Series

Introductions

Numbers arranged in a definite order a sequence. Each number in the sequence is derived from a particular rule.

The terms below are examples of sequences

(a) 1, 3, 5, 7, 9 ….. is a sequence of odd numbers
(b) 2, 3, 5, 7, 11 … is a sequence of prime number
(c) 4, 16, 64 …. is a sequence formed by multiplying the proceding number by 4 to give the next number

Series are categorized into two:

- Arithmetic progression (A.P)
- Geometric progression

Arithmetic progression (A.P)

This is a series in which each term is obtained from the preceding one by addition or subtraction of a constant quantity.

The series 1 + 3 + 5 + 7 + 9 ….. is an A.P

Note the following in an A.P

(i) The first term of an A.P is denoted a. the first letter of the English alphabet
(ii) There is a common difference d. in the progression, a = 1 and d = 2.
(iii) Given the first term, a and the common difference, d
1^{st} term = a
2^{nd} term = a + d
3^{rd} term = a + 2d
n^{th} term (U_n) = a + (n – 1)d

Example 1

Find the 30^{th} term of a series that has an n^{th} term given by $\frac{1}{2}(32 - n)$

Solution

$U_n = \frac{1}{2}(32 - n)$

$U_{30} = \frac{1}{2}(32 - 30) = 1$

Example 2

The first term of an arithmetic progression (A.P) is 73 and the 9^{th} term is 25. Determine the common difference

Solution

U_n = a + (n – 1)d

25 = 73 + (9 – 1)d

25 = 73 + 8d

d = -6

Example 3

The 3^{rd}, 5^{th} and 8^{th} terms of A.P are 3n + 8, n + 34, and n^3 + 15 respectively. Find the value of n and hence the common difference of the A.P

Solution

a + 2d = 3n + 8 ……………… (i)

a + 4d = n + 24 ………………(ii)

a + 7d = n^3 + 15 ……………(iii)

Eqn. (ii) – eqn. (i)

2d = -2n + 16

d = -n + 8…………………… (iv)

eqn. (iii) – eqn. (i)

5d = n^3 – 3n + 7 ………………… (v)

Substituting eqn. (iv) into eqn. (v)

5(-n + 8) = n^3 – 3n + 7

n^3 + 2n – 33 = 0

By factorizing the equation

(n- 3)(n^2 + 3n + 11) = 0

Either n – 3 = 0 or n^2 + 3n + 11 = 0

n = 3 since n^2 + 3n + 11 = 0 has no real roots

Substituting for n in eqn. (iv)

d = -n + 8

d = -3 + 8 = 5

Hence n = 3 and the common difference is 5

Example 4

An A.P has the first term 3, common difference -2 and nth term – 15. Find n and the (n -3)th term

Solution

Given, a = 3, d =-2 , U_n = -15

But U_n = a + (n – 1)d

-15 = 3 + (n – 1) x (– 2)

n = 10

substitute n – 3 for n

$U_{(n-3)}$ = 3 + [(10 – 3) – 1](-2)

= -9

Example 5

The nth term of a series is Un = a3^n + bn + c given that U_1 = 4, U_2 = 13 and U_3 = 46, find the values of a, b and c.

Solution

By substituting for n

n = 1:

3a + b + c = 4 ……………..(i)

n =2

9a + 2b + c = 13 …………(ii)

n = 3

27a + 3b + 2 = 46 …………(iii)

Eqn. (ii) – eqn. (i)

6a + b = 9 …………………. (iv)

Eqn.(iii) – eqn (ii)

18a + b = 33 ……………… (v)

Eqn. (v) – eqn. (iv)

12a = 24

a = 2

Substitute for a in (v)

36 + b = 33

b = -3

Substituting for a and b in eqn. (i)

3 x 2 – 3 + c = 4

c = 1

Hence a = 2, b = -3 and c = 1

The sum of the first n terms of an A.P

There are two formulas used for finding the sum of the first n terms of the A.P depending on the terms given

Formula A

If the first term, a and common difference are given, then sum (S_n) of the first n terms is given by

$S_n = \frac{n}{2}(2a + (n-1)d)$

Formula B

If the first term is a and the last term is L, then sum of the first n terms (S_n)is given by

$S_n = \frac{n}{2}(a + L)$

Example 6

The first term of A.P is 73 and the common difference is -6, find the number of terms that must be added to give a sum of 96

Solution

$S_n = \frac{n}{2}(2a + (n-1)d$

$\frac{n}{2}(2x\ 73 - 6(n-1) = 93$

n(73 – 3(n – 1)) = 96

$73n - 3n^2 + 3n = 96$

$3n^2 - 76n + 96 = 0$

$n = \frac{76 \pm \sqrt{76^2 - 4x\ 3\ x\ 96}}{2\ x\ 3}$

$n = \frac{76 \pm 68}{6}$

n = 24

Hence the number of terms that must be added to give a sum of 96 are 24

Example 7

The tenth term of an arithmetic progression (A.P) is 29 and the fifteenth term is 44. Find the value of the first term and the common difference. Hence find the sum of the first 30 terms

Solution

U_n = a + (n – 1)d

U_{10} = a + (10 – 1)d

a + 9d = 29 …………….(i)

U_{15} = a + (15 – 1)d

a + 14d = 44 ………… (ii)

Eqn (ii) – eqn. (i)

5d = 15

d = 3

Substituting d in eqn. (i0

a + 9 x 2 = 29

a = 2

Sum of the first 30 terms

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_n = \frac{30}{2}(2\ x\ 2 + 3(30-1))$ = 1365

Example 8

The 5th term of an arithmetic progression (A.P) is 12 and the sum of the first 5 terms is 80. Determine the first term and common difference.

Solution

U_n = a + (n – 1)d [Un= nth term, a = first term and d = common difference}

U_5 = a + (5 – 1)d

a + 4d = 12 …………………………….(i)

The sum of the first n terms,

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_5 = \frac{5}{2}(2a + (5-1)d)$

10a + 20d = 80 x 2 = 160 ………(ii)

Eqn. (ii) – 5eqn. (i)

5a = 100

a = 20

Substituting for a in eqn. (i)

20 + 4d = 12

d = -2

Hence the first term = 20 and the common difference = -2

Example 9

(a) Prove that $\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$

Solution

$S_n = 1 + 2 + 3 + ... + n$

$+ \ S_n = n + (n-1) + (n-2) + ... + 1$

$2S_n = (n+1) + (n+1) + (n+1) + ... + (n+1)$

$2S_n = n(n+1)$

$Sn = \frac{n}{2}(n+1)$

(b) Use your answer in (a) to deduce

(i) $\sum_{r=1}^{n}(3r-1) = \frac{n}{2}(3n+1)$

Note to deduce is to use the already existing result to work out other problems

$S_n = \frac{n}{2}[last\ term + first\ term]$

$\sum_{r=1}^{n}(3r-1) = \frac{n}{2}\left(3n-1\ + (3-1)\right)$

$= \frac{n}{2}(3n-1\ + 2)$

$= \frac{n}{2}(3n+1)$

(ii) $\sum_{r=0}^{n}(r+5) = \frac{1}{2}(n+1)(n+10$

$= \sum_{r=1}^{n}(r+5) + \sum_{r=0}(r+5)$

$= \frac{n}{2}\left((n+1)\ + (1+5)\right) + 5$

$= \frac{1}{2}(n^2 + 11n) + 5$

$= \frac{1}{2}(n^2 + 11n + 10)$

$= \frac{1}{2}(n+1)(n+10)$

Inserting geometric means

Like for A.Ps, the terms inserted between given two values of a G.P are known as geometric means.

If n terms are inserted, then the total number of terms will be n+ 2 with the two extreme values representing the first and last terms respectively

Example 10

(a) Insert two geometric means between 2 and 16
Solution
1^{st} term a = 2
4^{th} term, $ar^3 = 16$
$2(r^3) = 18$
r = 2
the second term, ar = 2 x 2 = 4
the third term, $ar^2 = 2 \times 2^2 = 8$

(b) Insert three geometric means between 1 and 81
a = 1
the 5^{th} term $ar^4 = 81$
$1(r^4) = 81$
r = 3
the second term, ar = 1 x 3 = 3
the third term, $ar^2 = 1 \times 3^2 = 9$
the third term, $ar^3 = 1 \times 3^3 = 27$

Revision exercise 1

1. Find the 5^{th} and 8^{th} terms of a series that has an nth term given by $(-1)^n(2n+1)$ [-11, 17]
2. The first term of an arithmetic progression (A.P) is $\frac{1}{2}$. The sixth term of th A.P is four times the fourth term. Find the common difference of the A.P $\left[\frac{-3}{14}\right]$
3. The sum of p terms of an arithmetic progression is q and the sum of q terms is p; find the sum of p + q terms
4. (a) the first four terms of an A.P are 5, 11, 17 and 23. Find the 30^{th} term and the sum of the first 30 terms [179, 2760]
   (b) the second term of an A.P is 7 and the 7^{th} term is -8. Find the first term, common difference and the sum of the first 14 terms [10, -3, -133]
5. (a) An A.P has the first term of 2 and common difference 5. Given that the sum of the first n terms of the progression is 119, calculate n [7]
   (b) the sum of the first five terms of an A.P is $\frac{65}{2}$. Also , five times the 7^{th} term is the same as six times the second term. Find the first term and the common difference $\left[a = 6, d = \frac{1}{4}\right]$

6. The sum of the first n terms of the a series is n(n + 2). Find the first three terms [3, 5, 7]
7. In an A.P, the 1^{st} term is 13 and 15^{th} term is 11. Find the common difference and sum of the first 20 terms [7, 1590]
8. (a) Show that $\ln 2^r$, r = 1, 2,3, is an arithmetic progression
   (b) find the sum of the first 10 terms of the progression [38.1231]
   (c) Determine the least value of m for which the sum of the first 2m terms exceeds 883.7 [25]
9. In an arithmetic progression $u_1 + u_2 + u_3 + u_4 = 15$ and $u_{16} = -3$. Find the greatest integer N such that $U_N \geq 0$. Determine the sum of the first N terms of the progression. [N = 14, S_{14} = 136.5]

Geometric progression (G.P)

It is a series in which each term is obtained from the preceding one by multiplication or division by a constant quantity.

Observations

- The first term of G.P is also denoted, a
- The common ratio is r
- Given a and r
  1^{st} term = a
  2^{nd} term = ar
  3^{rd} term = ar^2
  4^{th} term = ar^3
  nth term = ar^{n-1}

The sum (S_n) of the first n terms of G.P

The sum (S_n) of the first n terms of G.P is given by

$S_n = \frac{a(r^n-1)}{r-1}$ for r > 1 or

$S_n = \frac{a(1-r^n)}{1-r}$ for r< 1

The geometric mean (G.M)

Suppose three numbers a, b and c are consecutive terms of GP, then, the middle term is the geometric mean.

The common ratio, $r = \frac{b}{a} = \frac{c}{b}$

⇨ $b^2 = ac$
$b = \sqrt{ac}$, the geometric mean

Example 10

A GP has 3^{rd} term 7 and 5^{th} term 847. Find the possible values of the common ratio and the corresponding 4^{th} terms

Solution

$U_3 = ar^2 = 7$

⇨ $ar^2 = 7$ ………………..(i)

$U_5 = ar^4 = 847$

⇨ $ar^4 = 847$ ……………(ii)

Eqn. (ii) ÷ eqn. (i)

$r^2 = 121$

$r = \pm 11$

From eqn (i)

$a = \frac{7}{121}$

The 4^{th} term, $U_4 = ar^3$

If r = 11, $U_4 = \frac{7}{121}(11)^3 = 77$

If r = -11, $U_4 = \frac{7}{121}(-11)^3 = -77$

Example 11

In a G.P the 2^{nd} term is 15 and the 5^{th} term is -405. Find the sum of the first 8 terms

Solution

$U_2 = ar = 15$

⇨ ar = 15 …………………….(i)

$U_5 = ar^4 = -405$

⇨ $ar^4 = -405$

Eqn. (ii) ÷ eqn. (i)

$r^3 = -27$

r = -3

From eqn. (i), a = -5

Since r < 1

$S_n = \frac{a(1-r^n)}{r-1} = \frac{-5(1-(-3)^8)}{1-(-3)} = \frac{32800}{4} = 8200 =$

Example 12

In the geometric series $u_1 + u_2 + u_3 + \ldots$

$u_1 + u_3 = 26$ and $u_3 + u_5 = 650$.

Find the possible values of u_4

Solution

$u_1 + u_3 = 26$

$a + ar^2 = 26$

$a(1 + r^2) = 26$ ………………… (i)

$u_3 + u_5 = 650$.

$ar^2 + ar^4 = 650$

$ar^2(1 + r^2) = 650$ …………..(ii)

Eqn. (ii) ÷ eqn. (i)

$r^2 = 25$

$r = \pm 5$

From eqn. (i)

$a(1 + 25) = 26$

$a = 1$

$u_4 = ar^3$

If r = 5; $u_4 = a(5)^3 = 125$

If r = 5; $u_4 = a(-5)^3 = -125$

Example 13

In a Geometric Progression (G.P), the difference between the fifth and the second term is 156. The difference between the seventh and the fourth is 1404 Find the possible values of the common ratio.

$U_5 – U_2 = 156$
$ar^4 – ar = 156$
$ar(r^3 -1) = 156$ ……………. (i)
$U_7 – U_4 = 156$
$Ar^6 – ar^3 = 1404$
$ar^3(r^3 -1) = 156$ ……………. (ii)
Eqn. (ii) ÷ eqn. (i)
$\frac{ar^3(r^3-1)}{ar(r^3-1)} = \frac{1404}{156}$
$r^2 = 9$
$r = \pm 3$
∴ r = 3 and r = -3

Example 14

(a) The first three terms of a Geometric progression (G.P) are 4, 8 and 16. Determine the sum of the first ten terms of the G.P. (04marks)

solution

a = 4, ar = 6
4r = 8
r = 2
$S_n = \frac{a(1-r^n)}{r-1}$
$S_{10} = 4\left(\frac{2^{10}-1}{2-1}\right) = 4092$

(b) An Arithmetic Progression (A.P) has a common difference of 3. A Geometric Progression (G.P) has a common ration of 2. A sequence is formed by subtracting the term of the A.P from the corresponding terms of the G.P. The third term of the sequence is 4. The sixth term of the sequence is 79. Find the first term of the

(i) A.P (08 marks)

(ii) G.P (06 mars)

A.P
x, x+3, x +6, x + 9, x + 12, x+ 15, ……
G.P
y, 2y, 4y, 8y, 16y, 32y, ….
4y – (x + 6) = 4
4y – x = 10 ……………..(i)
32y – (x + 15) = 79
32y – x = 94…………. (ii)
Eqn. (ii) – Eqn. (i)
28y = 84, => y = 3
Substituting for y into eqn. (i)
12 – x = 10

x =2

(i) A.P, U_1 = 2

(ii) G.P, U_1 =3

Example 15

The sum of the first n terms of a geometric progression (G.P) is $\frac{4}{3}(4^n - 1)$. Find the nth term as an integral power of 2

Solution

$S_n = \frac{a(1-r^n)}{r-1}$

Comparing with $S_n = \frac{4}{3}(4^n - 1)$

a = 4

r – 1 = 3, r = 4

The nth term, $U_n = ar^{n-1}$

$= 4 \times 4^{n-1} = 2^2 \times 2^{2(n-1)} = 2^{2+2n-2} = 2^{2n}$

Example 16

Find three numbers in geometrical progression such that their sum is 26 and their product is 216

Solution

Let the numbers be $\frac{a}{r}$, a and ar

Product = $\left(\frac{a}{r}\right)(a)(ar)$ = 216

$a^3 = 216 = 6^3$

a = 6

∴ the terms are $\frac{6}{r}, 6\ and\ 6r$

Sum of terms $\frac{6}{r} + 6 + 6r$ = 26

⇨ $6r^2 - 26r + 6 = 0$
$3r^2 - 13r + 3 = 0$
(r – 3)(3r – 1) = 0
Either r – 3 = 0; r = 3
Or 3r - 1 = 0; r = $\frac{1}{3}$

When r = 3

the terms are $\frac{6}{3} = 2, 6\ and\ 6x3 = 18$

when r = $\frac{1}{3}$

the terms are $6 \div \frac{1}{3} = 18, 6\ and\ 6\ x\ \frac{1}{3} = 2$

Hence the terms in their order are 2, 6, 18 or 18, 6, 2

Inserting geometric means

Like for A.Ps, the terms inserted between given two values of a G.P are known as geometric means.

If n terms are inserted, then the total number of terms will be n+ 2 with the two extreme values representing the first and last terms respectively

Example 16

(c) Insert two geometric means between 2 and 16
Solution
1st term a = 2
4th term, $ar^3 = 16$
$2(r^3) = 18$
r = 2
the second term, ar = 2 x 2 = 4
the third term, $ar^2 = 2 \times 2^2 = 8$

(d) Insert three geometric means between 1 and 81
a = 1
the 5th term $ar^4 = 81$
$1(r^4) = 81$
r = 3
the second term, ar = 1 x 3 = 3
the third term, $ar^2 = 1 \times 3^2 = 9$
the third term, $ar^3 = 1 \times 3^3 = 27$

Mixed terms of A.P and G.P

These are problems involving both A.Ps and G.Ps. when handling we make use of their respective properties.

Example 17

A geometric progression (G.P) and an arithmetic progression (A.P) have the same first term. The sum of their first, second and third terms are 6, 10.5 and 18 respectively. Calculate the sum of their 5th terms.

Solution

Terms	G.P	A.P	Sum
1st	a	a	2a = 6 …………………(i)
2nd	a + d	ar	a + d + ar = 10.5 ….(ii)
3rd	a + 2d	ar^2	$a + 2d + ar^2 = 18$ ..(iii)

From eqn. (i): 2a = 6; a = 3

From eqn. (ii): 3 + d + 3r = 10.5

d + 3r = 7.5 ……………….. (iv)

From eqn. (iii) $3 + 2d + 3r^2 = 18$

$2d + 3r^2 = 15$ …………….(v)

Eqn. (v) – 2eqn. (iv)

$3r^2 - 6r = 0$

3r(r -2) = 0

r -2 = 0

r = 2

Substitute for r into eqn. (iv)

d + 6 = 7.5

d = 1.5

Sum of their fifth terms

$= (a + 4d) + ar^4$

$= (3 + 4 \times 1.5) + 3 \times 2^4 = 57$

Example 18

The 1st, 4thand 8th terms of A.P form a G.P. if the first term is 9, find the

(i) Common difference of the A.P
(ii) Common ratio of the G.P
(iii) Difference in sums of the first 6 terms of the progressions.

Solution

Given that a, a + 3d, a + 7d form a G.P

Substituting for a = 9, the terms are

9, 9 + 3d, 9 + 7d

For a G.P, r = $\frac{9+3d}{9} = \frac{9+7d}{9+3d}$

⇨ (9 + 3d)2 = 9(9 + 7d)
$81 + 54d + 9d^2 = 81 + 63d$
$9d^2 - 9d = 0$
9d(d – 1) = 0
Either d – 1 = 0; d = 1
Or d = 0

When d = 0 all terms of A.P are equal

Hence the common difference d = 1

Example 19

(a) The sum of the first m terms of a progression is m(2m + 11)
   (i) Show that the progression is an A.P
   (ii) Determine the nth term of the progression

Solution

Given $S_m = m(2m + 11)$

First term = $S_1 = 1(2 \times 1 + 11) = 13$

$S_2 = 2(2 \times 2 + 11) = 30$

Second term = 30 – 13 = 17

$S_3 = 3(2 \times 3 + 11) = 51$

Third term = 51 – 20 = 21

The progression is 13, 17, 21, ….. Hence A.P with the first term 13 and common difference, d= 4

(ii) $U_n = S_n - S_{n-1}$

= n(2n + 11) – (n – 1)(2(n -1) + 11)

= 9 + 4n

Example 20

(a) The first, second and last term of an A.P are a, b, c respectively. Prove that the sum of all terms is $\frac{(a+b)(b+c-2a)}{2(b-a)}$
Solution
If the 1st term is a and the second term is b; the common difference, d = (b – a)

Last term (nth terms c = a + (n – 1)d

c = a + (n – 1)(b – 1)

i.e. n – 1 = $\frac{c-a}{b-a}$ => n = $\frac{b+c-2a}{b-a}$

but $S_n = \frac{1}{2}n(a+L)$

$=\frac{1}{2}\left(\frac{b+c-2a}{b-a}\right)(a+c)$

$= \frac{(a+b)(b+c-2a)}{2(b-a)}$

(b) The first, second and last terms of a GP are a and b. show that the sum of the first n terms is $\frac{a^n-b^n}{a^{n-2}(a-b)}$

Solution

Common ratio $=\frac{b}{a}$

$S_n = \frac{a(1-r^n)}{1-r} = \frac{a\left(1-\left(\frac{b}{a}\right)^n\right)}{1-\left(\frac{b}{a}\right)}$

$= \frac{a(a^n-b^n)a}{a^n(a-b)} = \frac{a^n-b^n}{a^{n-2}(a-b)}$

Sum to infinity of a G.P

We have seen that the sum of n terms of a G.P for r < 1 is $S_n = \frac{a(1-r^n)}{1-r}$

Now for -1 < n < 1 i.e. $|r| < 1$, as n →∞, r^n →0

Therefore $S_n = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$

Hence the sum of a GP to infinity for $|r| < 1$ converges to $S_\infty = \frac{a}{1-r}$ and diverges for r > 1 and r < 1

Example 21

(a) Calculate the sum to infinity of the following terms

(i) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

Solution

a = 1 and r = $\frac{1}{2}$

$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

(ii) $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$

Solution

a = $\frac{1}{5}$ and r = $\frac{1}{5}$

$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{1}{4}$

(b) Work out the following

(i) $\sum_{r=0}^{\infty}\left(\frac{1}{3}\right)^r$

Solution

$\sum_{r=0}^{\infty}\left(\frac{1}{3}\right)^r = 1 + \frac{1}{3} + \frac{1}{9} + \cdots$

a = 1 and r = $\frac{1}{3}$

$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$

(ii) $\sum_{r=2}^{\infty}\left(-\frac{1}{8}\right)^r$

Solution

$\sum_{r=2}^{\infty}\left(-\frac{1}{8}\right)^r = \frac{1}{64} - \frac{1}{512} + \frac{1}{4096} - \cdots$

A = $\frac{1}{64}$ and r = $-\frac{1}{8}$

$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{64}}{1+\frac{1}{8}} = \frac{1}{72}$

(iii) $\sum_{r=0}^{\infty} a^r$

Solution

a = 1 and r = a

$S_\infty = \frac{a}{1-r} = \frac{1}{1-a}$

(iv) $\sum_{r=1}^{\infty}(3x)^{r+1}$

Solution

$\sum_{r=1}^{\infty}(3x)^{r+1}$ = $9x^2 + 27x^3 + 81x^4 + \ldots$

a = $9x^2$ and r = 3x

$S_\infty = \frac{a}{1-r} = \frac{9x^2}{1-3x}$

Example 22

(a) Express the following as fractions using approach of sum of a G.P to infinity

(i) $0.\dot{4}$

Solution

$0.\dot{4} = \frac{4}{10} + \frac{4}{100} + \frac{1}{1000} + \cdots$

$=\frac{4}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right)$

$=\frac{4}{10}\left(\frac{1}{1-\frac{1}{10}}\right) = \frac{4}{10} x \frac{10}{9} = \frac{4}{9}$

(ii) $3.1\dot{2}\dot{7}$

Solution

$3.1\dot{2}\dot{7} = 3 + \frac{1}{10} + \frac{27}{1000} + \frac{27}{10000} + \cdots$

$= 3 + \frac{1}{10} + \frac{27}{1000}\left(1 + \frac{1}{100} + \frac{1}{10000} + \cdots\right)$

$= \frac{31}{10} + \frac{27}{1000}\left(\frac{1}{1-\frac{1}{100}}\right) = \frac{31}{10} + \frac{27}{1000}\left(\frac{100}{99}\right)$

$= \frac{31}{10} + \frac{3}{110} = \frac{344}{110} = \frac{172}{55}$

Hence $3.1\dot{2}\dot{7} = \frac{172}{55}$

(b) The sum to infinity of a GP is 7 and the sum of the first two terms is $\frac{48}{7}$. Find the common ratio and the first term of the GP with positive common ratio

Solution

$S_\infty = 7$

$\Rightarrow \frac{a}{1-r} = 7$

$a = 7(1-r)$ ……………………(i)

But $S_2 = \frac{48}{7}$

$a + ar = \frac{48}{7}$

$a(1 + r) = \frac{48}{7}$ ………………………(ii)

substituting eqn. (i) into eqn. (ii)

$7(1-r)(1+r) = \frac{48}{7}$

$49\,(1 - r^2) = 48$

$49r^2 = 1$

$r^2 = \frac{1}{49}$

$r = \pm\frac{1}{7}$

Considering appositive ratio

From eqn. (i)

$a = 7(1 - \frac{1}{7}) = 6$

Revision exercise 2

1. The common ration of a GP is -5 and the sum of the first seven terms of the progression is 449. Find the first three terms. $\left[\frac{1}{29}, \frac{-5}{29}, \frac{25}{29}\right]$
2. In the geometrical series $\sum_{r=1}^{n} u_r$, $u_5 - u_2 = 156$ and $u_7 - u_4 = 1404$. Find the possible values of the common ratio and corresponding values of u_1
   [r = 3, a = 156; r = -3, a = $\frac{13}{7}$]
3. The sum of the second and third terms of a G.P is 9. It the seventh term is eight times the fourth term, find the
   (a) The first term and the common ration [a = $^3/_2$ and r = 2]
   (b) The sum of the fourth and first term [36]
4. Find the sum of ten terms of geometrical series 2, -4, 8 ….. [-682]
5. The second and the third terms of a G.P progression are 24 and 12(b + 1) respectively. Find b if the sum of the first three terms of the progression is 76
   $\left[\frac{1}{2}\ or\ 2\right]$
6. The sum of the 2nd and 3rd terms of a G.P is 12. The sum of the 3rd and 4th terms is -36. Find the 1st term and common ratio [a = 2, r = -3]
7. What is the smallest number of terms of GP 5, 10, 20 …. that can give a sum greater than 500, 000 [n= 17]
8. The first, fourth and eighth terms of Arithmetic progression (A.P) form a geometric progression. If the first term is 9, find
   (i) The common difference of A.P [1]
   (ii) The common ratio of the G.P $\left[\frac{4}{3}\right]$
   (iii) The difference in the sums of the first 6 terms of the progressions [55.7049]
9. The second, third and ninth terms of an A.P form a G.P. find the common ratio of the G.P [6]
10. (a) The sum of the first 10 terms of an AP is 120. The sum of the next 8 terms is 240. Find the sum of the next 6 terms [264]
    (b) the arithmetic mean of the a and b is three times their geometric mean. Show that $\frac{a}{b} = 7 \pm 12\sqrt{2}$
11. The first three terms of a geometric series are 1, p, and q. Given also that 10, q and p are the first three terms of an arithmetic series. Show that $2p^2 - p - 10 = 0$
    Hence find the possible values of p and q
    [p = -2 and q = 4 or p = $\frac{5}{2}$ and q = $\frac{25}{4}$]

Application of A.Ps and G.Ps to interest rates

If a sum of money P is invested at a simple interest rate of r% per annum, the amount received after n years is given by A = P + I where I = $\frac{P\,x\,r\,x\,n}{100}$

By substitution we have

A = P + $\frac{P\,x\,r\,x\,n}{100}$ = P$\left(1+\frac{nr}{100}\right)$

The interest for one year is $\frac{Pr}{100}$, for 2 years is $\frac{2Pr}{100}$, for n year = $\frac{nPr}{100}$. Therefore the various amounts of interest after one, two, three, etc. years form an AP

On the other hand, if the principal P is invested at compound interest rate of r% per annum, the interest being added annually, the amount after one year is $\left(1+\frac{r}{100}\right)$, after two years is $P\left(1+\frac{r}{100}\right)^2$, after 3 years is $P\left(1+\frac{r}{100}\right)^3$ and after n years $P\left(1+\frac{r}{100}\right)^n$

Hence the amounts after one, two, three, etc. years for a GP.

Note: if with compound interest is added half annually as much as when added yearly, but it is added twice as much. Hence amount A = $P\left(1+\frac{r}{100}\right)^{2n}$

Now suppose that instead of adding the interest annually, it is the principal, P which is added annually,

Amount after 1st year = $P\left(1+\frac{r}{100}\right)^1$

Amount after 2nd year = $P\left(1+\frac{r}{100}\right)^2$

Amount after 3rd year = $P\left(1+\frac{r}{100}\right)^3$

Amount after nth year = $P\left(1+\frac{r}{100}\right)^n$

Total amount after n years

= $P\left(1+\frac{r}{100}\right)^1 + P\left(1+\frac{r}{100}\right)^2 + \ldots + P\left(1+\frac{r}{100}\right)^n$

= P$\left[\left(1+\frac{r}{100}\right)^1+\left(1+\frac{r}{100}\right)^2+\cdots+\left(1+\frac{r}{100}\right)^n\right]$

This is a G.P with:

first term = $\frac{100+r}{100}$ = (100 + r)%

And common ratio $\frac{100+r}{100}$ = (100 + r)%

$s_n = P$(100 + r)%$\left[\frac{(100+\text{r})\%-1}{(100+\text{r})\%-1}\right]$

Example 23

(a) Find the amount at the end of ten years when 500000 shillings is invested at 5% compound interest

(i) the interest being added annually

Solution

A = $P\left(1+\frac{r}{100}\right)^n$

$= 500000\left(1+\frac{5}{100}\right)^{10}$

A = 814,447.3134

(ii) the interest being added twice a year

Solution

A = $P\left(1+\frac{r}{100}\right)^{2n}$

$= 500000\left(1+\frac{5}{100}\right)^{20}$

A = 1,326,648.853

(b) Find the amount at the end of ten years when 500000 shillings is invested at 5% simple interest

A = $\frac{nPr}{100} = \frac{10\,x\,500000\,x\,5}{100}$ =750,000

Proof by induction

This is a mathematical technique that uses the reasoning that if a statement is true for a particular value say n = 1, then it must be true for n = 2, 3, 4, …. This involves the proof that the series on the LHS must be equal to the terms on the RHS

Example 24

Prove by induction that

1 + 2 + 3 + …+ n = $\frac{1}{2}n(n+1)$

Solution

Here we need to show that the above series agrees for all values of n = 1, 2, 3 .., q and q +1

Suppose n = 1

LHS = 1 [taking only the 1st number]

RHs = $\frac{1}{2}(1)((1)+1)$= 1[substituting for n = 1]

∴LHS = RHS => the series hold for n = 1

Suppose n = 2

LHS = 1+ 2 = 3 [taking first 2 numbers]

RHs = $\frac{1}{2}(2)((2)+1)$= 3[substituting for n = 2]

∴LHS = RHS => the series hold for n = 2

Suppose n = q

1 + 2 + 3 + …+ q = $\frac{1}{2}q(q+1)$

For n = q +1(i.e. adding k + 1 on both sides)

1 + 2 + 3 + …+ q + (q+1) = $\frac{1}{2}q(q+1)$+ (q + 1)

=$(q+1)(\frac{1}{2}q+1)$

=$\frac{1}{2}(q+1)(q+2)$

The result is true for n = q + 1, hence true for all positive values of n

Example 25

Prove by induction

(a) $1^2 + 2^2 + 3^2 + …. + n^2 = \frac{1}{6}n(n+1)(2n+1)$
Solution
For n = 1
LHS = 1^2 = 1;
RHS = $\frac{1}{6}(1)((1)+1)(2(1)+1)$ = 1
∴ LHS = RHS => the series holds for n = 1
For n = 2
LHS = $1^2 + 2^2$ = 5;
RHS = $\frac{1}{6}(2)((2)+1)(2(2)+1)$ = 5
∴ LHS = RHS => the series holds for n = 2
For n = k
$1^2 + 2^2 + 3^2 + …. + k^2 = \frac{1}{6}k(k+1)(2k+1)$
For n = k + 1
$1^2 + 2^2 + 3^2 + …. + k^2 + (k+1)^2$

$$= \frac{1}{6}k(k+1)(2k+1)+(k+1)^2$$

=$[k\ +1]\left[\frac{1}{6}k(2k+1)+(k+1)\right]$

=$[k\ +1]\left[\frac{1}{6}(2k^2+7+6)\right]$

=$\frac{1}{6}(k+1)(k+2)(2k+3)$

∴ LHS = RHS => the series holds for n = k+1 hence for all positive values of n

(b) $\sum_{r=1}^{n=r} r^3 = \frac{1}{4}n^2(n+1)^2$

Solution

For n = 1

LHS = $\sum_{r=1}^{r=1} r^3 = 1^3$ = 1

RHS =$\frac{1}{4}(1)^2(1+1)^2$= 1

∴ LHS = RHS => the series holds for n = 1

For n = 2

LHS = $\sum_{r=1}^{r=2} r^3 = 1^3 + 2^3$ = 1

RHS =$\frac{1}{4}(2)^2(2+1)^2$= 9

∴ LHS = RHS => the series holds for n = 2

For n = k

$$\sum_{r=1}^{n=k} r^3 = \frac{1}{4}k^2(k+1)^2$$

For n = k+1

$\sum_{r=1}^{n=k+1} r^3 = \sum_{r=1}^{n=k} r^3 + (k+1)^3$

=$\frac{1}{4}k^2(k+1)^2+(k+1)^3$

=$(k+1)^2\left[\frac{1}{4}k^2+k+1\right]$

=$\frac{1}{4}(k+1)^2[k^2+4k+4]$

=$\frac{1}{4}(k+1)^2(k+2)^2$

(c) $p + pq + pq^2 + … + pq^{n-1} = p\left(\frac{1-q^n}{1-q}\right)$
For n = 1,
LHS = p
RHS =$p\left(\frac{1-q^1}{1-q}\right)$ = p
∴the identity is true for n = 1

For n = 1,

LHS = p + pq =p(1+q)

RHS =p$\left(\frac{1-q^2}{1-q}\right)$ = p$\left(\frac{(1+q)(1-q)}{1-q}\right)$=p(1+q)

∴the identity is true for n = 2

For n = k

p + pq + pq^2 +…+pq^{k-1} =p$\left(\frac{1-q^k}{1-q}\right)$

For n = k+1

p + pq + pq^2 +…+pq^{k-1}+pq^k = p$\left(\frac{1-q^k}{1-q}\right)$+pq^k

= p$\left(\frac{1-q^k+q^k+p^{k+1}}{1-q}\right)$

= p$\left(\frac{1-q^{k+1}}{1-q}\right)$

∴the identity is true for n = k+1, hence true for all positive values of n

Revision exercise 3

1. Five millions shillings is invested each year at a rate of 15% compound interest by a certain bank.
   (a) Find how much he will receive at the end of ten years [116.7464m]
   (b) How many years will it take to accumulate to more than 50m [6]
2. John opened an account in the bank and deposited200,000 shillings every month for ten months without withdrawing. Find how much money he accumulated after 10 months if the bank offered 10% compound interest per month. [3,506,233.412]
3. Peter deposited sh. 100,000 at the beginning of every year for 5 years; find how much he got at the end of the fifth year at the compound interest rate of 2% per annum. [530,812.1]
4. Prove by induction
   (i) $\sum_{r=1}^{n} 3^{r-1} = \frac{3^n-1}{2}$
   (ii) $1^3 + 2^3 + 3^3 + \ldots. + n^3$
   $= \frac{1}{4}n^2(3n+1)(5n+1)$
   (iii) $\sum_{r-1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$
   (iv) $\frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n(n-1)} = 1 - \frac{1}{n}$
   (v) $\sum_{r=1}^{n} \frac{1}{r(1+1)} = \frac{n}{n+1}$
   (vi) $\sum_{r=1}^{n} ap^{r-1} = \frac{a(p^n-1)}{p-1}$
5. The sum of the first n terms of a Geometric Progression (G.P) is $\frac{4}{3}(4^n - 1)$. Find its nth term as an integral power of 2 [2^{2n}]
6. Prove by mathematical induction the $3^{2n} - 1$ is a multiple of 8 for all positive integers n
7. Use the method of induction to prove that 6n – 1is divisible by 5 for all positive integral values of n
8. Prove by induction that
   $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$
   hence evaluate $\sum_{r=1}^{20} r(r+1)$

Thank you

Dr. Bbosa Science