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Polynomials

A polynomial in x is a function in the form

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$ where $a \neq 0$

(a) Quadratics

If α and β are the two roots, then, x = α and x = β

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

(b) Cubic

If α, β and λ are the three roots, then,

x = α, x = β and x = λ

⇨ $(x - \alpha)(x - \beta)(x - \lambda) = 0$

$[x^2 - (\alpha + \beta)x + \alpha\beta](x - \lambda) = 0$

$x^3 - \lambda x^2 - (\alpha + \beta)x^2 + (\alpha + \beta)\lambda x + \alpha\beta x - \alpha\beta\lambda = 0$

$x^3 - (\alpha + \beta + \lambda)x^2 + (\alpha\beta + \alpha\lambda + \beta\lambda)x - \alpha\beta\lambda = 0$

(c) Qurtic

If α, β, λ and ρ are the four roots, then

x = α, x = β, x = λ and x = ρ

⇨ $(x - \alpha)(x - \beta)(x - \lambda)(x - \rho) = 0$

$[x^3 - (\alpha + \beta + \lambda)x^2 + (\alpha\beta + \alpha\lambda + \beta\lambda)x - \alpha\beta\lambda](x - \rho) = 0$

$X^4 - (\alpha + \beta + \lambda + \rho)x^3 + (\alpha\beta + \alpha\lambda + \alpha\rho + \beta\lambda + \beta\rho + \lambda\rho)x^2 - (\alpha\beta\lambda + \alpha\beta\rho + \alpha\lambda\rho + \beta\lambda\rho)x + \alpha\beta\lambda\rho = 0$

The above illustration shows that:

(i) The signs of the terms alternate in the order: +, - , + ….. starting with the first term.

(ii) The coefficient of the second term is –(sum of the roots) and the last term with its appropriate sign is the product of the roots

Example 1

Find the sums and products of the roots of the following equations

(a) $5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 6 = 0$

Solution

Dividing through by 5

$x^5 + \frac{4}{5}x^4 - \frac{3}{5}x^3 + \frac{2}{5}x^2 - \frac{1}{5}x + \frac{6}{5} = 0$

sum of roots = $-\frac{4}{5}$

product of roots = $\frac{6}{5}$

(b) $3x^4 - 5x^3 + 2x^2 + 0x + 9 = 0$

Dividing through by 3

$x^4 - \frac{5}{3}x^3 + \frac{2}{3}x^2 + 0x + 3 = 0$

Sum of roots = $\frac{5}{3}$

Product of roots = 3

Addition and subtraction of polynomial

Polynomial are added or subtracted if they are of the same degree. This is done by adding or subtracting the coefficients of the corresponding term

Example 2

(a) Given the polynomial

$f(x) = 5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 6$ and

$g(x) = 3x^4 - 5x^3 + 2x^2 + 9$

Find (i) f(x) + g (x) (ii) f(x) – g(x)

Solution

(i) f(x) + g(x)

$5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 6$

$+ \quad 3x^4 - 5x^3 + 2x^2 + 0x + 9$

$= \quad 5x^5 + 7x^4 - 8x^3 + 2x^2 - x + 15$

f(x) - g(x)

$5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 6$

$- \quad 3x^4 - 5x^3 + 2x^2 + 0x + 9$

$= \quad 5x^5 + x^4 + 2x^3 + 0x^2 - x - 3$

(b) Given polynomials

$f(x) = 2x^3 + 4x^2 - 2x - 8$ and

$g(x) = x^3 - 2x^2 + 3x + 5$

Find (i) f(x) + g (x) (ii) f(x) – g(x)

Solution

(i) f(x) + g (x)

$= (2 + 1)x^3 + (4 - 2)x^2 + (-2 + 3)x + (-8 + 5)$

$= 3x^3 + 2x^2 + x - 3$

(ii) f(x) – g(x)

$= (2 - 1)x^3 + (4 - -2)x^2 + (-2 - 3)x + (-8 - 5)$

$= x^3 + 6x^2 - 5x - 13$

Multiplication of polynomials

When multiplying two functions together, the terms of the first function are multiplied by the terms of the second function

Example 3

(a) Given the polynomial

$f(x) = 5x^3 + 2x^2 + 9$ and

$g(x) = 4x^4 - 3x^3 + 2x^2 - x + 6$

Find f(x) x g (x)

Solution

$= 5x^3 (4x^4 - 3x^3 + 2x^2 - x + 6)$

$+ 2x^2 (4x^4 - 3x^3 + 2x^2 - x + 6)$

$+ 9 (4x^4 - 3x^3 + 2x^2 - x + 6)$

$= (20x^7 - 15x^6 + 10x^5 - 5x^4 + 30x^3)$

$+ (8x^6 - 6x^5 + 4x^4 - 2x^3 + 12x^2)$

$+ (36x^4 - 27x^3 + 18x^2 - 9x + 54)$

$= 20x^7 - 7x^6 + 4x^5 + 35x^4 + x^3 + 30x^2 - 9x + 54$

Hence f(x) .g(x) =

$20x^7 - 7x^6 + 4x^5 + 35x^4 + x^3 + 30x^2 - 9x + 54$

Example 4

Find coefficient of x^3 in the expansion

$(2x^3 + x^2 - 5x + 6)(2x + 4)$

Solution

$(2x^3 + x^2 - 5x + 6)(2x + 4)$

$= 2x(2x^3 + x^2 - 5x + 6) + 4(2x^3 + x^2 - 5x + 6)$

$= (2x^4 + 2x^3 - 10x^2 + 12x) + 8x^3 + 4x^2 - 20x + 16$

$= 2x^4 + 10x^3 - 6x^2 - 8x + 16$

The coefficient of x^3 is 10

Division of polynomials

The division of polynomial may be done by long division as follows.

Example 5

(a) Divide $x^3 - 7x - 6$ by (x+ 1)

Solution

$$\begin{array}{r l}
 & x^2 - x - 6 \\
(x+1) & \overline{)\, x^3 - 7x - 6} \\
 & -\ x^3 - x^2 \\
\hline
 & x^2 - 7x - 6 \\
 & -\ x^2 - x \\
\hline
 & -6x - 6 \\
 & -\ \ -6x - 6 \\
\hline
 & 0 + 0
\end{array}$$

Example 6

Find the remainder when $2x^3 + x^2 + 5x - 4$ is divided by 2x - 1

$$\begin{array}{r l}
 & x^2 + x + 3 \\
(2x-1) & \overline{)\, 2x^3 + x^2 + 5x - 4} \\
 & -\ 2x^3 - x^2 \\
\hline
 & 2x^2 + 5x - 4 \\
 & -\ 2x^2 - x \\
\hline
 & 6x - 4 \\
 & -\ \ 6x - 3 \\
\hline
 & -1
\end{array}$$

Hence the remainder is -1

Example 7

Show that x = -2 is a root of the equation $2x^3 - x2 - 8x + 4 = 0$. Hence find the other roots

Solution

If x =-2 is a root of the function, then its

Remainder must be equal to zero

Hence x = -2 is a root of $2x^3 - x^2 - 8x + 4 = 0$

x = -2 => x + 2 =0

$$
\begin{array}{r}
2x^2 - 5x + 2 \\
(x - 2)\overline{)\,2x^3 - x^2 - 8x + 4} \\
-\ 2x^3 + 4x^2 \qquad\qquad \\
\hline
-5x^2 - 8x + 4 \\
-\ 5x^2 - 10x \qquad \\
\hline
2x + 4 \\
-\qquad 2x + 4 \\
\hline
0
\end{array}
$$

Since a cubic equation has at most three roots; the remaining two roots are obtained by solving the quadratic equation by factorization

2x2 – 5x + 2 = 0

2x2 – 4x – 2x + 2 =0

2x(x – 2) – 1(x – 2) = 0

(2x – 1)(x – 2) = 0

2x – 2 = 0 or x – 2 = 0

$x = \frac{1}{2}$ or x = 2

∴ the other roots are $x = \frac{1}{2}$ and x = 2

Revision exercise 1

1. Find the degree of each of the following polynomials
   (a) $x^6 + 4x^4 - 2$ [6]
   (b) $2x^3 - x^2 - 8x + 4$ [3]
   (c) $5x^3 + 2x^2 + 9$ [3]
   (d) $2x^4 + 10x^3 - 6x^2 - 8x + 16$ [4]
2. Given that
   (a) $f(x) = x^3 + 2x^2 - 3x + 2$ and $g(x) = 2x^3 - x^2 + 5x - 4$, find f(x) – g(x)
   $[-x^3 + 3x^2 - 8x + 6]$
   (b) $f(x) = x^3 + 2x^2 - 3x + 2$ and $g(x) = 2x^3 - x^2 + 5x - 4$, find g(x) – f(x)
   $[x^3 - 3x^2 + 8x - 6]$
   (c) $f(x) = 2x^3 - 5x^2 + 6x$ and $g(x) = x^3 - 6x^2 + 5x + 1$, find f(x) – g(x)
   $[x^3 + x^2 + x - 1]$
   (d) $f(x) = 2x^3 - 5x^2 + 6x$ and $g(x) = x^3 - 6x^2 + 5x + 1$, find 2f(x) + g(x)
   $[5x^3 - 16x^2 + 7x + 1]$
3. Given that
   (a) $f(x) = x^2 - 2x + 5$ and $g(x) = x^3 + 6x - 4$, find xf(x) + 3g(x)
   $[4x^3 - 2x^2 + 23x - 12]$
   (b) $f(x) = x^3 + 6x - 4$ and $g(x) = x^2 - 2x + 5$, find 3f(x) + xg(x)
   $[2x^3 + 2x^2 + 13x - 12]$

Other methods of finding the remainder

Apart from using long division, the remainder when a function is divided by a certain factor can be obtained by **remainder theorem** and **synthetic approach**.

(a) The remainder and factor theorems

When a number say 186 is divided by 4, this can be represented simply as follow

$$
\begin{array}{r}
46 \\
4\overline{)186} \\
-\ 16\ \\
\hline
26 \\
-\ 24 \\
\hline
2
\end{array}
$$

∴ the quotient is 46 and the remainder is 2
The above alogarithm can be written as
$\frac{186}{4} = 46 + \frac{2}{4}$
Or simply 186 = 4Q + R, where the quotient Q = 46 and the remainder, R = 2. This is referred to as the remainder theorem

The **remainder theorem** states that, when a function f(x) is divided by by (x – a) and leaves a remainder, then the remainder of the function is f(a)
From f(x) = (x –a)Q(x) + R
When we substitute for x = a; f(a) = R

When R = 0, => f(a) = 0. This is referred to as the **factor theorem**,

Example 8

Find the remainder when

(a) $f(x) = x^3 + 3x^2 – 4x + 2$ is divided by x – 1
Solution
Let $x^3 + 3x^2 – 4x + 2 = (x – 1)Q(x) + R$
Putting x = 1: 1 + 3 – 4 + 2 = R
2 = R
(b) $f(x) = 3x^3 + 2x – 4$ is divided by x – 2
Solution
Let $3x^3 + 2x – 4 = (x – 2)Q(x) + R$
Putting 2: 24 + 4 – 4 = R
R = 24
(c) $f(x) = 2x^3 + 4x^2 – 6x + 5$ is divided by x – 1
Let $2x^3 + 4x^2 – 6x + 5 =(x – 1)Q(x) + R$
Putting 1: 2 + 4 – 6 + 5 = R
R = 5
(d) $8x^3 + 4x + 3$ is divided by 2x -1
Let $8x^3 + 4x + 3 = (2x – 1)Q(x) + R$
Putting $\frac{1}{2}$: 1 + 2 + 3 = R
R = 6

Synthetic approach for finding the remainder

The synthetic approach can be illustrated as follows
$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$ is divided by x – a, then the following steps are taken using a table
(i) The factor in question must be linear
(ii) The 1st row of the table contains the coefficients of x in f(x) in descending order.
(iii) The at most left hand side of the 3rd row contains a value a from the division, x – a i.e. if x – a = 0, then x = a
(iv) The first digit in the second row comes under the second digit in the first row and this digit is equal to a_0 x a, where a_0 is the coefficient of x^n.
(v) The second digit in the 3rd row is the same as the sum of the digits in the 1st and 2nd rows
(vi) The next digit in the 2nd row is equal to a_0 x the 2nd digit in the 3rd row
(vii) The corresponding numbers in the 1st and 2nd row are added to give the digit in the 3rd row and the process continues
(viii) The last digit in the 3rd row is the remainder of the polynomial and the digit to the left of the remainder are coefficients of the **quotient** provided the divisor is in the form tx + b, where t = 1. If t ≠ 0, then we divide the digit to left of the remainder by t to obtain the coefficients of the quotient.

Example 9

Find the remainder and quotient when the function

(a) $f(x) = x^3 + 3x^2 – 4x + 2$ is divided by x – 1
from $f(x) = x^3 + 3x^2 – 4x + 2$, the coefficients of x in a descending order are 1, 3, -4 and 2
From x – 1 = 0, => x = a = 1

1st row	1	3	-4	2	+; a=1
2nd row		1	4	0	
3rd row	1	4	0	(2)	←remainder

The remainder is 2
Quotient $x^2 + 4x$
(b) $x^5 + x – 9$ is divided by x + 1

$x^5 + x – 9 \equiv x^5 + 0x^4 + 0x^3 +0x^2 + x - 9$

1st row	1	0	0	0	1	-9	+; x = -1
2nd row		-1	1	-1	1	-2	
3rd row	1	-1	1	-1	2	(-11)	←remainder

Remainder: -11

Quotient: $x^4 – x^3 + x^2 –x + 2$

(c) $4x^5 – 3x^3 + 2x + 7$is divided by 2x – 1

$4x^5 - 3x^3 + 2x + 7 \equiv 4x^5 + 0x^4 - 3x^3 + 0x^2 + 2x + 7$

$$\begin{array}{lcccccc} 1^{st}\text{ row} & 4 & 0 & -3 & 0 & 2 & 7 \\ 2^{nd}\text{ row} & & 2 & 1 & -1 & -\frac{1}{2} & \frac{3}{4} \\ \hline 3^{rd}\text{ row} & 4 & 2 & -2 & -1 & \frac{3}{2} & \left(\frac{31}{4}\right) \end{array} \quad +; \quad x = \frac{1}{2}$$

$\leftarrow$remainder

The quotient is $\frac{1}{2}[4x^4 + 2x^3 - 2x^2 - x + \frac{3}{2}] = 2x^4 + x^3 - x^2 - \frac{1}{2}x + \frac{3}{4}$ and remainder is $\frac{31}{4}$

Example 10

The polynomial $x^4 + px^3 - x2 + qx - 12$ has factors $x + 1$ and $x + 2$.

Find the values of p and q, hence factorize the polynomial completely.

Solution

Let $f(x) = x^4 + px^3 - x^2 + qx - 12$

By factor theorem

Putting x = -1

$f(-1) = 1 - p - 1 - q - 12 = 0$

$p + q = -12$ ………………………… (i)

Putting x = -2

$f(-2) = 16 - 8p - 4 - 2q - 12 = 0$

$4p + q = 0$ …………………………(ii)

Eqn. (ii) – eqn. (i)

$3p = 12$

$p = 4$

substituting for p into eqn. (i)

$4 + q = -12$

$q = -16$

$\therefore f(x) = x^4 + 4x^3 - x^2 + 16x - 12$

Now $(x + 1)(x + 2) = x^2 + 3x + 2$

Since $(x + 1)(x + 2)$ is a factor, then $x^2 + 3x + 2$ is also a factor of f(x)

By long division to find other factors

$$\begin{array}{r} x^2 + x - 6 \\ x^2 + 3x + 2 \overline{) x^4 + 4x^3 - x^2 - 16x - 12} \\ -\ \underline{x^4 + 3x^3 + 2x^2} \\ x^3 - 3x^2 - 16x \\ -\ \underline{x^3 + 3x^2 + 2x} \\ -6x^2 - 18x - 12 \\ -\ \underline{-6x^2 - 18x - 12} \\ 0 \quad + 0 \quad + 0 \end{array}$$

$\therefore x^4 + 4x^3 - x^2 + 16x - 12$

$= (x^2 + 3x + 2)(x^2 + x - 6)$

Now $x^2 + x - 6 = (x + 3)(x - 2)$

Hence $x^4 + 4x^3 - x^2 + 16x - 12$

$= (x + 1)(x + 2)(x + 3)(x - 2)$

Example 11

When $f(x) = x^3 - ax + b$ is divided by $x + 1$, the remainder is 2 and $x + 2$ is a factor. Find a and b.

Solution

By substitution for x = -1 in the function

$(-1)3 - a(-1) + b = 0$

$-1 + a + b = 2$

$a + b = 3$ ……………… (i)

since x + 2 is a factor, substituting for x = -2 in the function gives zero

$(-2)^3 - a(-2) + b = 0$

$-8 + 2a + b = 0$

$2a + b = 8$ …………….(ii)

Eqn. (ii) – eqn. (i)

$a = 5$

substituting for a in eqn. (i)

$5 + b = 3 \Rightarrow b = -2$

Example 12

The function $f(x) = x^3 + px^2 – 5x + q$ has a factor $x – 2$ and has a value of 5 when $x = -3$. Find the values of p and q

Solution

By substitution for $x = 2$ in the function

$2^3 + p(2)^2 – 5(2) + q = 0$

$4p + q = 2$ ………….(i)

By substitution for $x = -3$ in the function

$(-3)^3 + p(-3)^2 – 5(-3) + q = 5$

$9p + q = 17$ ………. (ii)

Eqn. (ii) – eqn.(i)

$5p = 15$

$p = 3$

From eqn. (i)

$4 \times 3 + q = 2$

$q = -10$

Hence $p = 3$ and $q = -10$

Example 13

The polynomial $x^4 + px^3 – x^2 + qx – 12$ has a factor $x^2 + 3x + 2$.

Find the value of p and q and hence factorise completely

Solution

$x^2 + 3x + 2 = (x + 1) (x + 2)$

Let $f(x) = x^4 + px^3 – x^2 + qx – 12$

$f(-1) = (-1)^4 + p(-1)^3 – (-1)^2 + q(-1) – 12 = 0$

$1 – p – 1 + q – 12 = 0$

$-p + q = 12$……………..(i)

$f(-2) =(-2)^4 + p(-2)^3 – (-2)^2 + q(-2) – 12 = 0$

$16 – 8p – 4 – 2q – 12 = 0$

$-8p – 2q = 0$

$4p – q = 0$ ……………(ii)

Eqn. (i) + eqn. (ii)

$3p = 12$

$p = 4$

From eqn. (i)

$q = -4 – 12 = -16$

Hence $p = 4$ and $q = -16$

$\therefore x^4 + px^3 – x^2 + qx – 12= x^4 + 4x^3 – x^2 – 16x – 12$

By long division to find other factors

$$x^2 + x – 6$$
$$x^2 + 3x + 2\overline{)x^4 + 4x^3 –x^2 - 16x – 12}$$
$$– \underline{x^4 + 3x^3 + 2x^2}$$
$$x^3 – 3x^2 – 16x$$
$$– \underline{x^3 + 3x^2 + 2x}$$
$$– 6 x^2 –18x – 12$$
$$– \underline{– 6 x^2 –18x – 12}$$
$$0 \quad + 0 \quad + 0$$

$\therefore x^4 + 4x^3 –x^2 + 16x – 12$

$=(x^2 + 3x + 2)(x^2 + x – 6)$

Now $x^2 + x – 6 = (x + 3)(x – 2)$

Hence $x^4 + 4x^3 –x^2 + 16x – 12$

$= (x + 1)(x + 2) (x + 3)(x – 2)$

Example 14

If $4x^3 + ax^2+ bx + 2$ is divisible by $x^2 + k^2$, show that $ab = 8$

Solution

By long division

$$\begin{array}{r} 4x + a \\ (x^2 + k^2)\overline{)4x^3 + ax^2 + bx + 2} \\ -\underline{4x^3 + 4k^2} \\ ax^2 + (b - 4k^2)x + 2 \\ -\underline{ax^2 + ak^2} \\ (b - 4k^2)x + 2 - ak^2 \end{array}$$

Since the remainder is zero,

$(\lambda - 4k^2)x + 2 - ak^2 = 0$

Comparing coefficients of x:

$b - 4k^2 = 0$ ………………… (i)

$2 - ak^2 = 0$

$k^2 = \frac{2}{\mu}$……………………… (ii)

Eqn. (ii) into eqn. (i)

$b - 4\left(\frac{2}{a}\right) = 0$

⇨ $ab - 8 = 0$

∴ $ab = 8$

Revision exercise 2

1. Given that f(x) $3x^3 - 4x^2 - 5x + 2$, factorize f(x) completely and hence solve the equation f(x) = 0 $\left[x = -1, 2\ or\ \frac{1}{3}\right]$
2. Prove that a – b is a factor of $a^2(b - c) + b^2(c - a) + c^2(a - b)$ and write down two other factors of the expression. Hence or otherwise factorize the expression completely [-(a – b)(b – c)(c – a) or (b – a))(b – c)(c – a)]
3. The polynomial $ax^3 + bx^2 - cx - 2$ is divisible by x +2. When divided by x – 1 it leaves a remainder of 18 and when divided by x + 3, it leaves a remainder -50. Determine the values of a, b, and c. Hence factorize the polynomial completely [a = 6, b = 13, c = -1; (x + 2)(2x +1)(3x-1)]
4. Factorize completely $f(xyz) = (x + y)^3(x - y) + (y + z)^3(y - z) + (z + x)^3(z - x)$ [(x + y + z)(x – y)(y – z)(z – x)]
5. When the quadratic expression ax2 + bx + c is divided by x – 1, x – 2 and x + 1, the remainders are 1, 1, and 25 respectively, determine the factors of the expression [(2x – 3) and (2x – 3)]
6. The remainder when $px^3 + 2x^2 - 5x + 7$ is divided by x – 2 is equal to the remainder when the same expression is divided by x + 1. Find the value of p [p = 1]
7. Given that x – 4 is a factor of $2x^3 - 3x^2 - 7x + k$, where k is a constant, find the remainder when the expression is divided by 2x -1 [k = - 52, remainder = -56]
8. The expression $px^3 + qx^2 + 3x + 8$ leaves a remainder of -6 when divided by x – 2 and a remainder of -34 when divided by x + 2. Find the values of constants p and q. [p = 1, q = -7]
9. Show that x – 2 is a factor of $x^3 - 9x^2 + 26x - 24$. Find the set of values of x for which $x^3 - 9x^2 + 26x - 24 < 0$ [x < 2 and 3< x < 4]
10. The remainder when $x^3 - 2x^2 + kx + 5$ is divided by x – 3 is twice when the same expression is divided by x + 1. Find the value of the constant k [k = -2]

Thank you

Dr. Bbosa Science