Simple harmonic motion

This is a periodic motion of a body whose acceleration is directly proportional to the displacement from a fixed point and is directed towards the fixed point.

$a \propto -x$ or;

$a = -\omega^2 x$ where ω is angular velocity, x, is displacement from fixed point

The negative sign means that the acceleration and displacement are always in opposite direction

Maximum acceleration

$a_{max} = -\omega^2 r$ where r is the maximum displacement or amplitude.

Force F

F = ma = $m\omega^2 x$

Maximum force, $F_{max} = m\omega^2 r$

Velocity in terms of displacement

Velocity of a body executing S.H.M can be expressed as a function of displacement x.

$a = -\omega^2 x$

$a = \frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt}$

But $\frac{dx}{dt} = v$

$a = v\frac{dv}{dx}$

$v\frac{dv}{dx} = -\omega^2 x$

$vdv = -\omega^2 xdx$

Integrating both sides

$\int v\, dv = -\omega^2 \int xdx$

$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + c$ ....(i)

Where c is a constant of integration

At momentary rest v = 0

x = r (amplitude

$$\frac{0^2}{2} = -\frac{\omega^2 x^2}{2} + c$$

$c = \frac{\omega^2 r^2}{2}$

Substituting c in eqn. (i)

$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + \frac{\omega^2 r^2}{2}$

$v^2 = (\omega^2 r^2 - \omega^2 x^2)$

$\boldsymbol{v^2 = \omega^2(r^2 - x^2)}$

Velocity is maximum when x = 0

$v^2 = \omega^2 r^2$

$\boldsymbol{v_{max} = \omega r}$

Displacement at any time t

$\frac{dx}{dt} = v$

$\frac{dx}{dt} = \omega\sqrt{(r^2 - x^2)}$

$\int \frac{dx}{\sqrt{(r^2-x^2)}} = \int \omega dt$

$\sin^{-1}\frac{x}{r} = \omega t + \varepsilon$

x = rsin$(\omega t + \varepsilon)$

When timing at the centre, t = 0, x = 0

x = rsin$\boldsymbol{\omega t}$ particle moves away from the centre

When timing at the amplitude, t = 0, x = r

x = rsin$\boldsymbol{\omega t}$ particle moves towards from the centre

Period

This is the time taken for one complete oscillation

T = $\frac{distance}{speed} = \frac{2\pi r}{v}$

But v = rω

T = $\frac{2\pi r}{r\omega}$

T = $\frac{2\pi}{\omega}$

Example 1

A particle moves in a straight line with simple harmonic motion about mean position O with a periodic time of $\frac{\pi}{2}$s. Find the magnitude of acceleration of the particle when 1m from O.

Solution

From $a = -\omega^2 x$ and $\omega = \frac{2\pi}{T}$

Negative ignored

$a = \left(\frac{2\pi}{\pi/2}\right)^2 x1 = 16ms^{-2}$

Example 2

A particle moves with S.H.M about a mean position O. When the particle is 25cm from O, its acceleration is 1ms^{-1} towards O. Find the

(i) Periodic time of motion
(ii) Magnitude of acceleration of the particle when 20cm from O

Solution

$a = -\omega^2 x$

$1 = \omega^2(0.25)$

$\omega^2 = 4$

$\omega = 2rads^{-1}$

T = $\frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$s

(ii) $a = -\omega^2 x$

$a = 2^2(0.2)$ = 0.8ms^{-2}

Example 3

A particle move with S.H.M of the periodic time $\frac{\pi}{2}$s and has a maximum speed of 3ms^{-1}. Find the maximum acceleration experienced by the particle

$\boldsymbol{v_{max} = \omega r}$

3 = $\left(\frac{2\pi}{\pi/2}\right)r$; r = 0.75m

$a_{max} = -\omega^2 r$

$a_{max} = \left(\frac{2\pi}{\pi/2}\right)^2 x\ 0.75$ = 12ms^{-2}

Example 4

A particle moves with S.H.M about a mean position O. the amplitude of the motion is 5m and the period is 8πs. Find the

(i) maximum speed of the particle (ii) speed of the particle when 3m from O

Solution

(i) $\boldsymbol{v_{max} = \omega r} = \frac{2\pi}{8\pi} x 5$ = 1.25ms^{-1}

(ii) $v^2 = \omega^2(r^2 - x^2)$

$v = \sqrt{\left(\frac{2\pi}{8\pi}\right)^2 (5^2 - 3^2)}$ =1ms^{-1}

Example 5

A body of mass 200g executes S.H.M with amplitude of 20mm. The maximum force which acts on it is 0.064N, calculate (a) its maximum velocity (ii) it period of oscillation

Solution

$F_{max} = m\omega^2 r$

0.064 = 0.2 $x\ \omega^2 x$ 0.02

$\omega = 4rads^1$

$v_{max} = \omega r$ = 4 x 0.02 = 0.08ms^{-1}

(ii) T = $\frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$ s

Example 6

A particle moves with S.H.M about O with a period of 2π seconds. It passes a point A with a velocity of 4ms^{-1} away from O. Given that OA = 4m, find

(i) The amplitude
(ii) The speed at B where OB = 3m

Solution

$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1rads^{-1}$

From $v^2 = \omega^2(r^2 - x^2)$

$4^2 = 1^2(r^2 - 4^2)$

r = 5.66m

Amplitude = 5.66m

(ii) $v = \omega\sqrt{r^2 - x^2}$

$v = \omega\sqrt{5.66^2 - 3^2}$ = 4.8ms^{-1}

Example 7

A particle moving with S.H.M has velocities of 4ms^{-1} and 3ms^{-1} at distance of 3m and 4m respectively from equilibrium position. Find

(a) amplitude (b) period

$v^2 = \omega^2(r^2 - x^2)$

$4^2 = \omega^2(r^2 - 3^2)$

$16 = \omega^2(r^2 - 9)$ …….. (i)

Also,

$3^2 = \omega^2(r^2 - 4^2)$

$9 = \omega^2(r^2 - 16)$ ……..(ii)

(i) ÷ (ii) $\frac{16}{9} = \frac{\omega^2(r^2-9)}{\omega^2(r^2-16)}$

r = 5m

Amplitude = 5m

(b) eqn. (i)

$16 = \omega^2(5^2 - 9)$

ω=1rads^{-1}

Period,$T = \frac{2\pi}{\omega} = \frac{2\pi}{1} = 2\pi$s

Example 8

A particle moves with S.H.M about a mean position O. the particle is initially projected from O with speed $\frac{\pi}{6}ms^{-1}$ and just reaches a point A, 2m from O.

(a) Find how far the particle from O, 3 seconds after projection

(b) How many second after projection is the particle a distance of 1m from O

(i) For the first time, (ii) second time (iii) third time

Solution

(a) At equilibrium position, $v_{max} = \omega r$

$\frac{\pi}{6} = \omega\ x2; \omega = \frac{\pi}{12}rads^{-1}$

x = rsinωt since particle moves away from O

x =2sin$\left(\frac{\pi}{12}x3\right)$= 1.414m

x = rsinωt

1 = rsin$\left(\frac{\pi}{12}\right)$t

$\left(\frac{\pi}{12}\right)$t =$\sin^{-1}$ 0.2= 30^0, 150^0, 210^0

T = 2s, 10s, 14s

Example 9

A particle is released from rest at point A. 1m from a second point O. the particle accelerates towards O and moves with S.H.M of period 12s and O is the centre of oscillation

(a) Find how far the particle is from O, 1s after release

(b) How many seconds after release is the particle at the midpoint of OA

(i) For the first time (ii) second time

Solution

(a) x = rcosωt since particles moves towards centre

$\omega = \frac{2\pi}{12} = \frac{\pi}{6}rads^{-1}$

$x = 1cos\frac{\pi}{6}\ x\ 1 = \frac{\sqrt{3}}{2}m$

(b) x = rcosωt

$0.5 = 1cos\frac{\pi}{6}\ x\ t$

$\frac{\pi}{6}t = \cos^{-1}$ 0.5 =60^0, 300^0

t = 2s, 10s

Example 10

A particle of mass 2kgmoving with S.H.M along the x-axis, is attracted towards the origin O by a force of 32x newton. Initially the particle is at x = 20. Find

(a) amplitude and period of oscillation
(b) velocity of the particle at any time t > 0
(c) speed when t = $\frac{\pi}{4}$s

Solution

(a) $F = m\omega^2 x$

$32x = 2\omega^2 x$

$\omega = 4rads^{-1}$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = 1.571s$

$v^2 = \omega^2(r^2 - x^2)$

$0^2 = 4^2(r^2 - 20^2)$

r =20m

(b) x = rcosωt

$v = \frac{d}{dt}(rcos\omega t)$

v = -rωsinωt = -20 x 4sin4t

v = -80sin4t

(c) speed = -80sin$\left(4\, x \frac{\pi}{4}\right)$ = 0ms^{-1}

Example 11

A particle is initially released from rest at point A and performs S.H.M about mean position B. The particle just returns to A during each oscillation and AB = $2\sqrt{2}m$. If the particle passes through B with speed $\pi\sqrt{2}ms^{-1}$, find

(i) The time when the particle is first travelling with speed of πms^{-1}
(ii) How far from B the particle during this time

Solution

(i) At equilibrium position (B)

$v_{max} = \omega r$

$\pi\sqrt{2} = \omega\, x\, 2\sqrt{2}$

$\omega = \frac{\pi}{2} rads^{-1}$

$v^2 = \omega^2(r^2 - x^2)$

$\pi^2 = \left(\frac{\pi}{2}\right)^2\left(\left(2\sqrt{2}\right)^2 - x^2\right)$

x = 2m

(ii) x = rcosωt

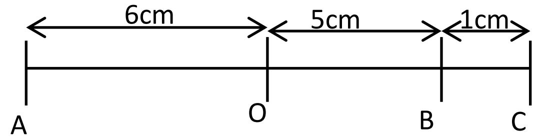
2 = $2\sqrt{2}cos\frac{\pi}{2}t$

$\frac{\pi}{2}t = \cos^{-1}\frac{1}{\sqrt{2}}$ = 45^0

t = 0.5s

Example 12

The points A, O, B, C lie in the that order on a straight line AO= OOC = 6cm and OB = 5c. A particle perform S.H.M of period 3s and amplitude 6cm between A and C. find the time taken for the particle from A to B



Time for AO is half the period = 1.5s

B is 5cm from O

x = rcosωt

5 = 6cos$\frac{2\pi}{3}$t

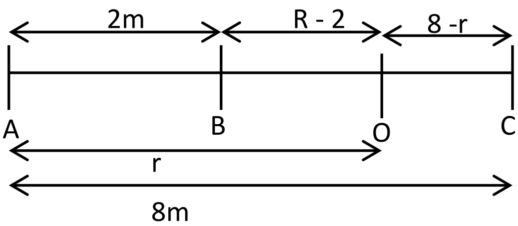
$\frac{2\pi}{3}$t = $\cos^{-1}\frac{5}{6}$ = 33.6^0

t = 0.28

Time for AB = 1.5 + 0.28 = 1.78s

Example 13

A particle passes through 3 points A, B and C in that order with velocity $0ms^{-1}$, $2ms^{-1}$ and $-1ms^{-1}$ respectively. The particle is moving with S.H.M in a straight line. What is the amplitude and period of the motion if AB = 2m and AC = 8m.



At B: v = $2ms^{-1}$, x = r-2

Using $v^2 = \omega^2(r^2 - x^2)$

$2^2 = \omega^2(r^2 - (r-2)^2)$

$1 = \omega^2(r-1)$ ……………………….. (i)

At C: v =$-1ms^{-1}$, x = 8-r

$(-1)^2 = \omega^2(r^2 - (8-r)^2)$

$1 = \omega^2(16r - 64)$ …………………. (ii)

(i)÷ (ii) $\frac{1}{1} = \frac{\omega^2(r-1)}{\omega^2(16r-64)}$

r = 4.2

Amplitude = 4.2m

Using equation (i)

$1 = \omega^2(4.2 - 1)$

ω = $0.56rads^{-1}$

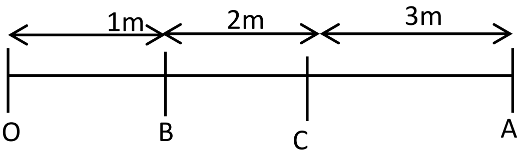
T = $\frac{2\pi}{\omega} = \frac{2\pi}{0.56} = 11.22s$

Example 14

A particle is performing S.H.M with centre O, amplitude 6m and period 2π. Points B and C lis between O and A with OB = 1m, OC = 3m and OA = 6m. find the least time taken while travelling from

(a) A to B (ii) A to C

Solution



$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

x = rcosωt

1 = 6cos (1x t)

t= 1.403s

(ii)

x = rcosωt

3 = 6cos (1x t)

t = 1.047s

Example 15

The velocity of a particle at any time is given by v(t) = -aωsinωt + bωcosωt.

(a) Find the expression for displacement x at any time that x = 0 when time t = 0

v(t) = -aωsinωt + bωcosωt.

But v(t) = $\frac{dx}{dt}$

⇨ $\frac{dx}{dt}$ =-aωsinωt + bωcosωt.

$\int dx = \int(-\text{a}\omega\sin\omega\text{t} + \text{b}\omega\cos\omega\text{t})dt$

x = $\frac{a\omega}{\omega}\cos\omega t + \frac{b\omega}{\omega}\sin\omega t + c$

Since a and b are expressed in terms of amplitude and phase angle ε

If at t =0, x = 0, this means that A = 0, hence a = b = 0

By substituting, 0= 0 + 0 + c

c = 0

hence : x = acosωt+ bsinωt

(b) Show that the motion of the particle is simple harmonic [x

$\frac{dx}{dt}$ =-aωsinωt + bωcosωt.

$\frac{d^2x}{dt^2}$ =-aω^2cosωt - bω^2sinωt

- $\omega^2(acos\omega t + bsin\omega t)$

But x = acosωt+ bsinωt

⇨ $\frac{d^2x}{dt^2}$ =- $\omega^2 x$, hence S.H.M

Example 16

A particle is moving with simple harmonic motion (SHM). When the particle is 15m from the equilibrium, its speed is 6ms^{-1}. When the particle is 13 m from equilibrium, its speed is 9ms^{-1}. Find the amplitude of the motion (05marks)

$v^2=\omega^2(A^2 – x^2)$

$6^2= \omega^2(A^2 – 15^2)$ ……. (i)

$9^2=\omega^2(A^2 – 13^2)$ …….. (ii)

Dividing (i) by (ii)

$\frac{36}{81} = \frac{A^2-225}{A^2-169}$

Amplitude A = 16.4256m

Revision exercise 1

1. A particle moving with S.H.M about a mean position O has velocities of 5ms^{-1} and 8ms^{-1} at distances of 16m and 12m respectively from O.
   (a) Amplitude [18.1m]
   (b) Period [10.63s]
2. A particle is describing S.H.M in a straight line directly towards a fixed point O. when it is a distance from O is 3m, its velocity is 25ms^{-1} and acceleration is 75ms^{-2}. Determine the
   (a) Period [$\left[\frac{2\pi}{5}\right]$ and amplitude [5.83m]
   (b) Time taken by the particle to reach O$\left[\frac{\pi}{10}\right]$
   (c) Velocity of the particle as it passes through O [29.15ms^{-1}]
3. A particle moving with simple harmonic motion about a mean position O has velocities of $3\sqrt{3}ms^{-1}$ and 3ms^{-1} at distances of 1m and 0.268m respectively. Find the amplitude of motion [2m]
4. A mass oscillates with S.H.M of period 1second and amplitude of oscillation is 5cm. Given that the particle begins from the centre of the motion, state the relationship between displacement x of the mass at any time t. Hence find the first two times when the is 3cm from its end position [x = rsinωt, 0.066s, 0.434s]
5. A particle moves in a straight line with S.H.M of period5s. the greatest speed is 4ms^{-1}, find the

(a) amplitude $\left[\frac{10}{2\pi}m\right]$.

(b) Speed when it is $\frac{6}{\pi}m$ from the centre. [3.2ms^{-1}]

6. A particle moves with S.H.M about mean position O with a periodic time $\frac{2\pi}{3}s$. When the particle is 0.8m from one extreme end, its speed is 3.6ms^{-1}. Find the amplitude of motion [1.3m]
7. A body of mass 0.30kg executes S.H.M with a period 2.5s and amplitude 0.04m. Determine
   (i) Maximum velocity of the body [0.101ms^{-1}]
   (ii) The maximum acceleration of the body [0.25ms^{-2}]
8. A particle moving with S.H.M about a mean position O has velocities of 1.6ms^{-1} and 1.2ms^{-1} at distances of 60cm and 80cm respectively from O. find
   (i) Amplitude [1m]
   (ii) Period [πs]
9. A particle moves with S.H.M in a straight line with amplitude 0.05m and period 12s. find
   (a) speed as it passes equilibrium position [0.026ms^{-1}]
   (b) maximum acceleration of the particle [0.014ms^{-2}]
10. A particle moves in straight line with S.H.M about mean position with periodic time $\frac{\pi}{2}s$ and amplitude 2m. Find the maximum speed of the particle [8ms^{-1}]
11. A body of mass 500g moves horizontally with S.H.M of periodic time $\frac{\pi}{2}s$ and amplitude 1m. Determine the magnitude of the greatest horizontal force experienced by the body during the motion. [8N]
12. A body of mass 100g moves horizontally with S.H.M about a mean position O. When the body is 50cm from O, the horizontal force on the body is of magnitude 5N, find the period of motion $\left[\frac{\pi}{2}s\right]$
13. A particle moves in a straight line with S.H.M about a mean position O with a periodic time $\frac{\pi}{4}s$ and amplitude 65cm. find how far the particle from O when its speed is 2ms^{-1} [60cm]
14. A particle moves in a straight line with S.H.M about a mean position O. The particle has zero velocity at a point which is 50cm from O and speed of 3ms^{-1} at O. Find
   (a) The maximum speed of the particle [3ms^{-1}]
   (b) The amplitude of motion[50cm]
   (c) The periodic time of motion[$\frac{\pi}{2}s$]
15. A particle moving with S.H.M about a mean position O has velocities 3ms^{-1} and 1.4ms^{-1} at distances of 2m and 2,4m respectively from point O. Find the
   (a) Amplitude of motion [2.5m]
   (b) Greatest speed attained by the particle [5ms^{-1}]
16. A particle is initially projected from a point A performs S.H.M about mean position A with periodic time of 3s and amplitude 50cm. find the
   (a) Maximum speed of projection [1.047ms^{-1}]
   (b) Speed of the particle 2s after projection [0.524ms^{-1}]
   (c) Distance of the particle from A 2s after projection[0.433m]
17. The head of piston moves with S.H.M of amplitude $\frac{\sqrt{3}}{10}$m about mean position O. How far from O the head of the piston when travelling with a speed equal to half of its maximum speed [15cm]
18. A particle is fastened to the midpoint of a stretched spring lying on a smooth horizontal surface. The particle is set in motion so that it moves with S.H.M about a mean position O. If one metre is the greatest distance the particle is from O during the motion. Find how far from O the particle is when it is travelling with speed equal to four fifth of its greatest speed. [60cm]

19. A particle performs S.H.M about mean position O with a periodic time of 3s and amplitude 6cm. Find time it takes the particle to travel from O to a point P, a distance of 3cm from O [0.25s]
20. A particle performs S.H.M about mean position O with a periodic time of 4s and amplitude 2cm. Find the time it takes the particle to travel from O to a point P, a distance of $\sqrt{2}$cm from O[0.5s]
21. A particle performs S.H.M about mean position O with a periodic time of 10s and amplitude 8cm. After passing through O, the particle moves through a point A which is 1cm from to a point B which is 2cm from O. find the time it takes the particle to move from A to B[0.186s]
22. A particle performs S.H.M about mean position O with a periodic time of 4.5s and amplitude 6cm. After passing through O, the particle moves through point P which is 3cm from O. Find the time that elapse before the particle passes through P [1.5s]
23. The points A, O, B, C lie in that order on a straight line with AO = OC = 4cm and OB = 2cm. A particle perform S.H.M of period 6s and amplitude 4cm between A and C. Find the time taken for the particle to travel from A to B [2s]

Thank you

Dr. Bbosa Science