Inequalities

An inequality is a logical statement that states relationship between two mathematical expressions.

The basic inequalities commonly used are

- Less than (<)
- More than (>)
- Less than or equal (≤)
- Greater than or equal (≥)

When solving for equations, the solutions or answers are individual values but when solving inequalities, the solutions are a range of possible real values.

Linear inequalities

Linear inequalities in one unknown given one in one equation

Solving linear inequalities in one unknown given one in one equation is done in the same way as solving for linear equation except

- The inequality symbols must be maintained
- The inequality symbol changes when dividing both sides of inequality equation by a negative number.

Example 1

Solve the following inequalities

(a) $4x - 2 > x + 7$
Solution
$4x - x > 7 + 2$
$3x > 9$
$x > 3$

(b) $3(2 - x) > 5(3 + 2x)$
Solution
$6 - 3x > 15 + 10x$
$-9 > 13x$
$x < \frac{-9}{13}$

(c) $\frac{x-2}{4} < \frac{2x-3}{3}$
Solution
Multiply both sides by 12
$12\left(\frac{x-2}{4}\right) < 12\left(\frac{2x-3}{3}\right)$
$3(x - 2) < 4(2x - 3)$
$3x - 6 < 8x - 12$
$-5x < -6$
$x > \frac{6}{5}$

(d) $\frac{1}{2}(x-1) + \frac{1}{3}(x-2) \leq \frac{1}{4}(x-3)$
Multiply both sides by 12
$6(x - 1) + 4(x - 2) \leq 3(x - 3)$
$6x - 6 + 4x - 8 \leq 3x - 9$
$7x \leq 5$
$x \leq \frac{5}{7}$

Linear inequalities involving indices

When solving inequalities involving indices such as $a^x > b$, where a and b are positive integers, introduce natural logarithms to both sides of the inequality. i.e.

$lna^x > b$

$xlna > lnb$

$x > \frac{lna}{lnb}$

Example 2

Solve the following inequalities correct to 3 decimal places

(a) $5^{2x} > 8$

Solution

$\ln 5^{2x} > \ln 8$

$2x\ln 5 > \ln 8$

$x > \frac{ln8}{2ln5} = 0.646$

$x > 0.646$

(b) $20^{-3x} < 15$

Solution

$\ln 20^{-3x} > \ln 15$

$-3x\ln 20 > \ln 15$

$x > -\frac{ln15}{3ln20} = -0.301$

$x > -0.301$

(c) $(0.8)^{-3x} > 2.4$

Solution

$\ln(0.8)^{-3x} > \ln 2.4$

-3xln0.8- > ln2.4

$x > \frac{ln2.4}{-3ln0.8} = 1.308$

$x > 1.308$

(d) $(0.8)^{3x} > 2.4$

Solution

$\ln(0.8)^{3x} > \ln 2.4$

$3x\ln 0.8 > \ln 2.4$

Note that logarithm of any number between 0 and 1 is negative; so ln0.8 is negative

$x < \frac{ln2.4}{3ln0.8} = -1.308$

$x < -1.308$

Linear inequalities in one unknown given two inequalities equations.

Linear inequalities in one unknown given two inequalities equations

The solution to two linear inequalities can be best handled by use of a number line. When finding a set of integers that satisfy the equations, we only take on integral (discrete) values.

Example 3

Find the set of integers which satisfy simultaneously both of the following equations

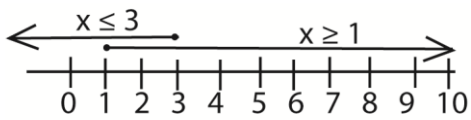
(a) $4x + 3 \geq 2x + 5$; $x + 4 \leq 7$

Solution

$4x + 3 \geq 2x + 5$; $x + 4 \leq 7$

$2x \geq 2$ $x \leq 3$

$x \geq 1$



The number line show that the set of integers that satisfies the two equations are {1, 2, 3}

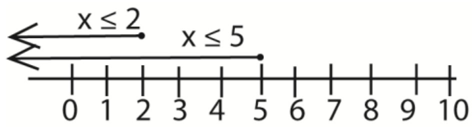
(b) $5 – 2x \geq 3 – x$; $1 -2x \leq 11 – 4x$

Solution

$5 – 2x \geq 3 – x$; $1 -2x \leq 11 – 4x$

$-x \geq -2$ $2x \leq 10$

$x \leq 2$ $x \leq 5$



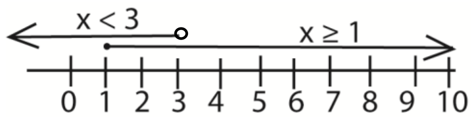
The number line show that the set of integers that satisfies the two equations are {x:x ≤ 2}

(c) $5x – 4 \geq 4x -3$, $\frac{1}{3}x < 1$

Solution

$5x – 4 \geq 4x -3$, $\frac{1}{3}x < 1$

$x \geq 1$ $x < 3$



The number line show that the set of integers that satisfies the two equations are {1, 2} (3 is not included

Example 4

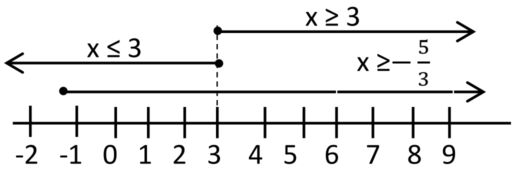
Show that there is just one integer which simultaneously satisfies the three inequalities and find that number

$\frac{1}{2}(x-1) \geq 1 \quad 2-3x \leq 7 \quad \frac{1}{3}x \leq 1$

Solution

$\frac{1}{2}(x-1) \geq 1 \quad 2-3x \leq 7 \quad \frac{1}{3}x \leq 1$

$x - 1 \geq 2$	$-3x \leq 5$	$x \leq 3$
$x \geq 3$	$x \geq -\frac{5}{3}$	$x \leq 3$



From the number line, there is only one point of intersection of the three inequalities, which is 3

Hence the set of integers that satisfy the three inequalities is {3}

Non-linear inequalities in one unknown

The following methods are employed

- Sign change
- Graphical method

When using graphical method, the set of values above the axis are positive and those below are negative

When using sign change method, a table describing specific regions of inequalities is used and the necessary tests are performed

If the inequality symbol is ≥ or ≤, care must be taken, because the critical values of the function and the numerator in case of fractions will always satisfy the inequalities

Before solving inequality, all terms must be taken to one side preferably the LHS

Method I: Graphical method

Example 5

Solve the following inequalities

(a) $2x^2 - 3x + 1 \leq 0$

Solution

Let $y = 2x^2 - 3x + 1$

The curve cuts the x – axis when y = 0

⇨ $2x^2 - 3x + 1 = 0$
$(x - 1)(2x - 1) = 0$

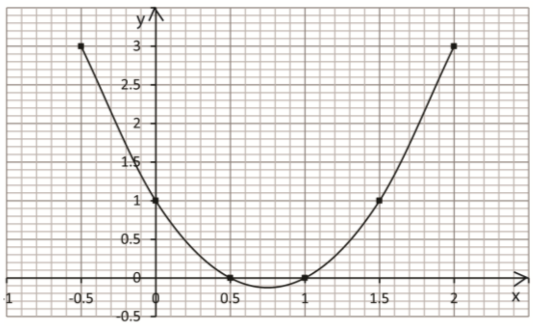
Either x – 1 = 0 or 2x – 1 = 0;

x = 1 or $x = \frac{1}{2}$

the curve cuts the y – axis when x = 0

⇨ y = 1

Since the coefficient of x^2 is positive, that the curve is U shaped



The solution set is $0.5 \leq x \leq 1$

(b) $7x^2 > 1 - 6x$

Solution

$7x^2 > 1 - 6x$

$7x^2 + 6x - 1 > 0$

Let $y = 7x^2 + 6x - 1$

The curve cuts the x – axis when y = 0

⇨ $7x^2 + 6x - 1 = 0$

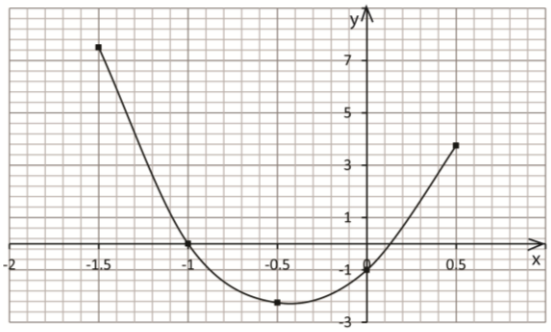
$(x + 1)(7x - 1) = 0$

Either x + 1 = 0 or 7x – 1 = 0

x = -1 or $x = \frac{1}{7}$

the curve cuts the y – axis when x = 0

⇨ y = -1



From the graph, the solution is x < -1 and x > $\frac{1}{7}$

(c) $2x^3 + 3x^2 \geq 2x$

Solution

$2x^3 + 3x^2 - 2x \geq 0$

Let $y = 2x^3 + 3x^2 - 2x$

The curve cuts the x – axis when y = 0

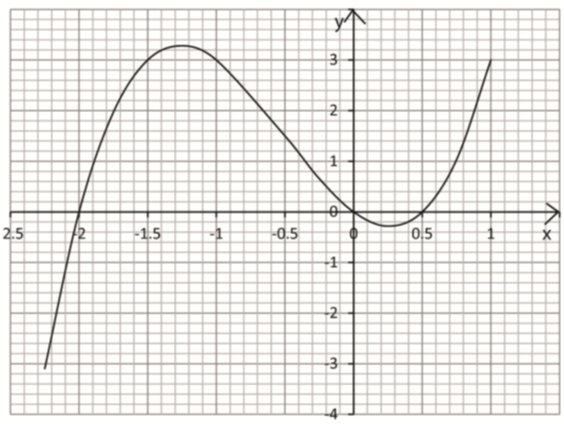
⇨ $2x^3 + 3x^2 - 2x = 0$
$x(2x - 1)(x + 2) = 0$

Either x = 0, 2x -1 = 0 or x + 2 = 0

x = 0, x = $\frac{1}{2}$ or x = -2

the curve cuts y – axis when x = 0

⇨ y = 0



From the graph the solution
-2 ≤ x ≤ 0 and x ≥ 0.5

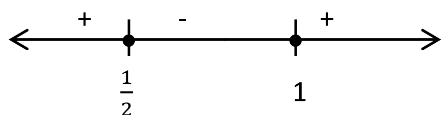
(d) Solve the inequality

$4x^2 + 2x < 3x + 6$ (06marks)

Method II: sign change

Example 6

(a) $2x^2 - 3x + 1 \leq 0$
(2x -1)(x-1) ≤ 0
The critical values of (2x -1)(x-1)= 0 are
x = 1 and x = $\frac{1}{2}$ respectively
The above illustration shows that the numbers $\frac{1}{2}$ and 1 subdivide the number line into three regions namely
$x \leq \frac{1}{2}$, $\frac{1}{2} \leq x \leq 1$, $x \geq 1$
The corresponding sign in the respective regions can be analysed by choosing any random value in each region, substitute it in the equation (2x -1)(x-1) and put the sign of the answer on the following number line



Note that the solution for (2x -1)(x-1) ≤ 0 is equal or less than zero or negative.
Closed circles indicate that the critical values are part of the solution.

Hence the solution set for (x – 1)(2x – 1) ≤ 0

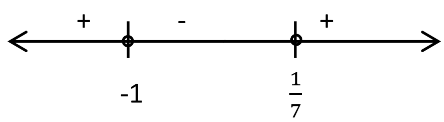
is $\frac{1}{2} \leq x \leq 1$

(b) $7x^2 + 6x - 1 > 0$

$7x^2 + 7x - x - 1 = 0$

(x + 1)(7x -1) = 0

The critical values x = -1 and $\frac{1}{7}$



The solution for (x + 1)(7x -1) > 0 is positive . Open circles indicate that the critical values are not part of the solution

Hence the solution set for (x + 1)(7x -1) > 0 is

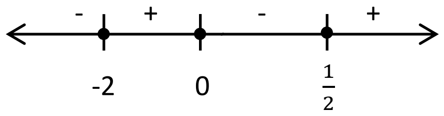
x < -1 and x > $\frac{1}{7}$

(c) $2x^3 + 3x^2 - 2x \geq 0$

$x(2x^2 + 3x - 2) \geq 0$

$x(x + 2)(2x - 1) \geq 0$

Critical values x = 0, x =-2, and x = $\frac{1}{2}$



The solution for $x(x + 2)(2x - 1) \geq 0$ is positive and the critical values are part of the solution

Hence the solution for $x(x + 2)(2x - 1) \geq 0$

$-2 \leq x \leq 0$ and $x \geq \frac{1}{2}$

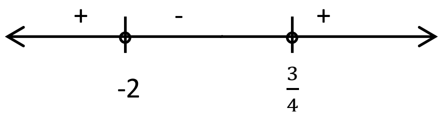
(d) $4x^2 + 5x - 6 < 0$

$4x^2 + 5x - 6 = 0$

Critical values

$x = \frac{-5 \pm \sqrt{5^2 - 4(4)(-6)}}{2(4)} = \frac{-5 \pm \sqrt{121}}{8} = \frac{-5 \pm 11}{8}$

$x = -2, \frac{3}{4}$



The solution $4x^2 + 5x - 6 < 0$ is negative and the critical values are not part of the solution

∴ the solution is $-2 < x < \frac{3}{4}$

(e) $\frac{3x^2-1}{x+2} \geq 2$

$\frac{3x^2-1}{x+2} - 2 \geq 0$

$\frac{3x^2-1-2(x+2)}{x+2} \geq 0$

$\frac{3x^2-1-2x-4}{x+2} \geq 0$

$\frac{(3x-5)(x+1)}{x+2} \geq 0$

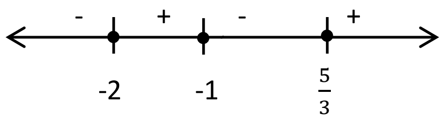
Finding critical values

$3x - 5 = 0$; $x = \frac{5}{3}$

$(x + 1) = 0$; $x = -1$

$(x + 2) = 0$, $x = -2$

Testing for correct region



The solution for $3x^2 - 2x - 5 \geq 0$ is positive and the critical values are part of the solution

Hence the solution for $3x^2 - 2x - 5 \geq 0$ is

$-2 \leq x \leq -1$ and $x \geq \frac{5}{3}$

The modulus of inequalities

The modulus of a number is the magnitude of that number (absolute value) which is always positive, e.g. $|1| = |-1| = 1$

When finding modulus of an inequality, the following must be considered

- The modulus on one side of the linear inequality is removed by introducing a negative number of the given value on the other side
  i.e. if $|x| < 3$, then $-3 < x < 3$
- The modulus on both sides of the linear inequality is removed by squaring both sides
- When the terms under modulus are fractional, square both sides of the inequality

Example 7

Solve the following inequalities

(a) $|x - 6| < 4$

$-4 < x - 6 < 4$

$-4 + 6 < x\ 4 + 6$

$2 < x < 10$

(b) $|3x + 4| < 6$

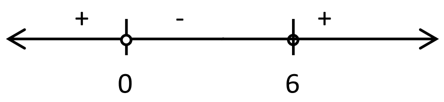
$-6 < 3x + 4 < 6$

$-6 - 4 < 3x < 6 - 4$

$-10 < 3x < 2$

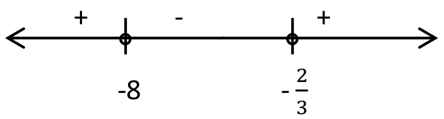
$-\frac{10}{3} < x < \frac{2}{3}$

(c) $|2x - 3| > |x + 3|$
$(2x – 3)^2 > (x + 3)^2$
$4x^2 – 12x + 9 > x^2 + 6x + 9$
$3x^2 – 18x > 0$
$3x(x – 6) = 0$
Critical values are x = 0 and x = 6



The solution x < 0 and x > 6

(d) $|2x + 5| < |x - 3|$
$(2x + 5)^2 < (x – 3)^2$
$4x^2 + 20x + 25 < x^2 – 6x + 9$
$3x^2 + 26x + 16 < 0$
$(3x + 2)(x + 8) < 0$
Critical values are x = -8 and x = $-\frac{2}{3}$



The solution is $-8 < x < -\frac{2}{3}$

Example 8

Find the range of value of x can take for the following inequality to be true

$\left|\frac{x}{x-3}\right| < 2$

Solution

Squaring both sides

$\frac{x^2}{x^2-6x+9} < 4$

$x^2 < 4(x^2 – 6x + 9)$

$x^2 < 4x^2 – 24x + 36$

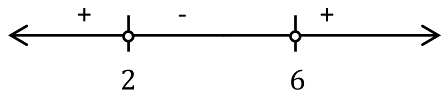
$0 < 3x^2 – 24x + 36$

Divide through by 3

$x^2 – 8x + 12 > 0$

$(x – 2)(x – 6) > 0$

Critical values are x = 2 and x = 6



Solution for (x – 2)(x – 6) > 0 (positive)

Hence the solution x < 2 and x > 6

Limits of inequality

This refers to interval within which the inequalities lies or does not lie.

This is done by expressing the function given as quadratic equation in x.

For real values of x, $b^2 \geq 4ac$

Example 9

(a) Given the function y = $\frac{3x-6}{x^2+6x}$, find the range of values within which y does not lies

Solution

$y = \frac{3x-6}{x^2+6x}$

$y(x^2 + 6x) = 3x – 6$

$yx^2 + (6y – 3)x + 6 = 0$

For real values of x, $b^2 \geq 4ac$

$(6y – 3)^2 = 24y$

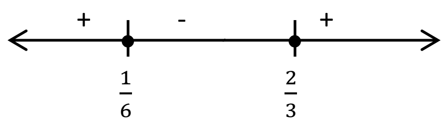
$36y^2 – 60y + 9 \geq 0$

Dividing through by 3

$12y^2 – 2y + 3 \geq 0$

$(6y – 1)(2y – 3) \geq 0$

The critical values are y = $\frac{1}{6}$ and y = $\frac{2}{3}$



Since the solution of the equation is positive; the required range $\frac{1}{6} < x < \frac{2}{3}$

(b) Find the range of values within which the function y =$\frac{3-2x}{4+x^2}$ lies

Solution

$y(4 + x^2) \geq 3 - 2x$

$yx^2 + 2x + 4y - 3 \geq 0$

For real values of x, $b^2 \geq 4ac$

$2^2 \geq 4y(4y - 3)$

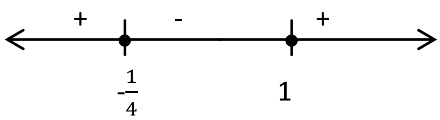
$1 \geq 4y^2 - 3y$

$0 \geq 4y^2 - 3y - 1$

$4y^2 - 3y - 1 \leq 0$

$(y - 1)(4y + 1) \leq 0$

Critical values y = 1 and y = $-\frac{1}{4}$



Solution for $(y - 1)(4y + 1) \leq 0$ is negative and critical values are part of the solution

Hence range of values is $-\frac{1}{4} \leq x \leq 1$

Simultaneous inequalities

Solving two simultaneous inequalities is best done by representing the inequalities on the graph. The unshaded (feasible) region represents the solution to the inequalities.

Example 10

Show by shading the unwanted regions; the region satisfying the inequalities y ≤ 2x +1 and y ≥ 3

Solution

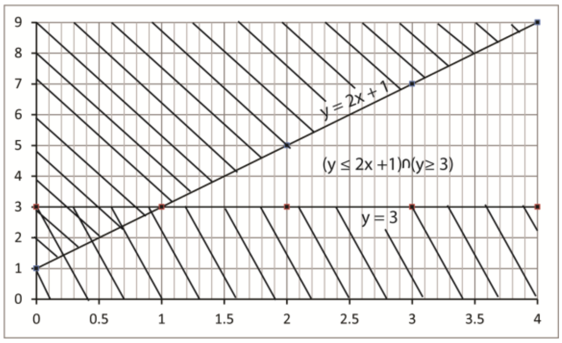
For y ≤ 2x +1, the boundary line is y = 2x +1

If x= 0, y= 1, (x, y) = (0, 1)

If x= 2, y= 5, (x, y) = (2, 5)

Testing for wanted region using point (0,0); 0 ≤ 1. Hence this point is in wanted region.

For y ≥ 3 boundary line is y = 3



Show by shading the unwanted regions; the region satisfying the inequalities x +2y ≥ 6, y > x, x< 5 and 3x + 5y ≤ 30

Solution

For x +2y ≥ 6 the boundary line is

x + 2y = 6

x	0	6
y	3	0

For x> y

The boundary line is x = y

x	0	5
y	0	5

For x < 5

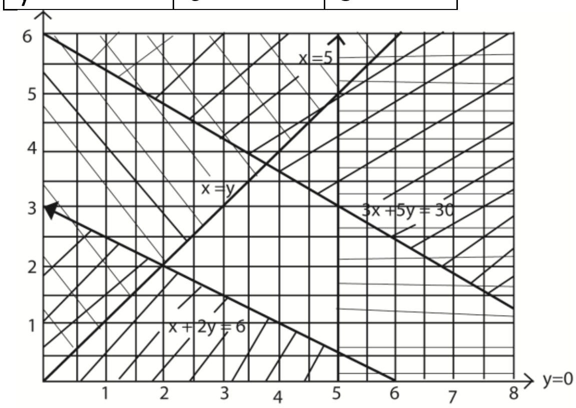
The boundary line is x = 5

For 3x + 5y ≤ 30

The boundary line

3x + 5y = 30

x	0	5
y	6	3



Revision exercise

1. Solve the following inequalities
   (a) $7x - 3 \geq 2x - 1$ $\left[x \geq \frac{2}{5}\right]$
   (b) $5(2 - x) - 2(3 - 6x) + 2(x - 1) > 0$ $\left[x > -\frac{2}{9}\right]$
   (c) $\frac{1}{2}(x+3) \leq \frac{1}{3}(x-5)$ $[x \leq -19]$
   (d) $\frac{1}{3}(x - 3) + \frac{1}{2}(3x - 1) > 2$ $\left[x \geq \frac{19}{11}\right]$
2. Solve the following inequalities. Correct 2 decimal places
   (a) $(0.8)^{-3x} > 4.0$ [x > 2.07 (2dp)]
   (b) $(0.6)^{-2x} < 3.6$ [x< 1.25 (2dp)]
3. Find the integers which simultaneously satisfy the following inequalities
   (a) $3x + 2 \geq 2x - 1$, $7x + 3 < 5x + 2$
   {-3, -2, -1}
   (b) $\frac{1}{2}(x+1) > 1$, $5x + 1 < 4(x + 2)$
   {2, 3, 4, 5, 6}
4. Find the set of values of x for which
   (a) $\frac{3x^2-1}{2+x} > 2$ $\left[x < -1, x > \frac{5}{3}\right]$
   (b) $\frac{3x^2-1}{1+x^2} > 1$[x < -1, x > 1]
   (c) $2(x^2 - 5) < x^2 + 6$ [-4 < x < 4]
   (d) $x^2 - x - 12 > 0$ [x < -3, x > 4]
   (e) $2x(x + 3) > (x + 2)(x - 3)$ [x < -6, x > -1]
5. Solve the following inequalities
   (a) $\frac{x}{x+1} \leq \frac{x-2}{x+3}$
   $\left[x \leq -3 \text{ and } -1 \leq x \leq -\frac{1}{2}\right]$
   (b) $\frac{x+2}{x-3} < \frac{x+5}{x-5}$ $[1 < x < 3 \text{ and } x > 5]$
   (c) $\frac{x-1}{2+2} > 2x$
   $\left[x < -2 \text{ and } -1 < x < -\frac{1}{2}\right]$
6. Solve the following inequalities
   (a) $|x - 3| < |2x + 3|$
   $[x < -18 \text{ and } x > 0]$
   (b) $\left|\frac{2x-4}{x+1}\right| < 4$ [x < -4 and x> 0]
   (c) $\left|\frac{x-4}{x+1}\right| > 3$ $\left[-\frac{7}{2} < x < \frac{1}{4}\right]$
   (d) $\left|\frac{x}{x-3}\right| < 2$ [x < 2, and x > 6]
7. If y = $\frac{x}{x^2+4}$, find the range of possible values of y for which x is real $\left[\frac{-1}{4} \leq x \leq \frac{1}{4}\right]$
8. Find the range of values of x can take for the following inequalities to be true
   (a) $\left|\frac{2x-4}{x+1}\right| < 4$ [x < -4, x > 0]
   (b) $|x + 1| > |x - 3|$ [x > 1]

Thank you

Dr. Bbosa Science