Surds

A logarithm is an exponent, an index or power

The logarithm of a positive quantity p to a given base q is defined as the index or power to which the bases q must be raised to make it equal to P. i.e. $\log_q p = x \text{ means that } q^x = p$ or x s the logarithm of p to base q

- x is the power (index, logarithm or exponent)
- q is the base
- p is the number (which must be positive)

Example 1

Find the values of x in the following

(a) $\log_2 8 = x$
(b) $\log_x 25 = 2$
Solution
(a) 8 = 2^3
$\therefore \log_2 8 = 3$; x = 3
(b) $x^2 = 25 = 5^2$

$$\therefore x = 5$$

Example 2

Evaluate

(a) $\log_{27} 9\sqrt{3}$
(b) $\log_{\frac{1}{2}} \frac{1}{4}$

Solution

Let $\log_{27} 9\sqrt{3}$ = x

$27^x = 9\sqrt{3}$

$3^{3x} = 3^2 . 3^{\frac{1}{2}} = 3^{\frac{5}{2}}$

Equating powers

3x = $\frac{5}{2}$

x = $\frac{5}{6}$

$\therefore \log_{27} 9\sqrt{3} = \frac{5}{6}$

(b) let $\log_{\frac{1}{2}} \frac{1}{4}$ = x

$\left(\frac{1}{2}\right)^x = \frac{1}{4} = \left(\frac{1}{2}\right)^2$

Equating powers x = 2

$\therefore \log_{\frac{1}{2}} \frac{1}{4} = 2$

Rules of logarithms

(a) (i) $\log_a a = 1$
Proof
Let $\log_a a = x$
$a^x = a^1$
x = 1
$\therefore \log_a a = 1$
(ii) $\log_a 1 = 0$
Proof
Let $\log_a 1 = x$
$a^x = a^0$
x = 0
$\therefore \log_a 1 = 0$

(b) The power rule
$\log_a P^q = \text{q}\log_a P$
Proof
Let $\log_a P = x$
$a^x = P$
Raising each to the power q
$a^{qx} = P^q$
⇨ $\log_a P = \log_a a^{qx} = qx$
$\therefore \log_a P^q = \text{q}\log_a P$

(c) The addition/multiplication rule

$\log_a pq = \log_a p + \log_a q$

Proof

Let $\log_a p = x \text{ and } \log_a q = y$

$p = a^x \text{ and } q = a^y$

$pq = a^x.a^y = a^{x+y}$

$\log_a pq = \log_a a^{x+y} = x + y$

$\therefore \log_a pq = \log_a p + \log_a q$

(d) The subtraction/division rule

$\log_a \left(\frac{p}{q}\right) = \log_a p - \log_a q$

Proof

Let $\log_a p = x \text{ and } \log_a q = y$

$p = a^x \text{ and } q = a^y$

$\frac{p}{q} = \frac{a^x}{a^y} = a^{x-y}$

$\log_a \left(\frac{p}{q}\right) = \log_a a^{x-y} = x - y$

$\therefore \log_a \left(\frac{p}{q}\right) = \log_a p - \log_a q$

(e) Change of base

$\log_a p = \frac{\log_q p}{\log_q a}$

Let $\log_a p = x$, then $a^x = p$

⇨ $\log_q a^x = \log_q p$

$x\log_q a = \log_q p$

$x = \frac{\log_q p}{\log_q a}$

$\therefore \log_a p = \frac{\log_q p}{\log_q a}$

Example 3

Evaluate

(a) $\log_2 8\sqrt{2}$

(b) $\log_a \frac{1}{a}$

Solution

(a) Either: let $\log_2 8\sqrt{2}$= x

⇨ $2^x = 8\sqrt{2} = 2^3.2^{\frac{1}{2}} = 2^{\frac{7}{2}}$

$x = \frac{7}{2}$

$\therefore \log_2 8\sqrt{2} = \frac{7}{2}$

Or $\log_2 8\sqrt{2} = \log_2 2^3.2^{\frac{1}{2}} = \log_2 2^{\frac{7}{2}}$

$= \frac{7}{2}\log_2$

$= \frac{7}{2}$

(b) Let $\log_a \frac{1}{a}$ = x

$a^x = a^{-1}$

$x = -1$

$\therefore \log_a \frac{1}{a}$ = -1

Example 4

Express each of the following as a single logarithm

(a) Log 4 + log3

(b) Log 5 + log 18 – log3

Solution

(a) Log 4 + log3 = log (4 x 3) = log 12

(b) Log 5 + log 18 – log3 = $\log\left(\frac{5\,x18}{3}\right)$ = log30

Example 5

Show that $\log_a p = \frac{1}{\log_p a}$. Hence solve the equation $\log_5 x + 2\log_x 5 = 3$

Solution

Let $\log_a p = x$

⇨ $a^x = p$

Introducing log to base p on both sides

$\log_p a^x = \log_p p$

$x\log_p a = 1$

$x = \frac{1}{\log_p a}$

$\therefore \log_a p = \frac{1}{\log_p a}$

Then,

$\log_5 x + 2\log_x 5 = 3$

$\log_5 x + \frac{2}{\log_5 x} = 3$

Let y = $\log_5 x$

⇨ $y + \frac{2}{y} = 3$

$y^2 - 3y + 2 = 0$

(y – 1)(y – 2) = 0

Either y = 1 or y = 2

When y = 1: $\log_5 x = 1$; x = $5^1 = 5$

When y = 2: $\log_5 x = 2$; x = $5^2 = 25$

x = 5 and x = 25

Example 6

Solve $\log_x 5 + 4\log_5 x = 4$
Expressing terms on LHS to $\log_5$.
$\frac{\log_5 5}{\log_5 x} + 4\log_5 x = 4$
$\frac{1}{\log_5 x} + 4\log_5 x = 4$
Let $\log_5 x = y$
$\frac{1}{y} + 4y = 4$
$4y^2 - 4y + 1 = 0$
(2y – 1)(2y – 1) = 0
2y = 1
y = $\frac{1}{2}$
⇨ $\log_5 x = \frac{1}{2}$

x = $5^{\frac{1}{2}} = \sqrt{5}$

Example 7

Show that 2log4 + $\frac{1}{2}\log 25 - \log 20 = 2\log 2$.
2log4 + $\frac{1}{2}\log 25 - \log 20$
$2\log 2^2 + \frac{1}{2}\log 5^2 - (\log 4 + \log 5)$
$2\log 2^2 + \frac{1}{2}\log 5^2 - \log 4 - \log 5$
4log2 + $\log 5 - 2\log 2 - \log 5$
2log 2

Example 8

(a) (i) Find $\log_9 27\sqrt{3}$ without using tables
(ii) Simplify $(\log_a b^2)(\log_b a^3)$

(b) Express $\log_{25} xy$ in terms of $\log_5 x$ *and* $\log_5 y$. Hence solve the simultaneous equation s:

$$\log_{25} xy = 4\frac{1}{2}$$

$\frac{\log_5 x}{\log_5 y}$ = -10

Solution

(a)(i) Changing the base from 9 to 3

$$\log_9 27\sqrt{3} = \frac{\log_3 27\sqrt{3}}{\log_3 9}$$

$$= \frac{\log_3 27 + \log_3 \sqrt{3}}{\log_3 9}$$

$$= \frac{\log_3 3^3 + \log_3 3^{\frac{1}{2}}}{\log_3 3^2} = \frac{3+\frac{1}{2}}{2} = \frac{7}{4} = 1.75$$

Or

Let $\log_9 27\sqrt{3}$ = x

$9^x = 27\sqrt{3}$

$(3^2)^x = 3^3.3^{\frac{1}{2}}$

$3^{2x} = 3^{\frac{7}{2}}$

Equating indices

2x = $\frac{7}{2}$

x =1.75

(ii) $(\log_a b^2)(\log_b a^3) = (\log_a b^2)\frac{(\log_a a^3)}{\log_a b}$

$= (2\log_a b)\frac{(3\log_a a)}{\log_a b}$

= 2 x 3 = 6

Or$(\log_a b^2)(\log_b a^3) = (2\log_a b)(3\log_b a)$

$= \left(\frac{2\log_{ba} b}{\log_b a}\right)(3\log_b a)$

= 2 x 3 = 6

(b) By change of base rule

$$\log_{25} xy = \frac{\log_5 xy}{\log_5 25} = \frac{\log_5 x + \log_5 y}{\log_5 5^2}$$

$$= \frac{\log_5 x + \log_5 y}{2}$$

$$\therefore \log_{25} xy = \frac{\log_5 x + \log_5 y}{2}$$

Hence solving

$\log_{25} xy = 4\frac{1}{2}$
$\frac{\log_5 x + \log_5 y}{2} = \frac{9}{2}$
$\log_5 x + \log_5 y = 9$ ……………….. (i)

$\frac{\log_5 x}{\log_5 y}$ = -10
$\log_5 x = -10\log_5 y$ …………….(ii)
Substituting eqn. (ii) into eqn. (i)
$-10\log_5 y + \log_5 y = 9$
$\log_5 y = -1$

y = $5^{-1} = \frac{1}{5}$
Substituting $\log_5 y$ into equation (ii)
$\log_5 x = 10$
$x = 5^{10}$
$\therefore\ x = 5^{10}$ and y = $\frac{1}{5}$

Example 9

(a) Given that $\log_b a = x$ show that b =$a^{\frac{1}{x}}$ and deduce $\log_b a = \frac{1}{\log_a b}$
(b) Find the value of x and y such that
(i) $\log_{10} x + \log_{10} y = 1.0$
$\log_{10} x - \log_{10} y = \log_{10} 2.5$

(ii) Simplify $2^x.2^y = 432$
(c) Simplify $\frac{1+\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt{3}}$

Solution

$\log_b a = x$

$b^x = a$

$\sqrt[x]{b^x} = \sqrt[x]{a}$

$b = a^{\frac{1}{x}}$

Taking log to bas a on both sides

$\log_a b = \ \log_a a^{\frac{1}{x}}$

$\log_a b = \ \frac{1}{x}\log_a a = \frac{1}{x}$

But x = $\log_b a$

$\therefore \log_{ba} b = \frac{1}{\log_b a}$

(b)(i) $\log_{10} x + \log_{10} y = 1.0$ ……………(i)

$\log_{10} x - \log_{10} y = \log_{10} 2.5$ …(ii)

Eqn. (i) + eqn. (ii)

2$\log_{10} x$ = $\log_{10} 10 + \log_{10} 2.5$

$\log_{10} x^2 = \log_{10} 25$

⇨ $x^2 = 25$
x = 5

Substituting x into eqn. (i)
$\log_{10} 5 + \log_{10} y = 1.0$

$\log_{10} y = \log_{10} 10 - \log_{10} 5$

$\log_{10} y = \log_{10} 10 \div 5 = \log_{10} 2$

y = 2

Hence x = 5 and y = 2

(ii) $2^x.2^y = 432 = 2^4.3^3$

Comparing

x = 4 and y = 3

(c) By rationalizing

$$\frac{(1+\sqrt{2}+\sqrt{3})(\sqrt{2}+\sqrt{3})}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})} = 1+\sqrt{3}-\sqrt{2}$$

Example 10

Prove that $\log_6 x = \frac{\log_3 x}{1+\log_3 2}$. Given that $\log_3 2 = 0.631$, find without using tables or calculator $\log_6 4$ correct to 3significant figures

Solution

$$\log_6 x = \frac{\log_3 x}{\log_3 6} = \frac{\log_3 x}{\log_3(2\ x 3)} = \frac{\log_3 x}{\log_3 3+\log_3 2}$$

$$= \frac{\log_3 x}{1+\log_3 2}$$

Substituting for $\log_3 2 = 0.631$

$$\log_6 x = \frac{\log_3 2^2}{1+\log_3 2} = \frac{2\log_3 2}{1+\log_3 2} = \frac{2\ x\ 0.631}{1+0.631}$$

$$= 0.774$$

Revision exercise

1. Evaluate
   (a) $\log_{\frac{1}{5}} 25\sqrt{5}$ $\left[-\frac{5}{2}\right]$
   (b) $\log_3 27$ [3]
2. Express the following as a single logarithm
   (i) Log15 - $\frac{1}{2}log9$ [log 5]
   (ii) 3log2 + 2log5 – log 20 [log 10]
3. Given that $\log_b a$ and $\log_c b = a$, show that $\log_c a = ac$
4. (a) solve the equation
   (i) $\log_a 4 + \log_4 a^2$
   $[a = 2\ and\ a = 4]$
   (ii) $\log_{14} x = \log_7 4x$ $\left[\frac{1}{196}\right]$

5. Without using tables or calculator show that $\frac{2log9 + log8 - \log 375}{\frac{1}{3}log6 - log5^{\frac{1}{3}}}$= 9
6. If$\log_2 x + \log_4 x + \log_{16} x = \frac{21}{16}$. Find the value of x [x = 1.6818]
7. Given$\log_a b = \log_d c$, show that $\log_c a = \log_d b$. Hence or otherwise solve the equation$\log_{9x} 64 = \log_x 4$. [x=3]
8. Solve the simultaneous equations
   $\log_{10}(y - x)$= 0
   $2\log_{10}(21 + x)$ [(x,y) =(-5, -4) or (4,5)
9. Given that $\log_2 x + 2\log_4 y = 4$. Show that xy = 16. Solve simultaneous equations
   $10\log_{10}(x + y) = 1$
   $\log_2 x + 2\log_4 y = 4$. [(x, y) =(2,8)or (8,2)
10. (a) If $\log_b a = x$, show that b = $a^{\frac{1}{x}}$ and deduce that $\frac{1}{\log_a b}$.

(b) Solve

(i) $\log_x 4 + \log_4 x^2 = 3$ $[x = 2\ or\ 4]$

(ii) $2^{2x-1} + \frac{3}{2} = 2^{x+1}$[x=0 or 1.585]

11. Prove that $\log_8 x = \frac{2}{3}\log_4 x$. Hence without using tables or calculator, evaluate $\log_8 6$ correct to three significant figure, if $\log_4 3 = 0.7925$ [0.862]
12. Given that $\log_3 x$ = p and $\log_{18} x = q$, show that $\log_6 3 = \frac{q}{p-q}$
13. Solve for x in the equation
    $\log_4(6 - x) = \log_2 x$
    [x = 2 since there is no negative log]
14. Solve the equation $\log_2 x - \log_x 8 = 2$
    [x =8 or x = $\frac{1}{2}$]
15. Solve the equation
    $\log_{25} 4x^2 = \log_5(3 - x^2)$ $[x = 1]$

Thank you

Dr. Bbosa Science