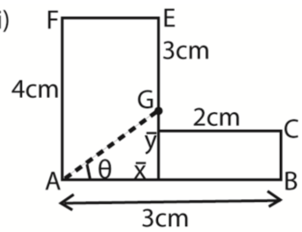
Lamina	Area (cm^2)	Weight	C.O.G from	
			AF	AB
AFEH	4	4w	1	2
HBCD	2	2w	1+1 =2	0.5
Composite	6	6w	$\bar{x}$	$\bar{y}$

C.O.G from AF: $6w\bar{x}$= 4w x 1 + 2w x 2; $\bar{x}$ = 1cm

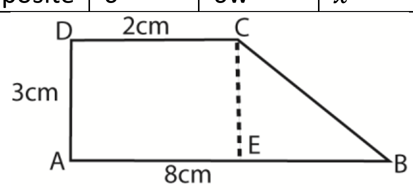
C.O.G from AB: $6w\bar{y}$= 4w x 2 + 2w x 0.5;

$\bar{y}$ = 1.5cm

(i)



$\theta = \tan^{-1}\left(\frac{1.5}{1}\right) = 56.3^0$



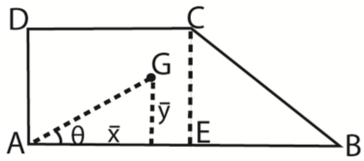
(ii)

Let w= weight per cm^2

Lamina	Area (cm^2)	Weight	C.O.G from	
			AD	AB
ADCE	6	6w	1	1.5
EBC	9	9w	$2+\frac{6}{3}=4$	1
Composite	15	15w	$\bar{x}$	$\bar{y}$

C.O.G from AD: $15w\bar{x}$= 6w x 1 + 9w x 4; $\bar{x}$ = 2.8cm

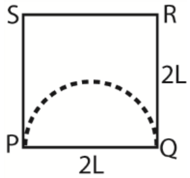
C.O.G from AB: $15w\bar{y}$= 6w x 1.5 + 9w x 1; $\bar{y}$ = 1.2cm



$$\theta = \tan^{-1}\left(\frac{1.2}{2.8}\right) = 23.2^0$$

Example 24

The figure below shows a uniform lamina PQRS of side 2L with semi-circular lamina cut off as show below



(a) Show that the distance of the centre of gravity of the from PQ is $\frac{20L}{3(8-\pi)}$.

(b) The figure is freely suspended from point R. Find the angle RS makes with the vertical.

Let w= weight per cm^2

Lamina	Area (cm^2)	Weight	C.O.G from PQ
PQRS	$4L^2$	$4L^2w$	L
Semi-circle	$\frac{1}{2}\pi L^2$	$\frac{1}{2}\pi L^2$ w	$\frac{4L}{3\pi}$
Composite	$\left(4-\frac{\pi}{2}\right)L^2$	$\left(4-\frac{\pi}{2}\right)L^2$ w	$\bar{y}$

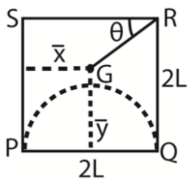
C.O.G from PQ:

$\left(4-\frac{\pi}{2}\right)L^2$ $w\bar{y}$= $4L^2$w x L + $\frac{1}{2}\pi L^2$ w x $\frac{4L}{3\pi}$;

$$\left(\frac{8-\pi}{2}\right)\bar{y} = 4L - \frac{2L}{3}$$

$$\left(\frac{8-\pi}{2}\right)\bar{y} = \frac{10L}{3}$$

$$\bar{y} = \frac{20L}{3(8-\pi)}$$



C.O.G from RS

$$2L - \frac{20L}{3(8-\pi)} = \frac{28L-6L\pi}{3(8-\pi)}$$

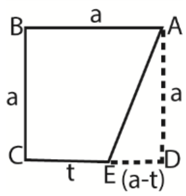
$$\theta = \tan^{-1}\left(\frac{28L-6L\pi}{3(8-\pi)} \Big/ L\right) = 32.12^0$$

Example 25

ABCD is a uniform square lamina of side a from which a triangle ADE is removed, E being a point of distance t from C

(i) Show that the centre of gravity of the remaining lamina is at a distance $\frac{a^2+at+t^2}{3(a+t)}$ from BC

(ii) If the lamina is placed in a vertical plane with CE resting on a horizontal table show equilibrium will not be possible if $t < \frac{a(\sqrt{3}-1)}{2}$



Let w= weight per cm^2

Lamina	Area (cm^2)	Weight	C.O.G from BC
ABCD	a^2	a^2w	$\frac{a}{2}$
DAE	$\frac{1}{2}a(a-t)$	$\frac{1}{2}a(a-t)$ w	$a-\frac{1}{3}(a-t)=\frac{2a-t}{3}$
remainder	$\frac{1}{2}a(a+t)$	$\frac{1}{2}a(a+t)w$	$\bar{x}$

$\frac{1}{2}a(a+t)w\bar{x} = a^2w\ x\ \frac{a}{2} - \ \frac{1}{2}a(a-t)$ w x $\frac{2a-t}{3}$

$\bar{x}(a+t) = \frac{3a^2-(a-t)(2a+t)}{3}$

$\bar{x} = \frac{a^2+at+t^2}{3(a+t)}$

(ii) t < $\bar{x}$

$t < \frac{a^2+at+t^2}{3(a+t)}$

3at + 3t^2 < a^2 + at + t^2

2t^2 +2at –a^2 < 0

$t < \frac{-2a\pm\sqrt{(2a)^2-4\ x\ 2\ x-a^2}}{2\ x\ 2}$

$t < \frac{-a\pm a\sqrt{3}}{2}$

$t < \frac{a(-1-\sqrt{3})}{2}$ or $t < \frac{a(\sqrt{3}-1)}{2}$

Since $\frac{a(-1-\sqrt{3})}{2}$ give a-t < 0 then

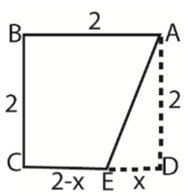
$t < \frac{a(\sqrt{3}-1)}{2}$

Example 26

ABCD is a uniform square lamina of side 2cm from which a triangle EDC is removed. E being a point on AD such that ED = x cm

(i) Show that the centre of gravity of the remaining lamina is at a distance $\frac{12-6x+x^2}{3(4-x)}$ from AB

(ii) If the lamina is placed in a vertical plane with AE resting on a rough horizontal table, show that it will topple if x> 3- $\sqrt{3}$



Let w= weight per cm^2

Lamina	Area (cm^2)	Weight	C.O.G from BC
ABCD	4	4w	1
DAE	$\frac{1}{2}(2x)$ =x	x w	$2-\frac{1}{3}x = \frac{6-x}{3}$
remainder	$(4-x)$	$(4-x)\ w$	$\bar{x}$

$\bar{x}(4-x)w = 4w\ x\ 1 - \frac{(6-x)wx}{3}$

$\bar{x}(4-x) = \frac{12-6x+x^2}{3}$

$\bar{x} = \frac{12-6x+x^2}{3(4-x)}$

(ii) 2-x > $\bar{x}$

$2-x > \frac{12-6x+x^2}{3(4-x)}$

$24-18x + 3x^2 > 12 - 6x + x^2$

$x^2-6x + 6 > 0$

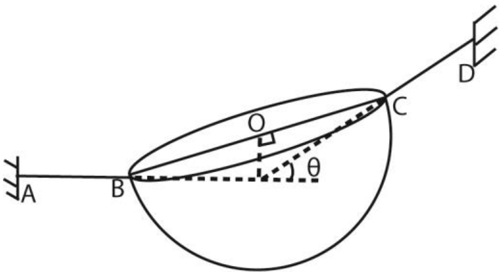
$x > \frac{6\pm\sqrt{6^2-4\ x\ 1\ x\ 6}}{2x1}$; $x > \frac{6\pm\sqrt{12}}{2}$

$x > 3-\sqrt{3}$ or $x > 3+\sqrt{3}$

Since $3+\sqrt{3}$ then $x > 3-\sqrt{3}$

Example 26

A uniform solid hemisphere is in equilibrium in space by two inelastic strings AB which is horizontal and CD which makes an angle θ with the horizontal. The strings and the diameter BC lie in the same vertical plane.

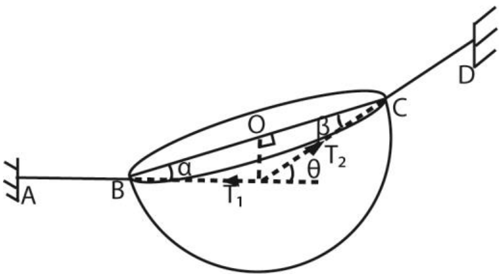


Given that the weight of the hemisphere is 24N, show that

(i) $\tan\theta = \frac{48}{55}$

(ii) Tension in strings AD and AB are 36.5N and 27.5N respectively

Solution



$\tan\alpha = \frac{3r}{8}\Big/ r = \frac{3}{8}$ and $\tan\beta\ \frac{3r}{8}\Big/ r = \frac{3}{8}$

$\theta = \alpha + \beta$

$\tan\theta = \tan(\alpha+\beta) = \frac{tan\alpha + tan\beta}{1-tan\alpha tan\beta}$

$\tan\theta = \frac{\frac{3}{8}+\frac{3}{8}}{1-\frac{3}{8}x\frac{3}{8}} = \frac{48}{55}$

(→) $T_2\cos\theta = T_1$ …… (i)

(↑) $T_2\sin\theta = 24$ ……..(ii)

$\frac{T_2 sin\theta}{T_2 cos\theta} = \frac{24}{T_1}$

$\tan\theta = \frac{24}{T_1}$

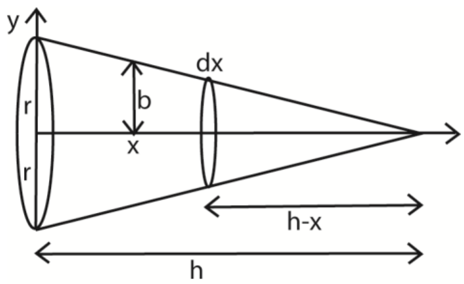
$T_1 = \frac{24}{\tan\left(\frac{48}{55}\right)}$ = 27.5N

$T_2 = \frac{24}{sin\theta} = \frac{24}{\sin\left(\tan^{-1}\frac{48}{55}\right)}$ 36.5N

Example 27

Prove that the centre of mass of a solid cone is $\frac{1}{4}$ of the vertical height from the base

By subdividing the cone into small discs



Let w= weight per unit volume

Volume of a disc = $\pi r^2h = \pi b^2dx$

By using similarity of figures

$\frac{b}{r} = \frac{h-x}{h}$; b = $\frac{(h-x)}{h}r$

By substitution

Volume of disc = $\pi\left(\frac{(h-x)}{h}\right)^2 r^2dx$

Volume of a cone = $\frac{1}{3}\pi r^2h$

Weight of a cone =$\frac{1}{3}\pi r^2hw$

Moment of the whole cone about the y-axis

=$\frac{1}{3}\pi r^2hwx$

Equating moments of the whole solid with the sum of moments of discs about y-axis

$\frac{1}{3}\pi r^2hw\bar{x} = \pi\int_0^h\left(\frac{(h-x)}{h}\right)^2 r^2wxdx$

$\frac{1}{3}\pi r^2hw\bar{x} = \frac{\pi}{h^2}\int_0^h(h^2 - 2hx + x^2)\,r^2wxdx$

$\frac{1}{3}h^3\bar{x} = \int_0^h(h^2x - 2hx^2 + x^3)\,dx$

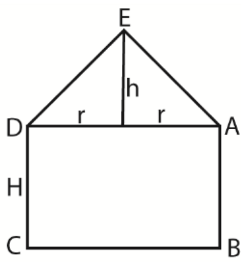
$\frac{h^3\bar{x}}{3} = \left[\frac{h^2x^2}{2} - \frac{2hx^3}{3} + \frac{x^4}{4}\right]_0^h$

$\frac{h^3\bar{x}}{3} = \frac{h^4}{12}$

$\bar{x} = \frac{h}{4}$

Example 28

The figure ABCDE below shows a solid cone of radius r, height, h joined to solid cylinder of the same material with the same radius and height, H.



(i) If the centre of gravity of the whole solid lies in the plane of the cone where the two solids are joined, find H.
Let w = weight per unit volume

Lamina	volume	Weight	C.O.G from BC
cone	$\frac{1}{3}\pi r^2 h$	$\frac{1}{3}\pi r^2 hw$	$H + \frac{h}{4}$
cylinder	$\pi r^2 H$	$\pi r^2 Hw$	$\frac{1}{2}H$
composite	$\pi r^2\left(H + \frac{h}{3}\right)$	$\pi r^2\left(H + \frac{h}{3}\right)w$	H

Equating moments

$$\pi r^2\left(H + \frac{h}{3}\right)wH = \frac{1}{3}\pi r^2 hw\ x\ \left(H + \frac{h}{4}\right) + \pi r^2 H\ w\ x\ \frac{H}{2}$$

$$\left(H + \frac{h}{3}\right)H = \frac{1}{3}h\left(H + \frac{h}{4}\right) + \frac{H^2}{2}$$

$$H^2 + \frac{Hh}{3} = \frac{Hh}{3} + \frac{h^2}{12} + \frac{H^2}{2}$$

$$\frac{H^2}{2} = \frac{h^2}{12}$$

$$H = \frac{h}{\sqrt{6}}$$

(ii) If instead H= h and r = ½r, find the angle AB makes with the horizontal, if the body is hanged from A.

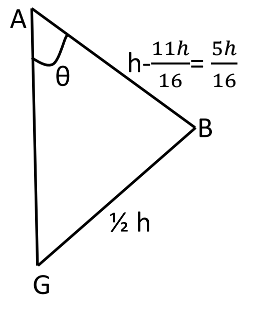
Let distance from BC to the centre of gravity be $\bar{y}$ and substituting H and r

$$\pi\left(\frac{r}{2}\right)^2\left(h + \frac{h}{3}\right)w\bar{y} = \pi\left(\frac{r}{2}\right)^2 hw\ x\ \frac{1}{2}h + \frac{1}{3}\pi\left(\frac{r}{2}\right)^2 hw\ x\ \left(h + \frac{h}{4}\right)$$

$$\frac{4h}{3}\bar{y} = \frac{h^2}{2} + \frac{5h^2}{12} = \frac{11h^2}{12}$$

$$\bar{y} = \frac{11h}{16}$$

If the body is hang from A



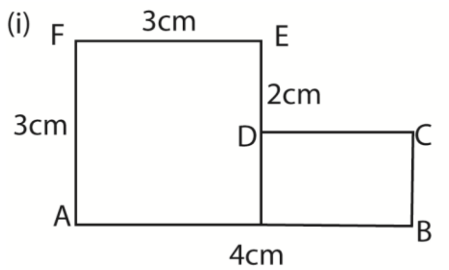
tanθ = $\left(\frac{1}{2}h\right) \div \left(\frac{5h}{16}\right)$

θ = 57.99^0

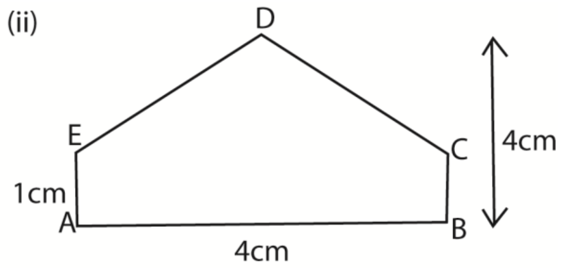
Therefore AB makes an angle 90 -57.99^0 = 32.01^0 with horizontal

Revision exercise 4

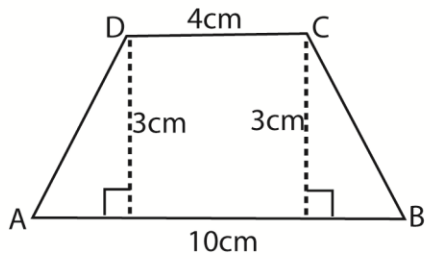
1. The uniform laminas below are freely suspended from point A. In each case find the angle that line AB makes with the vertical.



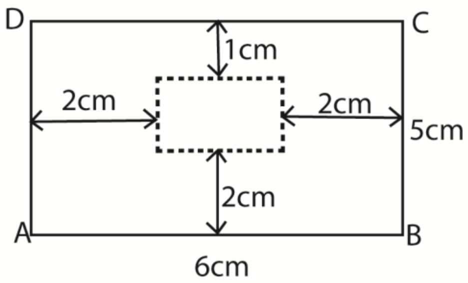
[39.5⁰]



[35⁰]

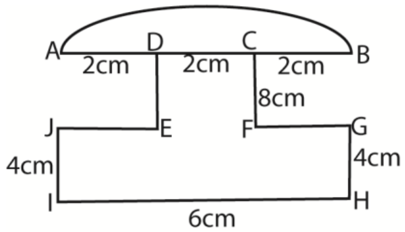


[14.4⁰]



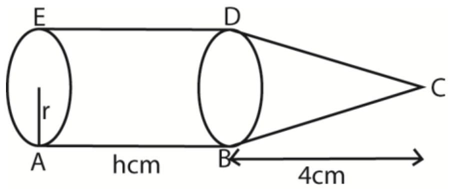
[38.9⁰]

2. The figure ABCDEFGHIJ shows a symmetrical composite lamina made up of a semi-circle of radius 3cm. a rectangle CDEF 2cm by 8cm and another rectangle GHIJ 6cm by 4cm.



Find the distance of the centre of gravity of this lamina from HI. If the lamina is suspended from H by means of a peg through a hole, calculate the angle of inclination of HG to the vertical [6.72cm, 24.1^0]

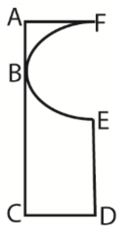
3. A, B, C and D are points (0, 0), (10, 0), (7, 4) and (3, 3) respectively. If AB, BC, CD and DA are made of a thin wire of uniform mass, find the coordinates of the centre of gravity.[5, 1.5]
4. A body consists of a uniform solid cylinder of base radius r and height h, attached to a plane face of a uniform solid cone of base radius r and height 4cm.



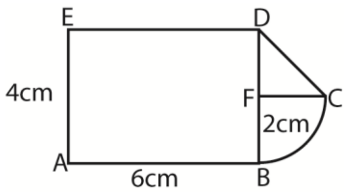
Show that the distance of the centre of gravity of the solid from AE is $\frac{3h^2+8h+8}{6h+8}$.

5. ABCD is a uniform square lamina of side 2a from which an isosceles triangle ABE is cutaway. In this triangle AB = BE and the distance of E from AB= d.
   (i) Show that G, the centre of gravity of the remaining body is a distance $\frac{12a^2-d^2}{3(4a-d)}$ from AB.
   (ii) If d = $\frac{a}{2}$ and the remaining body is suspended by the vertical string from A, find the angle which AD makes with the vertical. [41.8^0]

6. ABCD is a uniform rectangular lamina in which AB = p and BC = 3p. The point E is on AD such that ED = 3q.
   (i) Show that G the centre of gravity of the trapezium ABCE is a distance $\frac{3p^2-3pq+q^2}{(2p-q)}$ from AB and find its distance from BC.
   (ii) When the trapezium is suspended from E, the edge BC is horizontal, prove that q = $\frac{1}{2}p(3-\sqrt{3})$.
7. A child's toy consists of a solid uniform hemisphere of radius r and a solid right circular cone of base radius r and height h. If the density of the hemisphere is k time that of a cone;
   (i) Show that the distance from the vertex of the cone to the centre of gravity of the toy is $\frac{4r(3r+8h)+3h^2}{4(2kr+h)}$
   (ii) If the toy is suspended from a point on the rim of the common base and rests in equilibrium with the axis of the cone inclined at an angle θ to the downward vertical. Show that tanθ = $\frac{4r(2kr+h)}{h^2-3kr^2}$
8. A semi-circular lamina of radius r and bas AB is cut from a large semi-circular lamina of radius 2r with diameter AC.
   (i) Determine the centre of gravity of the remainder[2r, 0.99r]
   (ii) If the remainder is freely suspended from A, find the inclination of AB to the vertical[26.3^0]
9. The figure ABCDEF below shows a uniform lamina in form of a rectangle from which a hole in form of a semi-circle was made. The diameter of the semicircle is 6cm.



   (a) Find the centre of gravity from AC and CD if the semicircle is removed.[2.47cm, 4.06cm]
   (b) If the remaining lamina is suspended at D, find the angle CD makes with the vertical (82.56^0]
10. The figure below shows a uniform rectangular lamina ABCD with a triangle DFC and quarter circular lamina section BFC all of the same density. F is the centre of the circle from which the quadrant forms part.

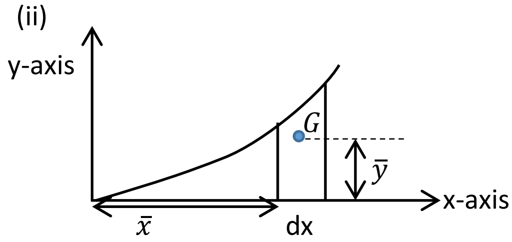


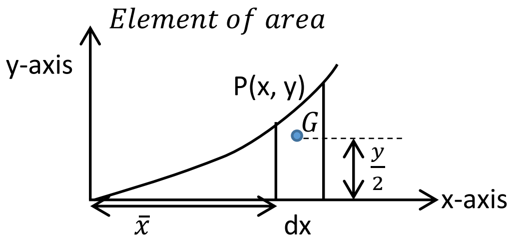
   (a)Find the coordinates of the centre of gravity from sides AB and AE taken as x and y-axes respectively
   (b) If the lamina is suspended about A, find the angle that AB makes with the horizontal.

Centre of gravity of the lamina whose area is bounded.

(i) C.O.G of the area bounded in the first quadrant

(ii)





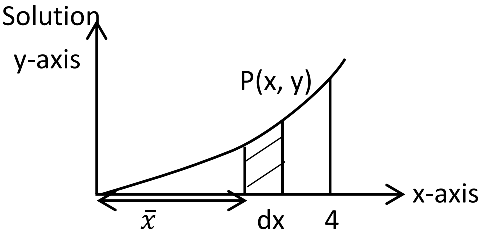
C.O.G from y-axis: w$\bar{x}\int ydx = w\int xydx$ C.O.G from the x-axis: w$\bar{y}\int ydx = w\int \frac{y}{2}y\,dx$

Where w = weight per unit area

Example 28

Find the coordinates of the centre of gravity of the uniform lamina enclosed by the curve x^2, the x-axis and the line x = 4

Solution



C.O.G from y-axis: w$\bar{x}\int ydx = w\int xydx$

$\bar{x}\int_0^4 ydx = \int_0^4 xy\,dx$

$\bar{x}\int_0^4 x^2dx = \int_0^4 x(x^2)\,dx$

$\bar{x}\left[\frac{x^3}{3}\right]_0^4 = \left[\frac{x^4}{4}\right]_0^4$

$\bar{x} = 3$

w$\bar{y}\int ydx = w\int \frac{y^2}{2}dx$

$\bar{y}\int_0^4 x^2\,dx = \frac{1}{2}\int_0^4 x^4dx$

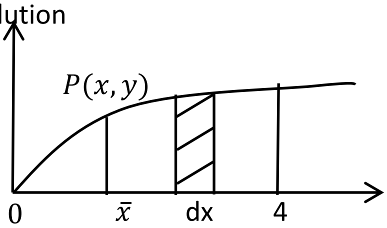
$\bar{y}\left[\frac{x^3}{3}\right]_0^4 = \frac{1}{2}\left[\frac{x^5}{5}\right]_0^4$

$\bar{y} = 4.8$

Example 29

Find the coordinates of the centre of gravity of uniform lamina enclosed by the curve $y^2 = 9x$, the x-axis and the line x =1 and x = 4 and lying in the first quadrant

Solution



$\bar{x}\int_1^4 ydx = \int_1^4 xy\,dx$

$\bar{x}\int_1^4 3x^{\frac{1}{2}}dx = \int_1^4 x\left(3x^{\frac{1}{2}}\right)dx$

$3\bar{x}\left[\frac{x^{\frac{3}{2}}}{2}\right]_1^4 = 5\left[\frac{x^{\frac{5}{2}}}{2}\right]_1^4$

$\bar{x} = 2.66$

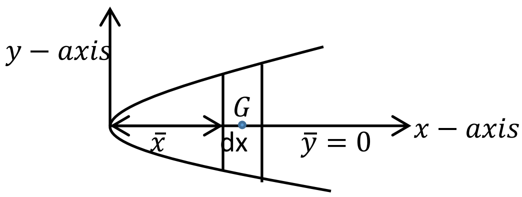
$w\bar{y}\int ydx = w\int \frac{y^2}{2}dx$

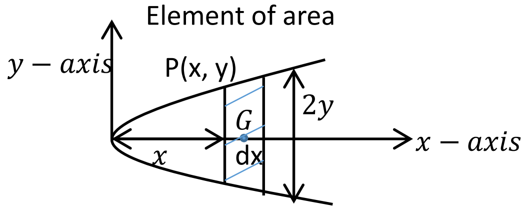
$\bar{y}\int_1^4 3x^{\frac{1}{2}}\,dx = \frac{1}{2}\int_1^4 9xdx$

$\bar{y}\left[\frac{x^{\frac{3}{2}}}{2}\right]_1^4 = \frac{3}{2}\left[\frac{x^2}{2}\right]_1^4$

$\bar{y} = 2.41$

(ii) C.O.G of the area bounded in the first and fourth quadrant



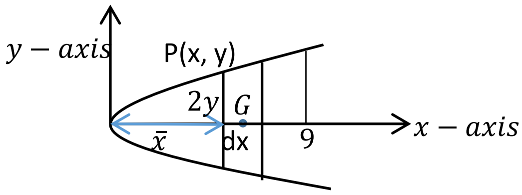


Taking moments about the y-axis: $w\bar{x}\int 2ydx = w\int x(2y)dx$

Where w- weight per unit area

Example 30

Find the coordinates of the centre of gravity of the uniform lamina enclosed by the curve $y^2 = 4x$ and the line x = 9



$w\bar{x}\int_0^9 2ydx = \int_0^9 x(2y)dx$

$\bar{x}\int_0^9 x^{1/2}\,dx = \int_0^9 x.x^{1/2}dx$

$3\bar{x}\left[\frac{x^{\frac{3}{2}}}{2}\right]_0^9 = 5\left[\frac{x^{\frac{5}{2}}}{2}\right]_0^9$

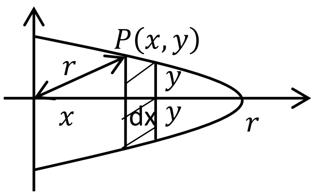
$\bar{x} = 5.4$

$(\bar{x}, \bar{y}) = (5.4, 0)$

Example 30

Show that the position of C.O.G of a uniform semi-circular lamina of radius r is $\frac{4r}{3\pi}$ from th straight edge.

Solution



Area of semi-circle = Area of element of semi-circle

$W\frac{1}{2}\pi r^2\bar{x} = w\int_0^r x(2y)dx$

But a semi-circle is part of a circle of radius r whose

Equation is $x^2 + y^2 = r^2$

$y = (r^2 - x^2)^{\frac{1}{2}}$

$\frac{1}{2}\pi r^2\bar{x} = 2\int_0^r x(r^2 - x^2)^{\frac{1}{2}}dx$

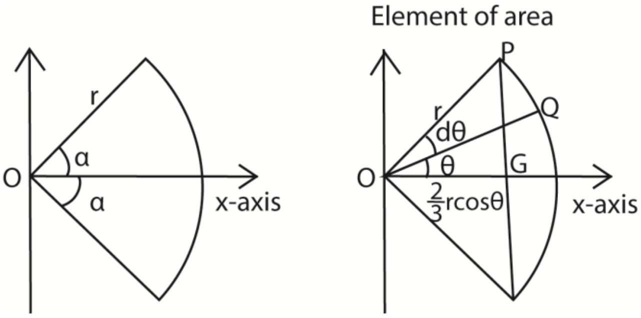
$\frac{1}{2}\pi r^2\bar{x} = 2\left[-\frac{(r^2-x^2)^{\frac{3}{2}}}{3}\right]_0^r$

$\frac{1}{2}\pi r^2\bar{x} = \frac{2r^3}{3}$

$\bar{x} = \frac{4r}{3\pi}$

Example 31

Show that the centre of gravity of a uniform lamina in shape of a sector of a circle of radius r and subtending an angle 2α at the centre O is given by $\frac{2r\sin\alpha}{3\alpha}$ from O.



The strip OPQ approximates a triangle and its C.OG

is at a distance $\frac{2}{3}r$ from O. The distance of C.O.G

from O is therefore $\frac{2}{3}r\cos\theta$

Area of sector = element of area of sector

$$w\frac{1}{2}r^2(2\alpha)\bar{x} = w\int_{-\alpha}^{\alpha} xy\,dx$$

$$\frac{1}{2}r^2(2\alpha)\bar{x} = \int_{-\alpha}^{\alpha} \frac{2}{3}r\cos\theta\left(\frac{1}{2}r\right)d\theta$$

$$\alpha\bar{x} = \frac{1}{3}\int_{-\alpha}^{\alpha} r\cos\theta\,d\theta$$

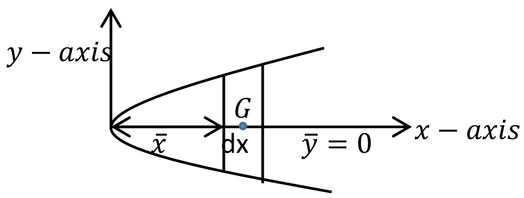
$$\alpha\bar{x} = \frac{r}{3}[\sin\theta]_{-\alpha}^{+\alpha}$$

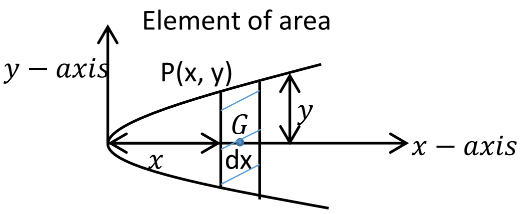
$$\bar{x} = \frac{2r\sin\alpha}{3\alpha}$$

For a complete semi-circle, $\alpha = \frac{\pi}{2}$

$$\therefore \bar{x} = \frac{4r}{3\pi}$$

Centre of gravity of solids of revolution

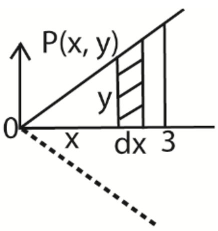




Taking moments about y-axis: $\bar{x}w\pi\int y^2dx = w\pi\int xy^2dx$; where w = weight per unit volume

Example 32

Find the centre of gravity of a solid generated by rotating about the x-axis, the area under y = x from x =0 and x = 3.



$$\bar{x}w\pi\int y^2dx = w\pi\int xy^2dx$$

$$\bar{x}\int_0^3 x^2dx = \int_0^3 x(x^2)dx$$

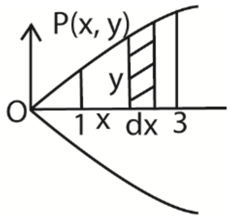
$$\bar{x}\left[\frac{x^3}{3}\right]_0^3 = \left[\frac{x^4}{4}\right]_0^3$$

$$\bar{x} = 2.25$$

$$(\bar{x}, \bar{y}) = (2.25, 0)$$

Example 33

Find the centre of the solid generated by rotating about the x-axis, the area bounded by $y^2 = 5x$ the x-axis, the lines x = 1 and x = 3 and lies in the first quadrant.



$\bar{x}w\pi \int y^2 dx = w\pi \int xy^2 dx$

$\bar{x} \int_1^3 5x dx = \int_1^3 x(5x) dx$

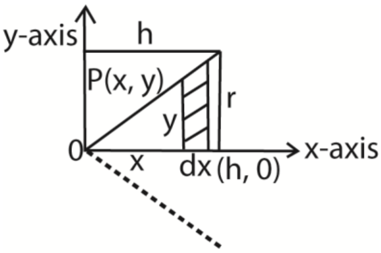
$\bar{x} \left[\frac{x^2}{2}\right]_1^3 = \left[\frac{x^3}{3}\right]_1^3$

$\bar{x} = 2.17$

$(\bar{x}, \bar{y}) = (2.17, 0)$

Example 34

Show that the position of centre of gravity of a uniform solid right circular cone of base radius r and height h is given by $\frac{h}{4}$ from the straight edge.



$\bar{x}w\frac{1}{3}\pi r^2 h = w\pi \int xy^2 dx$

$\bar{x}\frac{1}{3}r^2 h = \int xy^2 dx$

From similarity $\frac{y}{x} = \frac{r}{h}$

$y = \frac{r}{h}x$

$\bar{x}\frac{1}{3}r^2 h = \int x\left(\frac{r}{h}x\right)^2 dx$

$\bar{x}\frac{1}{3}r^2 h = \left(\frac{r^2}{h^2}\right)\left[\frac{x^4}{4}\right]_0^h$

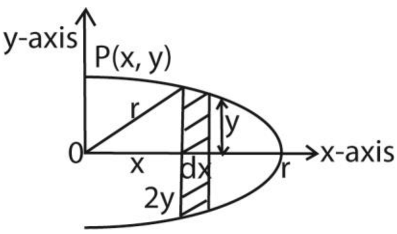
$\bar{x} = \frac{3h}{4}$

From the straight edge

$\bar{x} = h - \frac{3h}{4} = \frac{h}{4}$

Example 35

Show that the position of the centre f gravity of a <u>uniform solid hemisphere</u> of radius r is $\frac{3r}{8}$ from the straight edge.



$\frac{2}{3}\pi r^3 \bar{x} w = w\pi \int_0^r xy^2 dx$

But $x^2 + y^2 = r^2 => y^2 = r^2 - x^2$

$\frac{2}{3}r^3\bar{x} = \int_0^r x(r^2 - x^2)dx$

$\frac{2}{3}r^3\bar{x} = \left[r^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^r$

$\frac{2}{3}r^3\bar{x} = \frac{r^4}{2} - \frac{r^4}{4}$

$\bar{x} = \frac{3r}{8}$

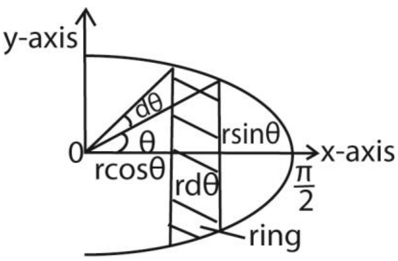
Revision exercise 5

1. Find the coordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = x^2$, the x-axis and the line x= 2 [1.5, 1.2]
2. Find the coordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = 2x – x^2$ and the x-axis. [1.0, 0.4]
3. Find the coordinates of the centre of gravity of a uniform lamina enclosed by the curve $y = x^2 + 2$, the x-axis and the lines x = 2 and x = 2 [1.56, 2.25]
4. Find the coordinates of the centre of gravity of the uniform lamina lying in the first quadrant and enclosed by the curve $y^2 = 8x$, the x-axis and the lines x = 2 and x = 8 [5.31, 3.21]
5. Find the coordinates of the centre of gravity of the uniform lamina lying in the first quadrant and enclosed by the curve $y^2 = x - 8$ and the axes [0.4, 0.375]
6. Find the coordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = x^2$ and the line y = 3x. [1.5, 3.6]
7. Find the coordinates of the centre of gravity of the uniform lamina lying in the first quadrant and enclosed by the curve $y = 4 – x^2$, $y = 3x^2$ and the y-axis. [0.375, 2.2]
8. Find the coordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = x^3$, the x-axis and the line x = 3 [2.4, 7.14]
9. The area enclosed by the curve $y^2 = x$, the x-axis, the line x =4 and lying in the first quadrant is rotated about the x-axis through one revolution. Find the coordinates of the centre of gravity of the uniform solid formed. [2.67, 0]
10. The area enclosed by the curve $y^2 = x$, the x-axis, the line x = 2, x=4 and lying in the first quadrant is rotated about the x-axis through one revolution. Find the coordinates of the centre of gravity of the uniform solid formed. [3.39, 0]
11. The area enclosed by the curve $y = x^2 + 3$, the x-axis, the y-axis and the line x = 2 is rotated about the x-axis through one revolution. Find the coordinates of the centre of gravity of the uniform solid formed. [1.3, 0]
12. The area enclosed by the curve $y = x^3$, the x-axis and the line x = 3 is rotated about the x-axis through one revolution. Find the coordinates of the centre of gravity of the uniform solid formed.[2.625, 0]

Surface of revolution

Example 36

Show that the centre of gravity of a uniform thin hemispherical cup of radius r is at a distance $\frac{r}{2}$ from the base



Surface area of = element of surface area

a hemisphere

$w2\pi r^2\bar{x} = w2\pi \int_0^{\frac{\pi}{2}} xydx$

$2r^2\bar{x} = 2\int_0^{\frac{\pi}{2}}(rsin\theta \ x \ rcos\theta)rd\theta$

$2r^2\bar{x} = r^3\int_0^{\frac{\pi}{2}} 2sin\theta \ x \ rcos\theta d\theta$

$\bar{x} = \frac{r}{2}\int_0^{\frac{\pi}{2}} sin2\theta d\theta$

$\bar{x} = \frac{r}{2}\left[-\frac{1}{2}cos2\theta\right]_0^{\frac{\pi}{2}}$

$\bar{x} = -\frac{r}{4}\left[cos\left(\frac{\pi}{2}\right) - cos2(0)\right]$

$\bar{x} = -\frac{r}{4}(-1-1) = \frac{r}{2}$

Thank you

Dr. Bbosa Science