Quadratic equations

These are equations expressed in the form $ax^2 + bx + c = 0$ where a, b and c are constants and $a \neq 0$

They have at most two roots which may be real or complex.

The latter roots are handled exclusively under complex numbers

Example of quadratic equations

$2y^2 + 3y + 5 = 0$; a = 2, b = 3 and c = 5

$x^2 + 4x – 10 = 0$; a = 1, b = 4, c = -10

Forming quadratic equations

Suppose that the roots of a quadratic equation are α and β,

then x – α = 0 and x – β = 0

When forming a quadratic equation

$(x – \alpha)(x – \beta) = 0$

$x^2 – (\alpha + \beta)x + \alpha\beta = 0$

This means that if the roots of a quadratic equation are given, its equation in terms of x is x^2-(sum of roots)x + product of roots = 0

Example 1

Form quadratic equation I terms of x with roots

(a) (2, 3)
Solution
Let x = 2 or x = 3
x – 2 = 0 x – 3 = 0
⇨ (x – 2)(x – 3) = 0
$x^2 – 5x – 6 = 0$

(b) (p, q)
Solution
Let x = p or x = q
x – p = 0 x – q = 0
⇨ (x – p)(x – q) = 0
$x^2 – (p + q)x – pq = 0$

(c) (-3, -2)
Solution
Let x = -3 or x = -2
x + 3 = 0 x + 2 = 0
⇨ (x + 3)(x + 2) = 0
$x^2 + 5x + 6 = 0$

Example 2

Given that the root of the equation $x^2 + px + q = 0$ are α and β, express $(\alpha - \beta^2)(\beta - \alpha^2)$ in terms of p and q. Deduce that for one root to be the square of another $p^3 – 3pq + q^2 + q = o$ must hold

Solution

For $x^2 + px + q = 0$

α + β = -p

αβ = q

$(\alpha - \beta^2)(\beta - \alpha^2) = \alpha\beta - \alpha^3 - \beta^3 + (\alpha\beta)^2$

$= \alpha\beta - (\alpha^3 + \beta^3) + (\alpha\beta)^2$

But $(\alpha^3 + \beta^3) = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$(\alpha - \beta^2)(\beta - \alpha^2)$

$= \alpha\beta - [(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)\,] + (\alpha\beta)^2$

$= q - [(-p)^3 - 3q(-p)] + q^2$

$= q + p^3 - 3pq + q^2$

$(\alpha - \beta^2)(\beta - \alpha^2) = p^3 - 3pq + q^2 + q$

If $\alpha = \beta^2$

$(\beta^2 - \beta^2)(\beta - \beta^4) = p^3 - 3pq + q^2 + q$

$0 = p^3 - 3pq + q^2 + q$

Or

$p^3 - 3pq + q^2 + q = 0$ As required

Example 3

Given that the root of the equation $x^2 + px + q = 0$ are α and β, Form quadratic equations with roots

(a) $\frac{\alpha}{\beta}$ *and* $\frac{\beta}{\alpha}$

Solution

Sum of roots $= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$

$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$

Product of roots $= \frac{\alpha}{\beta} x \frac{\beta}{\alpha} = 1$

Required equation

$x^2 - \left(\frac{p^2-2q}{q}\right)x + 1 = 0$

Or $qx^2 - (p^2 - 2q)x + q = 0$

(b) pα + qβ and qα + pβ

Solution

Sum of roots

pα + qβ + qα + pβ = p(α +β) + q(α +β)

=(p + q)(α +β)

=p(p + q)

Product of roots= (pα + qβ)(qα + pβ)

$= pq\alpha^2 + p^2\alpha\beta + q^2\alpha\beta + pq\beta^2$

$= pq(\alpha^2 + \beta^2) + \alpha\beta(p^2 + q^2)$

$= pq[(\alpha + \beta)^2 - 2\alpha\beta] + \alpha\beta(p^2 + q^2)$

$= pq(p^2 - 2q) + q(p^2 + q^2)$

Required equation

$x^2 - p(p + q)x + pq(p^2 - 2q) + q(p^2 + q^2) = 0$

Example 4

Given the equation $x^3 + x - 10 = 0$.

(a) Show that x = 2 is a root of the equation

Let $f(x) = x^3 + x - 10$

Substituting for x = 2

$f(2) = 2^3 + x - 10$

= 8 + 2 – 10

= 10 – 10 = 0

Hence x = 2 is a root of $x^3 + x - 10 = 0$

(b) Deduce the values of α + β and αβ where α and β are roots of the equation.

Hence form a quadratic equation whose roots are α^2 and β^2.

⇨ $x^3 + x - 10 = (x-2)(x^2 + 2x + 5)$

Either x – 2 = 0

Or $(x^2 + 2x + 5) = 0$

α + β = 2

αβ = 5

Sum of roots $= \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (-2)^2 - 2(5) = 4 - 10 = -6$

Product$= \alpha^2\beta^2 = (\alpha\beta)^2 = 5^2 = 25$

The equation become

$x^2 - (-6)x + 25 = 0$

$x^2 + 6x + 25 = 0$

Solving quadratic equations

Quadratic equations may be solved by

(a) Factorization method
(b) Completing square method
(c) Graphical method

(a) Factorization method

It is used for quadratic equations that are easy to factorise

Example 5

(a) $4x^2 + 7x + 3 = 0$

Solution	side work
4x(x + 1)+ 3(x + 1) = 0	Product = 4 x 3 = 12
(x + 1)(4x + 3) = 0	Sum = 7
Either x + 1 = 0; x = -1	Factor = (4, 3)

Or 4x + 3 = 0; x = $-\frac{3}{4}$

(b) $2x^2 + 5x + 3 = 0$

Solution	side work
2x(x + 1)+ 3(x + 1) = 0	Product = 3 x 2 = 6
(x + 1)(2x + 3) = 0	Sum = 5
Either x + 1 = 0; x = -1	Factors 3, 2

Or 2x + 3 = 0; x = $-\frac{3}{2}$

(c) $x^2 + x – 20 = 0$

Solution	Side work
x(x – 4) + 5(x – 4) = 0	Product = -20
(x – 4)(x + 5) = 0	Sum = 1
Either x – 4 = 0; x = 4	Factors (5, - 4)

Or x + 5 = 0; x = -5

(d) $10x^2 + x – 3 = 0$

Solution Side work

5x(2x -1) + 3(2x – 1) = 0 product 10 x – 3 = -30

(5x + 3)(2x – 1) = 0 sum = 1

Either 5x + 3 = 0; x = $-\frac{3}{5}$ Factors (6, -5)

Or 2x – 1 = 0; x = $\frac{1}{2}$

(b) Method II: Completing squares approach

The idea is to create a perfect square on one side of the equation:

Given the equation $ax^2 + bx + c = 0$

- Dividing the equation by and transposing the constant term to the RHS
  $x^2 + \frac{b}{a}x = -\frac{c}{a}$
- Marking the LHS a perfect square, add a half the coefficient of x squared on both sides

  $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$

- Factorise the terms on the LHS
- $\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2-4ac}{4a^2}$
- Taking square root on both sides of the equation
- $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$
- Solving $x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$
- $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

This is the general quadratic equation formula for finding the square root of any quadratic equation. This formula is locally known as ***bull dozer formula***

Example 6

Solve the following equations by completing squares

(a) $2x^2 – x – 3 = 0$

Solution

$2x^2 – x – 3 = 0$

$x^2 - \frac{1}{2}x = \frac{3}{2}$

$x^2 - \frac{1}{2}x + \left(-\frac{1}{4}\right)^2 = \frac{3}{2} + \left(-\frac{1}{4}\right)^2$

$\left(x - \frac{1}{4}\right)^2 = \frac{25}{16}$

$x - \frac{1}{4} = \sqrt{\frac{25}{16}} = \pm\frac{5}{4}$

Either $x = \frac{5}{4} + \frac{1}{4} = \frac{6}{4} = \frac{3}{2}$

Or $x = -\frac{5}{4} + \frac{1}{4} = \frac{-4}{4} = -1$

Hence x = -1 and x = $\frac{3}{2}$

(b) $18x^2 + 7x – 1 = 0$

Solution

$18x^2 + 7x – 1 = 0$

$x^2 + \frac{7}{18}x = \frac{1}{18}$

$x^2 + \frac{7}{18}x + \left(\frac{7}{36}\right)^2 = \frac{1}{18} + \left(-\frac{7}{36}\right)^2$

$\left(x + \frac{7}{36}\right)^2 = \frac{1}{18} + \frac{49}{1296} = \frac{121}{1296}$

$x + \frac{7}{36} = \sqrt{\frac{121}{1296}} = \pm\ \frac{11}{36}$

Either $x = \frac{11}{36} - \frac{7}{36} = \frac{4}{36} = \frac{1}{9}$

Or $x = -\frac{11}{36} - \frac{7}{36} = -\frac{18}{36} = -\frac{1}{2}$

Hence x = $\frac{1}{9}$ or x =$-\frac{1}{2}$

(c) $3x^2 + 7x + 2 = 0$

Solution

$3x^2 + 7x + 2 = 0$

$x^2 + \frac{7}{3}x = -\frac{2}{3}$

$x^2 + \frac{7}{3}x + \left(\frac{7}{6}\right)^2 = -\frac{2}{3} + \left(\frac{7}{6}\right)^2$

$\left(x + \frac{7}{6}\right)^2 = -\frac{2}{3} + \frac{49}{36} = \frac{25}{36}$

$x + \frac{7}{6} = \sqrt{\frac{25}{36}} = \pm\frac{5}{6}$

Either x =$\frac{5}{6} - \frac{7}{6} = -\frac{2}{6} = -\frac{1}{3}$

Or x =$-\frac{5}{6} - \frac{7}{6} = -\frac{12}{6} = -2$

Hence x = -2 and x =$-\frac{1}{3}$

Example 7

Solve the following equations by using the quadratic formula

(a) $7x^2 - 5x - 2 = 0$

Solution

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{5 \pm \sqrt{-5^2 - 4\ x\ 7\ x - 2}}{2\ x\ 7} = \frac{5 \pm 9}{14}$

Either $x = \frac{14}{14} = 1$ or $x = \frac{-4}{14} = -\frac{2}{7}$

Hence $x = 1\ and\ x = -\frac{2}{7}$

(b) $3x^2 - 7x - 6 = 0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{7 \pm \sqrt{-7^2 - 4\ x\ 3\ x - 6}}{2\ x\ 3} = \frac{7 \pm 11}{6}$

Either $x = \frac{18}{6} = 3$ or $x = \frac{-4}{6} = -\frac{2}{3}$

Hence $x = 3\ and\ x = -\frac{2}{3}$

(c) $6x^2 - 5x - 6 = 0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{5 \pm \sqrt{-5^2 - 4\ x\ 6\ x - 6}}{2\ x\ 6} = \frac{5 \pm 13}{12}$

Either $x = \frac{18}{12} = \frac{3}{2}$ or $x = \frac{-8}{12} = -\frac{2}{3}$

Hence $x = \frac{3}{2}\ and\ x = -\frac{2}{3}$

Method 3: Graphical approach

Here, the roots of the equations are established by plotting suitable graphs. It may done by:-

(i) Single graph plot
(ii) Two graph plot

Single graph method

Given the function y = f(x), tabulate the selected values of x closer to zero within a given range which must be substituted in the function given to obtain the corresponding values of y. These values are plotted on the same x and y axes and joined by a curve. The roots of the function f(x) = 0 are the values of x where the function y = f(x) crosses the x-axis.

Two graphs method

Here the function y = f(x) is split into two and the equations plotted separately on the same axes. The points of intersection of the two graphs are noted and the corresponding values of x are read off on the x – axis.

Example 8

Solve the following equation by using graphical approach

(a) $2x^2 + 3x - 3 = 0$

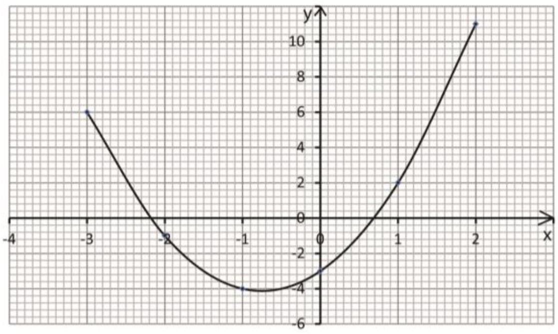
Solution

Using a single graphs

Let $y = 2x^2 + 3x - 3$ taking $-3 \leq x \leq 2$

Table of values

X	-3	-2	-1	0	1	2
y	6	-1	-4	-3	2	11



From the graph, the roots of the equation are x = -2.2 and x = 0.7

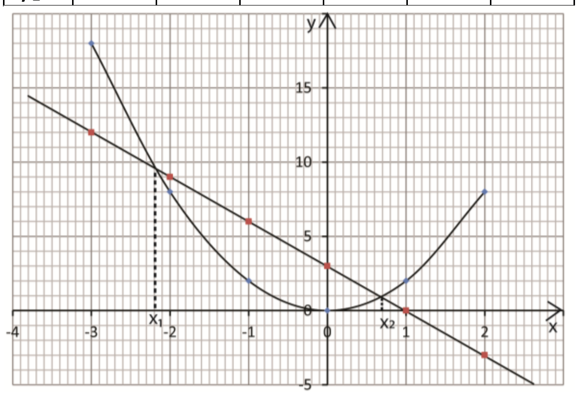
Using two graphs

By splitting the equation $2x^2 + 3x - 3 = 0$ into two we have $2x^2 = -3x + 3$

Let $y_1 = 2x^2$ and $y_2 = 3 - 3x$

Table of results

X	-3	-2	-1	0	1	2
y_1	18	8	2	0	2	8
y_1	12	9	6	3	0	-3



From the graph, the roots of the equation are x = -2.2 and x = 0.7

(b) $x^2 + x - 6 = 0$

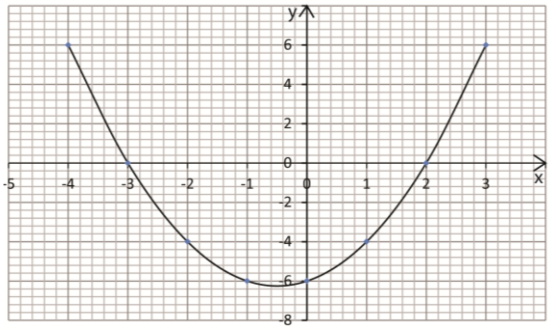
Solution

Using a single graphs

Let $y = x^2 + x - 6 = 0$, taking $-4 \leq x \leq 3$

Table of values

X	-4	-3	-2	-1	0	1	2	3
y	6	0	-4	-6	-6	-4	0	6



From the graph, the roots of the equation are x = -3 and x = 2

Using two graphs

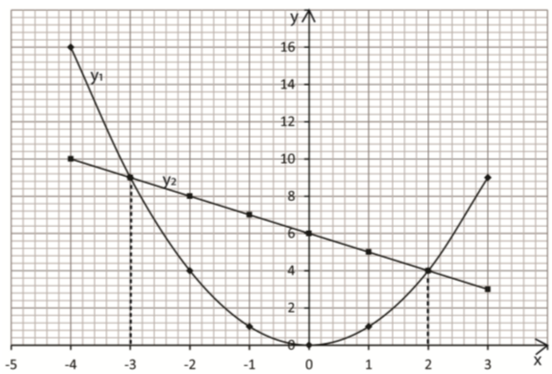
By splitting the equation $x^2 + x - 6 = 0$

into two we have $x^2 = 6 - x$

Let $y_1 = x^2$ and $y_2 = 6 - x$

Table of results

X	-4	-3	-2	-1	0	1	2	3
y_1	16	9	4	1	0	1	4	9
y_1	10	9	8	7	6	5	4	3



From the graph, the roots of the equation are x = -3 and x = 2

Minimum and maximum values of quadratic expression

The methods of completing squares and graphing can be used to obtain minimum and maximum values of quadratic expression

Completing square method

The general form of quadratic equation
y = $ax^2 + bx + c$ can be expressed as

$y = ax^2 + bx + c$

$y = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$

$y = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right)$

$y = a\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac-b^2}{4a}\right)$

The minimum/maximum value is the constant $\frac{4ac-b^2}{4a}$ which is attained when $x + \frac{b}{2a} = 0$

Note

- if a < 0, the value of the function is maximum
- if a > 0, the value of the function is minimum

Example 9

Determine the minimum or maximum values of the following expressions using completing squares

(a) $2x^2 - x - 3 = 0$

Solution

Let $y = 2x^2 - x - 3 = 0$

$y = 2\left(x^2 + \frac{2}{2}x - \frac{3}{2}\right)$

$y = 2\left(x^2 + x + \left(\frac{1}{2}\right)^2 - \frac{3}{2} - \left(\frac{1}{2}\right)^2\right)$

$y = 2\left(\left(x + \frac{1}{2}\right)^2 - \frac{7}{4}\right)$

$y = 2\left(x + \frac{1}{2}\right)^2 - \frac{7}{4}$

Since a > 0, the function has got a minimum value at x = $-\frac{1}{2}$. Hence $y_{min} = -\frac{7}{4}$

(b) $-4 + 6x - x^2$

Let $y = -4 + 6x - x^2$

$y = -(x^2 - 6x + 4)$

$y = -(x^2 - 6x + (-3)^2 + 4 - (-3)^2)$

$y = -((x - 3)^2 + 4 - 9)$

$y = -((x - 3)^2 + -5)$

Since a = -1 < 0, the expression has got a maximum value when x = 3. Hence $y_{max} = 5$

Using graphical method

After graphing, y =f(x) the minimum value of the expression is lowest point if a trough/valley like or curving upwards and the maximum value is the maximum point if downwards

Example 10

Determine the maximum or minimum values of the following expression using graphical method

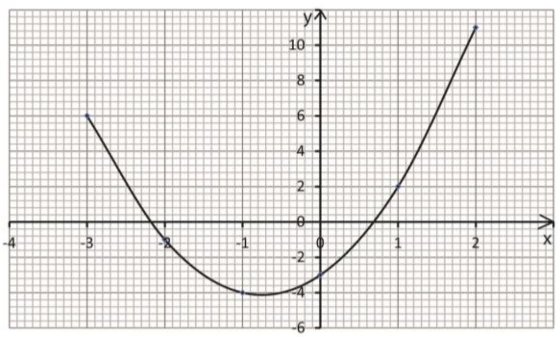
(a) $2x^2 + 3x - 3 = 0$

Solution

Let $y = 2x^2 + 3x - 3$ taking $-3 \leq x \leq 2$

Table of values

X	-3	-2	-1	0	1	2
y	6	-1	-4	-3	2	11



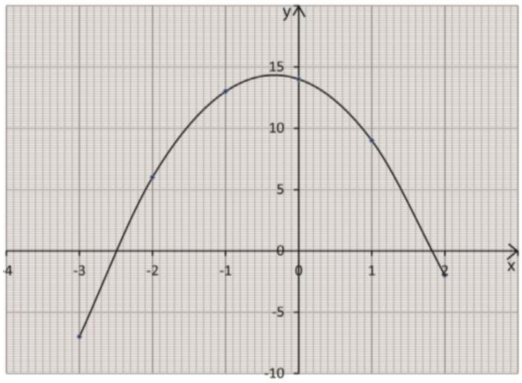
y_{min} = - 4.1 at -0.75

(b) $14 - 2x - 3x^2$

Ley $y = 14 - 2x - 3x^2$; $-3 \leq x \leq 2$

Table of results

x	-3	-2	-1	0	1	2
y	-7	6	13	14	9	-2



y_{max} = 14.3 at x =-0.3

Revision exercise

1. Given that the roots of the equation $x^2 + px + q = 0$ are α and β, find the values of:
   (a) $\alpha^2 + \beta^2 [p^2 - 2q]$
   (b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} \left[\frac{p^2-2q}{q}\right]$
   (c) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} \left[\frac{p}{q}(3q - p^2)\right]$
   (d) $\alpha - \beta \left[\pm\sqrt{p^2 - 4q}\right]$
   (e) $\alpha^2 - \beta^2 \left[-p\sqrt{(p^2 - 4q)}\right]$
   (f) $\alpha^3 + \beta^3 [p(3q - p^2)]$
2. If the roots of the equation $x^2 + 2x + 3 = 0$ are α and β, form an equation whose roots are $\alpha^2 - \beta$ and $\beta^2 - \alpha$. $[x^2 + 2 = 0]$

3. Given that the roots of the equation $x^2 - 2x + 10 = 0$ are α and β, form an equation whose roots are

   $\frac{1}{(2+\alpha)^2}$ and $\frac{1}{(2+\beta)^2}$ $[324x^2 + 1 = 0]$

4. If α and β are roots of the equation $x^2 - px + q = 0$, find the equation whose roots are $\frac{\alpha^3-1}{\alpha}$ and $\frac{\beta^3-1}{\beta}$
   $[qx^2 - (p^2q - 2q^2 - p)x + (q^3 - p^3 + 3pq)+1=0]$
5. The roots of the equation $3x^2 - ax + 6b = 0$ are α and β. Find the condition for one root to be
   (a) Twice the other $[81b = a^2)$
   (b) The cube of the other
   $[a^4 - 648(b - 1)b^2 - 18(4a^2 + 9)b = 0]$
6. By Factorization method solve the following quadratic equations
   (a) $x^2 + 9x + 14$ [x = -7, x= -2]
   (b) $x^2 + 2x - 8 = 0$ [x = -4, x = 2)
   (c) $2x^2 + x - 10 = 0$ [x= 2, x = $-\frac{5}{2}$]
   (d) $6x^2 - 19x + 10$ $\left[x = \frac{5}{2},\ x = \frac{2}{3}\right]$
7. Solve the following equations by completing squares
   (a) $2x^2 + 5x + 3 = 0$ $\left[x = -1\ and\ x = -\frac{3}{2}\right]$
   (b) $x^2 + 9x + 20 = 0$ [x = -5 and x = -4]
   (c) $x^2 + x - 20 = 0$ [x = -5 and x = 4]
   (d) $x^2 - x - 20 = 0$ [x = 4 and x = 5]
8. Solve the following equations using the quadratic formula
   (a) 2x2 + 5x + 3 = 0 $\left[\frac{-3}{2}, -1\right]$
   (b) $x^2 + 9x + 20$ [-5, -4]
   (c) $x^2 + x - 20$ [-5, 4]

9. Determine the maximum or minimum values of the following expression
   (a) $3x^2 - 2x + 1$ $\left[y_{min} = \frac{2}{3} at\ x = \frac{1}{3}\right]$
   (b) $4 - x - x^2$ $\left[y_{max} = \frac{33}{8} at\ x = -\frac{1}{4}\right]$
10. Determine the maximum or minimum values of the following expression using graphical method
    (a) $14 - 2x - 2x^2$

Thank you

Dr. Bbosa Science