Hooke's law, elastic strings and Simple harmonic motion

Hooke's law states that the tension in a stretched string is proportional to the extension,e from its natural (unstretched) length, L.

$T = \lambda\frac{e}{L}$ where λ is modulus of elasticity.

Example 1

An elastic string is of natural length 4m and modulus 25N. Find

(i) The extension in the string when the extension is 20cm

$T = \lambda\frac{e}{L}$ = $25\,x\,\frac{0.2}{4}$= 1.25N

(ii) The extension of the string when the tension is 6N

$e = T\frac{L}{\lambda} = 6\,x\,\frac{4}{25} = 0.96m$

Example 2

A spring is of natural length 1.6 and modulus 20N. Find the thrust in the spring when it is compresses to a length of 1m

$T = \lambda\frac{e}{L}$ = $20\,x\,\frac{0.6}{1.6}$= 7.5N

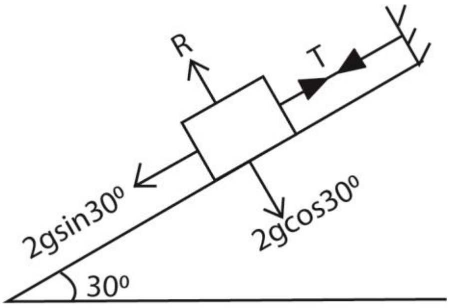
Example 3

A body of mass mkg lies on a smooth horizontal surface and is connected to a point O on the surface by a light elastic string of natural length 60cm and modulus 90N. When the body moves in a horizontal circular path about O with constant speed of 3ms^{-1}, the extension in the string is 30cm. find the mass of the body.

$T = \lambda\frac{e}{L}$ = $90\,x\,\frac{0.3}{0.6}$= 45N	T = $m\frac{v^2}{r}$	45 = $m\frac{3^2}{(0.6+0.3)}$	m = 4.5kg

Example 4

A smooth surface inclined at 30^0 to the horizontal has a body A of mass 2kg that is held at rest on the surface by a light elastic string which has one end attached to A and the other to a fixed point on the surface 1.5m away from A up to the line of greatest slope. If the modulus of elasticity of the string is 2gN, find the natural length



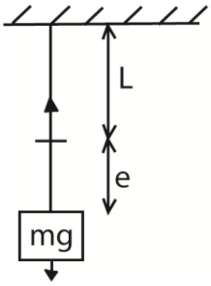
$T = \lambda \frac{e}{L}$

2gsin30 = 2g x $\frac{1.5-L}{L}$

L = 1m

Equilibrium of a suspended body

When an elastic string one end fixed and a mass attached to its other end so that the mass is suspended in equilibrium, the string is stretched due to the mass.



T = mg

mg = $\lambda \frac{e}{L}$ at equilibrium

Example 5

A light elastic string of natural length 65 has one end fixed and a mass of 500g freely suspended from the other end. Find the modulus of elasticity of the string if the total length of the string in equilibrium position is 85cm

mg = $\lambda \frac{e}{L}$ | 0.5 x 9.8 = $\lambda\ x\ \frac{0.85-0.65}{0.65}$ | λ = 15.93N

Example 6

A light spring of natural length 1.5m has one end fixed and a mass of 400g freely suspended from the other. The modulus of the spring is 44.1N

(a) Find the extension of the spring when the body hangs in equilibrium
mg = $\lambda \frac{e}{L}$ | 0.4 x 9.8 = $44.1\ x\ \frac{e}{1.5}$ | e= 0.13m

(b) The mass is pulled vertically downwards a distance of 10cm and released, find the acceleration of the body when released

$T_1 = \lambda \frac{e+x}{L}$

$T_1 = 44.1\ x \frac{0.13\ +0.1}{1.5} = 6.762N$

F =ma

$T_1 - T = ma$

6.762- 3.92 = 0.4a

A = 7.11ms^{-2}

Potential energy stored in an elastic string

Work done = average force x extension = $\lambda \frac{e^2}{2L}$

Example 7

An elastic string is of natural length 6.4m and modulus 55N. Find the work done in stretching it from 6.4m to 6.8m

Work done = $\lambda\frac{e^2}{2L} = 55\ x\frac{0.4^2}{2\ x\ 6.4}$ = 0.688J

Example 8

An elastic string of natural length 4m is fixed at one end and stretched to 5.6m by a force of 8N. Find the modulus of elasticity and the work done

$T = \lambda\frac{e}{L}$

8 = $\lambda\frac{5.6-4}{4}$

λ = 20N

Work done = $\lambda\frac{e^2}{2L} = 20\ x\frac{1.6^2}{2\ x\ 4}$ = 6.4J

Example 9

An elastic string of natural length 1.2m and modulus of elasticity 8N is stretched until the extending force is 6N. Find the extension and work done

$T = \lambda\frac{e}{L}$

6 = $8\ x\frac{e}{1.2}$

e = 0.9m

Work done = $\lambda\frac{e^2}{2L} = 8\ x\frac{0.9^2}{2\ x1.2}$ = 2.7J

Example 10

A light elastic string of natural length 1.2m has one end fixed at A and a mass of 5kg freely suspended from the other. The modulus of the string is such that a 5kg mass hanging vertically would stretch the string by 15cm. the mass is held at A and allowed to fall vertically, how far from A it comes to rest.

$mg = \lambda\frac{e}{L}$

$5\ x\ 9.8 = \lambda\ x\ \frac{0.15}{1.2}$

λ = 392N

At A; u = $0ms^{-1}$, s = 1.2m, g = $9.8ms^{-2}$

$v^2 = u^2 + 2gs = 0^2 + 2 \times 9.8 \times 1.2 = 23.52m^2s^{-2}$

w = k.e + p.e

$\lambda\frac{e^2}{2L} = \frac{mv^2}{2} + mge$

$392\ x\frac{e^2}{2\ x\ 1.2} = \frac{5\ x\ 23.52}{2} + 5\ x\ 9.8\ x\ e$

e = 0.769m

Depth = 1.2 + 0.769 = 1.969m

Revision exercise

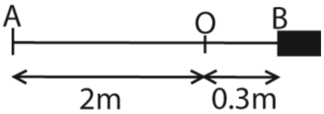
1. An elastic string of natural length 1m and modulus 20N. find the tension in the string when the extension is 20cm [4N]
2. A spring of natural length 50cm and modulus 10N. Find the thrust in the spring when it is compressed to a length 40cm [2N]
3. When the length of a spring is 60% of its original length, the thrust in the spring is 10N. find the modulus of the spring [25N]

4. An elastic string of natural length 60cm and modulus 18N. Find the extension of the string when the tension in the string is 6N [20cm]
5. The tension in an elastic string is 8N when the extension in the string is 25cm. If the modulus of the string is 8N. Find the un-stretched length. [25cm]
6. A light elastic string of natural length 20cm and modulus 2gN has one end fixed and a mass of 500g freely suspended from the other. Find the total length of the string [25cm]
7. When a mass of 5kg is freely suspended from one end of a light elastic string, the other end of it fixed, the string extends to twice its natural length. Find the modulus of the string [49N]
8. A body of mass 4kg lies on a smooth horizontal surface and is connected to point O of the surface by a light elastic string of natural length 64cm and modulus 25N. when the body moves in a horizontal circular path about O with constant speed of vms^{-1}, the extension in the string is 36cm. Find v [$1.875ms^{-1}$]
9. A body of mass 5kg lies on a horizontal surface and is connected to a point O on the surface by a light elastic string of natural length 2m and modulus 30N. When the body moves in a horizontal circular path about O with a constant speed $3ms^{-1}$, find the extension in the string[1m]
10. An elastic string of natural length 2m and modulus 10N. Find the energy stored when it is extended to a length of 3m [2.5J]
11. An elastic string of natural length 1m and modulus 20N. Find the energy stored when it is extended by a length of 30cm [0.9J]

Simple harmonic motion in strings

Example 11

One end of a light elastic string of natural length 2m and modulus 10N is fixed to point A on a smooth horizontal surface. A body of mass 200g is attached to the other end of the string and is held at rest at point B on the surface causing the spring to extend by 30cm. Show that when released, the body will move with S.H.M. State its amplitude and find the maximum speed.



$T = \lambda\frac{x}{L} = 10\,x\frac{x}{2} = 5x$

F = ma

5x = -0.2a

a = -25x

Since $a = -\omega^2 x$

$\omega^2 = 25$

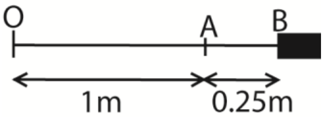
$\omega = 5rads^{-1}$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{2} s$ and r = 0.3m

$v_{max} = \omega r = 5 \times 0.3 = 1.5ms^{-1}$

Example 12

One end of a light elastic string of natural length 1m and modulus 5N is fixed to a point O on a smooth horizontal surface. A body of mass 1kg is attached to the other end, A of the string and is held at rest at point B where OB = 1.25m. Show that when released, the body will move with S.H.M. Find the maximum speed and the total time taken from B to O



$T = \lambda\frac{x}{L} = 5\,x\frac{x}{1} = 5x$

F = ma

5x = -1a

a = 5x

Since a = $-\omega^2 x$

$\omega^2 = 5$

$\omega = 2.236 rads^{-1}$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{2.236} s = 2.81s$ and r = 0.25m

$v_{max} = \omega r = 2.236 \times 0.25 = 0.559ms^{-1}$

$t_{OA} = \frac{distance}{speed} = \frac{1}{0.559} = 1.788s$

$\frac{distance}{time} = \frac{1}{T} = \frac{0.25}{t_{AB}}$

$t_{AB} = \frac{0.25\ x\ 2.81}{1} = 0.7025s$

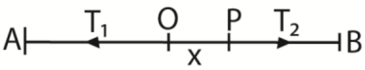
$t_{OB} = t_{OA} + t_{AB}$

$t_{OB} = 1.788 + 0.7025 = 2.491s$

Example 13

A light elastic string of natural length 2.4m and modulus 15 is stretched between two points A and B, 3m apart on a smooth horizontal surface. A body of mass 4kgis attached to the midpoint of the string is pulled 10 cm towards B and released

(i) Show that the subsequent motion is simple harmonic
(ii) Find the speed of the body when it is 158cmfrom A



At equilibrium $T_1 = T_2$

$\lambda \frac{e_1}{L} = \lambda \frac{e_2}{L}$

$e_1 = e_2$ ………. (i)

$e_1 + e_2 + 2.4 = 3$

$e_1 = 0.3m$

F = ma

$T_2 - T_1 = ma$

$\lambda\left(\frac{0.3-x}{1.2}\right) - \lambda\left(\frac{0.3+x}{1.2}\right) = 4a$

$4.8a = -2\lambda x$

$a = -\frac{2\ x\ 15x}{4.8} = -6.25x$

It is in form of a $=\omega^2 x$ hence S.H.M

$\omega^2 = 6.25$

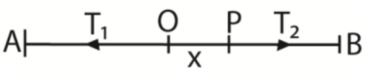
$\omega = 2.5 rads^{-1}$

$v^2 = \omega^2(r^2 - x^2)$ when 158 from A, x = 8cm

$v = \sqrt{0.1^2 - 0.08^2} = 0.15ms^{-1}$

Example 14

A particle of mass m is attached by means of a light string AP and BP of the same natural length a m and modulus of elasticity mgN and 2mgN respectively, to point A and B on a smooth horizontal surface. The particle is released from the midpoint AB where AB = 3a. Show that the subsequent motion is simple harmonic with period $T = \left(\frac{4\pi^2 a}{3g}\right)^{\frac{1}{2}}$.



At equilibrium $T_1 = T_2$

$mg\frac{e_1}{a} = 2mg\frac{e_2}{a}$

$e_1 = 2e_2$ ………. (i)

$e_1 + e_2 + 2a = 3a$

$2e_2 + e_2 + 2a = 3a$

$e_2 = \frac{a}{3}$ and $e_1 = \frac{2a}{3}$

F = ma

$T_2 - T_1 = ma$

$2mg\left(\frac{a/3 - x}{a}\right) - mg\left(\frac{2a/3 + x}{a}\right) = ma$

$a = -\frac{3g}{a}x$

It is in form of $a = \omega^2 x$ hence S.H.M

$\omega^2 = \frac{3g}{a}$

$\omega = \left(\frac{3g}{a}\right)^{1/2}$ rads^{-1}

$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{3g}{a}\right)^{1/2}} = \left(\frac{4\pi^2 a}{3g}\right)^{\frac{1}{2}}$

Example 15

A particle of mass 1.5kglies on a smooth horizontal table and attached to two light elastic string fixed at points P and Q 12m apart. The strings are of natural length 4m and 5m and their modulus are λ and 2.5λ respectively.

(a) Show that the particles stays in equilibrium at a point R midway between P and Q

At equilibrium $T_1 = T_2$

$\lambda\frac{e_1}{4} = 2.5\lambda\frac{e_2}{5}$

$e_1 = 2e_2$ ………. (i)

$e_1 + e_2 + 4 + 5 = 12$

$2e_2 + e_2 + 9 = 12$

$e_2 = 1$

$e_1 = 2$

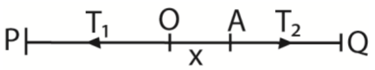
At R (midpoint) $4 + e_1 = 5 + e_2$

4 + 2 = 5 + 1 = 6

(b) If the particle is held at some point S in the line PQ with PS = 4.8m and then released. Show that the particle performs S.H.M and find the

(i) Period of oscillation

(ii) Velocity when the particle is 5.5m from P



F= ma

$T_2 - T_1 = ma$

$2.5\lambda\left(\frac{1-x}{5}\right) - \lambda\left(\frac{2+x}{4}\right) = 15a$

$a = -\frac{\lambda}{2}x$

It is in form of $a = \omega^2 x$ hence S.H.M

$\omega^2 = \frac{\lambda}{2}$

$\omega = \left(\frac{\lambda}{2}\right)^{1/2}$ rads^{-1}

$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{\lambda}{2}\right)^{1/2}} = \left(\frac{8\pi^2}{\lambda}\right)^{\frac{1}{2}}$

$v^2 = \omega^2(r^2 - x^2)$ when 5.5m from P, x = 0.5m

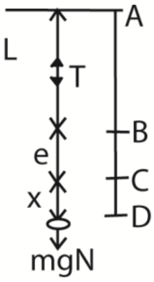
$v = \sqrt{\frac{\lambda}{2}((6 - 4.8)^2 - 0.5^2)}$

$v = \sqrt{\frac{\lambda}{2}((1.2)^2 - 0.5^2)} = \sqrt{0.595\lambda}\ ms^{-1}$

Elastic strings or springs hanging vertically

Example 16

A particle of mass m is suspended by a string from a fixed point A and has a natural length L. If the spring is extended from B to C where BC = e and this extension is due to weight of the body (mg), CD = x is the length a particle is pulled vertically downwards.



At equilibrium T = mg

T = $\lambda\frac{e}{L}$

When pulled a distance x:

$T - T_1$ = ma

$$\lambda\frac{e}{L} - \lambda\frac{e+x}{L} = ma$$

$$a = -\frac{\lambda}{mL}x$$

It is in form of a =ω^2x hence S.H.M

$\omega^2 = \frac{\lambda}{mL}$

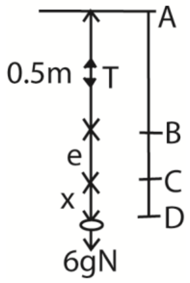
$\omega = \left(\frac{\lambda}{mL}\right)^{1/2}$ rads^{-1}

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{\lambda}{mL}\right)^{1/2}} = 2\pi\sqrt{\frac{mL}{\lambda}}$$

Example 17

A light elastic spring of natural length 50cm and modulus 20gN, hangs vertically with its upper and fixed and the body of mass 6kg attached to its lower end. The body initially rests in equilibrium and then pulled down a distance of 25cm and released.

(a) Show that the ensuing motion will simple harmonic and
(b) Find the period of motion and maximum speed of the body



At equilibrium T = mg

6g = $20g\,\frac{e}{0.5}$

e = 0.15m

When pulled a distance x:

$T - T_1$ = ma

$$6g - \lambda\frac{e+x}{0.5} = 6a$$

$$a = -\frac{196}{3}x$$

It is in form of a =ω^2x hence S.H.M

$\omega^2 = \frac{196}{3}$

$\omega = \left(\frac{196}{3}\right)^{1/2} = 8.083$ rads^{-1}

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{196}{3}\right)^{1/2}} = 0.773s$$

$$v_{max} = \omega r = 8.083\ x\ 0.25 = 2.021ms^{-1}$$

Thank you

Dr. Bbosa Science