Simultaneous equations

Simultaneous equations in two unknowns

These are equations containing two unknowns. The equations may all be linear (equations of straight lines) or one of them may be linear and the other non-linear.

Linear simultaneous equations

Simultaneous equations may solved by any of the three methods

- Elimination method
- Substitution method
- Graphical method
- Solving simultaneous equations using matrixes

Solving simultaneous equations using elimination method

This involves elimination of one of the unknown variables so as to be in position to find the other.

Example 1

(a) 5x + 3y = 7
2x -4y = 3

Solution

5x + 3y = 7 ………… (i)

2x -4y = 3 …………(ii)

To eliminate y the equation are multiplied by relevant factors to make the coefficients of y in both equations equal. Thus eqn. (i) is multiplied by 4 and eqn. (ii) by 3

i.e. 4 x eqn. (i) + 3 x eqn. (ii)

20x + 12y =28

\+ 6x – 12y = 9

26x = 37

$x = \frac{37}{26}$

Substituting x in eqn.(i)

5x + 3y = 7

$3y = 7 - 5 \times \frac{37}{26}$

$Y = \frac{-1}{26}$

(b) 3x + 2y = 8
3y + 4x = 11

Solution

Rearrange the eqns.

3x + 2y = 8 ………… (i)

4x + 3y = 11 ……….. (ii)

Eliminate x as follows

4 x eqn. (i) – 3 x eqn. (ii)

12x + 8y = 32

\- 12x + 9y = 33

y = 1

Substituting y in eqn. (i)

3x + 2y = 8

3x + 2 x 1 = 8

3x = 6

x = 2

Solving simultaneous equations using substitution method

This involves the expression of one of the unknown variable in terms of the other.

Example 2

Solve the following pairs of simultaneous equation for x and y by substitution method

(a) 5x + 3y = 7

2x -4y = 3

Solution

5x + 3y = 7
5x = 7 – 3y
x = $\frac{7-3y}{5}$ ………… (i)

2x -4y = 3 ……… (ii)

Substituting x in eqn. (ii)

$2\left(\frac{7-3y}{5}\right) - 4y = 3$

Multiply 5 through

2(7 – 3y) – 20y = 15

14 – 6y – 20y = 15

-26y = 1

y = $\frac{-1}{26}$

Substituting y into eqn. (i)

x = $\frac{7-3y}{5}$ = $\frac{7-3\left(\frac{-1}{26}\right)}{5} = \frac{26\ x\ 7+-3\ x-1}{5\ x\ 26} = \frac{37}{26}$

∴ x = $\frac{37}{26}$ and y = $\frac{-1}{26}$

(b) 3x + 2y = 8
y = $\frac{8-3x}{2}$…………………. (i)

3y + 4x = 11 ………….. (ii)

Substituting y in equation (ii)

$3\left(\frac{8-3x}{2}\right) + 4x = 11$

Multiplying 2 through

3(8 – 3x) + 8x = 22

24 – 9x + 8x = 22

-1x = = -2

x = 2

substituting x into equation (i)

y = $\frac{8-3x}{2}$ = $\frac{8-3\ x\ 2}{2} = \frac{2}{2} = 1$

∴ x =2 and y = 1

Solving simultaneous equations using graphical method

This method involve drawing graphs of the two linear equations and finding the coordinates of their points of intersection

Establish at least two possible points with known coordinates satisfying the equations. The coordinates of the point of intersection of the lines drawn are the solutions to the equations

Example 3

Solve the following pairs of simultaneous equations for x and y using graphical method

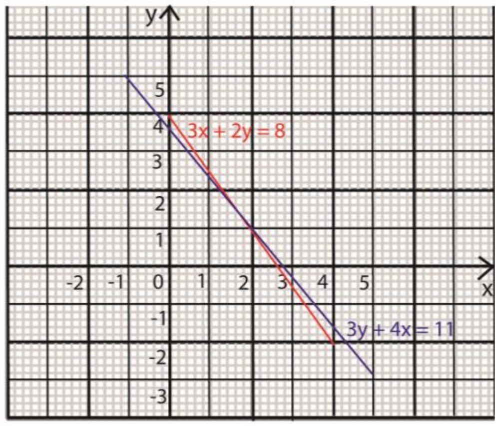
(a) 3x + 2y = 8
3y + 4x = 11

For 3x + 2y = 8

x	0	4
y	4	-2

For 3y + 4x = 11

x	-1	2
y	5	1



From the graph the point of intersection is (2, 1) Hence x = 2 and y = 1

(b) x + y = 4

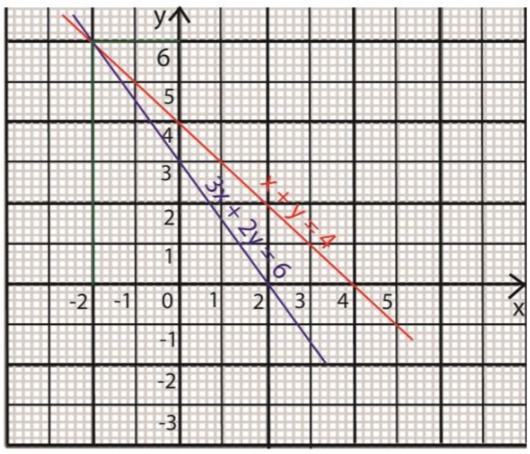
3x + 2y = 6

For x + y = 4

x	0	4
y	4	0

For 3x + 2y = 6

x	0	0
y	3	2



From the graph, the point of intersection is (-2, 6). Hence x = -2, y = 6

Solving simultaneous equations using matrixes

A. Determinant method

Consider the simultaneous equations

$a_1x + b_1y = c_1$

$a_2x + b_1y = d_1$

⇨ $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ d_1 \end{pmatrix}$ .... (i)

$B = \begin{pmatrix} c_1 & b_1 \\ d_2 & b_2 \end{pmatrix}$

Matrix B is obtained by interchanging the coefficients of x in eqn. (i) with the column matrix on the right hand side

$$C = \begin{pmatrix} a_1 & c_1 \\ a_2 & d_1 \end{pmatrix}$$

Matric C is obtained by interchanging the coefficients of y in equation (i) with the column matrix on the right and side of eqn.(i)

For the determinant method

$x = \frac{|B|}{|A|}$ and y = $\frac{|C|}{|A|}$

Example 4

(a) x + 3y – 15 = 0

3x = 17 – 2y

Re-arranging the equation

x + 3y = 15

3x + 2y = 17

Expressing the equation in matrix form

$$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 17 \end{pmatrix}$$

$$x = \frac{\begin{pmatrix} 15 & 3 \\ 17 & 2 \end{pmatrix}}{\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}}$$

$$x = \frac{(15\,x\,2)-(17\,x\,3)}{(1\,x\,2)-(3\,x\,3)} = \frac{-21}{-7} = 3$$

$$y = \frac{\begin{pmatrix} 1 & 15 \\ 3 & 17 \end{pmatrix}}{\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}}$$

$$y = \frac{(1\,x\,17)-(3\,x\,15)}{(1\,x\,2)-(3\,x\,3)} = \frac{-28}{-7} = 4$$

Hence x = 3 and y = 4

(b) 5x + 3y = 7

2x -4y = 3

Expressing the equation in matrix form

$$\begin{pmatrix} 5 & 3 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

$$x = \frac{\begin{pmatrix} 7 & 3 \\ 3 & -4 \end{pmatrix}}{\begin{pmatrix} 5 & 3 \\ 2 & -3 \end{pmatrix}}$$

$$x = \frac{(7\,x-4)-(3\,x\,3)}{(5\,x-4)-(2\,x\,3)} = \frac{-37}{-26} = \frac{37}{26}$$

$$y = \frac{\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}}{\begin{pmatrix} 5 & 3 \\ 2 & 3 \end{pmatrix}}$$

$$y = x = \frac{(5\ x\ 3) - (2\ x\ 7)}{(5\ x - 4) - (2\ x - 3)} = \frac{-1}{26}$$

(c) 34 + 3y = 3x
 3x – 4y- 16 = 0

Rearranging the equations

-3x + 3y = -34

3x – 4y = 16

Expressing the equations in matrix form

$$\begin{pmatrix} -3 & 3 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -34 \\ 16 \end{pmatrix}$$

$$x = \frac{\left|\begin{pmatrix} -34 & 3 \\ 16 & -4 \end{pmatrix}\right|}{\left|\begin{pmatrix} -3 & 3 \\ 3 & -4 \end{pmatrix}\right|}$$

$$x = \frac{(-34\ x - 4) - (16\ x\ 3)}{(-3\ x - 4) - (3\ x\ 3)}$$

$$x = \frac{136 - 48}{12 - 9} = \frac{88}{3}$$

$$y = \frac{\left|\begin{pmatrix} -3 & -34 \\ 3 & 16 \end{pmatrix}\right|}{\left|\begin{pmatrix} -3 & 3 \\ 3 & -4 \end{pmatrix}\right|}$$

$$x = \frac{(-3\ x16) - (3\ x - 34)}{(-3\ x - 4) - (3\ x\ 3)}$$

$$x = \frac{-48 + 102}{12 - 9} = \frac{54}{3} = 18$$

Hence x = $\frac{88}{3}$ and y = 18

B. Adjunct method

Consider the simultaneous equations

$a_1x + b_1y = c_1$

$a_2x + b_1y = d_1$

Arranging in matrix form

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} \text{.............. (i)}$$

If A= $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$

Adjunct A =$\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}$

To obtain the values of x and y, the adjunct is pre-multiplied on the both sides, i.e.

$$\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}\begin{pmatrix} c_1 \\ d_1 \end{pmatrix}$$

Example 5

Solve the following equations using matrix method

(a) 4x – 3y = 2
 x + 2y = 1
(b) 4x + 3y = 17
 5x – 2y = 4
(c) 7x – y = -1
 3x – 2y = - 24

Solution

(a) 4x – 3y = 2
 x + 2y = 1

Express the equation in matrix form

$$\begin{pmatrix} 4 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{.................... (i)}$$

Let $A = \begin{pmatrix} 4 & -3 \\ 1 & 2 \end{pmatrix}$

Adjunct $A = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$

Pre- multiply both sides of eqn. (i) the adjunct matrix A

$$\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2x4 + 3x1 & 2x - 3 + 3x2 \\ -1x4 + 1x4 & -1x - 3 + 4x2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x2 + 3x1 \\ -1x2 + 4x1 \end{pmatrix}$$

$$\begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

11x = 7

$$x = \frac{7}{11}$$

11y =2

$$y = \frac{2}{11}$$

Hence $= \frac{7}{11}, y = \frac{2}{11}$

(b) 4x + 3y = 17
5x – 2y = 4

Express in matrix form

$\begin{pmatrix}4 & 3\\5 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}17\\4\end{pmatrix}$ .... (i)

Let $A = \begin{pmatrix}4 & 3\\5 & -2\end{pmatrix}$

Adjunct A$\begin{pmatrix}-2 & -3\\-5 & 4\end{pmatrix}$

Pre- multiply both sides of eqn. (i) the adjunct matrix A

$$\begin{pmatrix}-2 & -3\\-5 & 4\end{pmatrix}\begin{pmatrix}4 & 3\\5 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-2 & -3\\-5 & 4\end{pmatrix}\begin{pmatrix}17\\4\end{pmatrix}$$

$$\begin{pmatrix}-2\ x4 + -3x5 & -2\ x\ 3 + -3x - 2\\ -5x4 + 4x5 & -5x3 + 4x - 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-2\ x\ 17 + -3x4\\ -5x17 + 4x4\end{pmatrix}$$

$$\begin{pmatrix}-23 & 0\\0 & -23\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-46\\-69\end{pmatrix}$$

-23x = -46

$x = \frac{-46}{-23} = 2$

-23y = -69

$y = \frac{-69}{-23} = 3$

Hence x = 2, y = 3

(c) 7x – y = -1
3x – 2y = - 24

Arrange in matrix form

$\begin{pmatrix}7 & -1\\3 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-1\\-24\end{pmatrix}$ ....(i)

Let $A = \begin{pmatrix}7 & -1\\3 & -2\end{pmatrix}$

Adjunct A= $\begin{pmatrix}-2 & 1\\-3 & 7\end{pmatrix}$

Pre- multiply both sides of eqn. (i) the adjunct matrix A

$$\begin{pmatrix}-2 & 1\\-3 & 7\end{pmatrix}\begin{pmatrix}7 & -1\\3 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-2 & 1\\-3 & 7\end{pmatrix}\begin{pmatrix}-1\\-24\end{pmatrix}$$

$$\begin{pmatrix}-2\ x\ 7 + 1\ x\ 3 & -2\ x - 1 + 1\ x - 2\\ -3\ x\ 7 + 7\ x\ 3 & -3\ x - 1 + 7\ x - 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-2\ x - 1 + 1\ x - 24\\ -3\ x - 1 + 7x - 24\end{pmatrix}$$

$$\begin{pmatrix}-11 & 0\\0 & -11\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-22\\-165\end{pmatrix}$$

-11x = -22

$x = \frac{-22}{-11} = 2$

-23y = -69

$y = \frac{-165}{-11} = 15$

Hence x = 2 and y = 15

C. Inverse method

Example 6

(a) 4x – 3y = 2
x + 2y = 1
(b) 4x + 3y = 17
5x – 2y = 4
(c) 7x – y = -1
3x – 2y = - 24

Solution

(a) 4x – 3y = 2
x + 2y = 1

Arrange n matrix form

$\begin{pmatrix}4 & -3\\1 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$

Let $A = \begin{pmatrix}4 & -3\\1 & 2\end{pmatrix}$

det A = (4 x 2) – (1 x -3) = 11

Inverse, $A^{-1} = \frac{1}{11}\begin{pmatrix}2 & 3\\-1 & 4\end{pmatrix} = \begin{pmatrix}\frac{2}{11} & \frac{3}{11}\\ \frac{-1}{11} & \frac{4}{11}\end{pmatrix}$

Pre-multiply both sides of eqn. (i) by the inverse matrix A

$$\begin{pmatrix}\frac{2}{11} & \frac{3}{11}\\ \frac{-1}{11} & \frac{4}{11}\end{pmatrix}\begin{pmatrix}4 & -3\\1 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{2}{11} & \frac{3}{11}\\ \frac{-1}{11} & \frac{4}{11}\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}$$

$$\begin{pmatrix} \frac{2}{11}x4+\frac{3}{11}x1 & \frac{2}{11}x-3+\frac{3}{11}x2 \\ \frac{-1}{11}x4+\frac{4}{11}x1 & \frac{-1}{11}x-3+\frac{4}{11}x2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{11}x2+\frac{3}{11}x1 \\ \frac{-1}{11}x2+\frac{4}{11}x1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{7}{11} \\ \frac{2}{11} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{7}{11} \\ \frac{2}{11} \end{pmatrix}$$

Hence x = $\frac{7}{11}$, y=$\frac{2}{11}$

Note: when we pre-multiply a matrix by its inverse, we obtain an identity matrix, i.e. $AA^{-1} = 1$

$$\begin{pmatrix} \frac{2}{11}x4+\frac{3}{11}x1 & \frac{2}{11}x-3+\frac{3}{11}x2 \\ \frac{-1}{11}x4+\frac{4}{11}x1 & \frac{-1}{11}x-3+\frac{4}{11}x2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{11}x2+\frac{3}{11}x1 \\ \frac{-1}{11}x2+\frac{4}{11}x1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{11}x2+\frac{3}{11}x1 \\ \frac{-1}{11}x2+\frac{4}{11}x1 \end{pmatrix} = \begin{pmatrix} \frac{7}{11} \\ \frac{2}{11} \end{pmatrix}$$

Hence x = $\frac{7}{11}$, y=$\frac{2}{11}$

(b) 4x + 3y = 17
5x – 2y = 4

Arrange in matrix form

$$\begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 17 \\ 4 \end{pmatrix} \text{.....................(i)}$$

Let $A = \begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix}$

det A = 4 x -2 – (5 x 3) = -23

Adjunct A = $\begin{pmatrix} -2 & -3 \\ -5 & 4 \end{pmatrix}$

Inverse A (A^{-1})= $\frac{-1}{23}\begin{pmatrix} -2 & -3 \\ -5 & 4 \end{pmatrix} = \begin{pmatrix} \frac{2}{23} & \frac{3}{23} \\ \frac{5}{23} & \frac{-4}{23} \end{pmatrix}$

Pre-multiply both sides of eqn. (i) by the inverse matrix A

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{23} & \frac{3}{23} \\ \frac{5}{23} & \frac{-4}{23} \end{pmatrix}\begin{pmatrix} 17 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{23}\ x17+ & \frac{3}{23}\ x4 \\ \frac{5}{23}x17+ & \frac{-4}{23}x4 \end{pmatrix} = \begin{pmatrix} \frac{46}{23} \\ \frac{69}{23} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Hence x = 2, y = 3

(c) 7x – y = -1
3x – 2y = - 24

Express in matrix form

$$\begin{pmatrix} 7 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -24 \end{pmatrix}$$

Let $A = \begin{pmatrix} 7 & -1 \\ 3 & -2 \end{pmatrix}$

det A = (7 x -2)- (3 x -1) = -11

Adjunct A= $\begin{pmatrix} -2 & 1 \\ -3 & 7 \end{pmatrix}$

Inverse of matrix A (A^{-1}) =$\frac{-1}{11}\begin{pmatrix} -2 & 1 \\ -3 & 7 \end{pmatrix}$

$$= \begin{pmatrix} \frac{2}{11} & \frac{-1}{11} \\ \frac{3}{11} & \frac{-7}{11} \end{pmatrix}$$

Pre-multiply both sides of eqn. (i) by the inverse matrix A

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{11} & \frac{-1}{11} \\ \frac{3}{11} & \frac{-7}{11} \end{pmatrix}\begin{pmatrix} -1 \\ -24 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{11}\ x-1+ & \frac{-1}{11}x-24 \\ \frac{3}{11}x-1 & \frac{-7}{11}x-24 \end{pmatrix} = \begin{pmatrix} \frac{22}{11} \\ \frac{165}{11} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 15 \end{pmatrix}$$

Hence x = 2 and y 15

Non-linear simultaneous equation

These are solved basically by using substitution method

Example 7

(a) $x^2 + 2x - y = 14$
$2x^2 - 3y = 47$
(b) $2x^2 - xy + y^2 = 32$
$y = -\frac{5}{x}$

Solution

(a) $x^2 + 2x - y = 14$
$y = x^2 + 2x - 14$……………… (i)
$2x^2 - 3y = 47$ …………………. (ii)

Substituting eqn. (i) into eqn. (ii)

$2x^2 - 3(x^2 + 2x - 14) = 47$

$2x^2 - 3x^2 - 6x + 42 = 47$

$-x^2 - 6x - 5 = 0$

$x^2 + 6x + 5 = 0$

$(x+1)(x+5) = 0$

x = -1 or x = -5

Substituting x into eqn. (i)

When x = -1, $y = (-1)^2 + 2(-1) - 14 = -15$

When x = -5, $y = (-5)^2 + 2(-5) - 14 = 1$

Hence (x, y) = (-1, -15) and (-5, -1)

(b) $2x^2 - xy + y^2 = 32$ ………………. (i)
$y = -\frac{5}{x}$ ……………………………… (ii)

Substituting equation (ii) into eqn. (i)

$2x^2 - x\left(-\frac{5}{x}\right) + \left(-\frac{5}{x}\right)^2 = 32$

$2x^2 + 5 + \frac{25}{x^2} = 32$

$2x^2 + \frac{25}{x^2} - 27 = 0$

$2x^4 - 27x^2 + 25 = 0$

$(x^2 - 1)(2x^2 - 25) = 0$

$x^2 = 1$

$x = \pm 1$

$2x^2 - 25 = 0$

$2x^2 = 25$

$x \pm \frac{5}{\sqrt{2}}$

Substituting for x into eqn. (i)

$y = -\frac{5}{x}$

When x = 1, $y = -\frac{5}{1} = -5$

When x = -1, $y = -\frac{5}{-1} = 5$

When x = $\frac{5}{\sqrt{2}}$, $y = -\frac{5}{\frac{5}{\sqrt{2}}} = -\frac{5\sqrt{2}}{5} = -\sqrt{2}$

When x = $-\frac{5}{\sqrt{2}}$, $y = -\frac{5}{-\frac{5}{\sqrt{2}}} = \frac{5\sqrt{2}}{5} = \sqrt{2}$

Hence the solution to simultaneous equations are (x, y)= (1, -5), (-1, 5), $\left(\frac{5}{\sqrt{2}}, -\sqrt{2}\right)$, $\left(\frac{5}{\sqrt{2}}, \sqrt{2}\right)$,

(c) $(x - 4y)^2 = 1$
3x = 8y = 11 (06marks)
Solving equations
$(x - 4y) = 1$ …………………… (i)
3x = 8y = 11…………………… (ii)
Eqn. (ii) - 3Eqn. (i)
20y = 8
$y = \frac{8}{20} = \frac{2}{5}$
From eqn. (i)
$x = 1 + 4\left(\frac{2}{5}\right) = \frac{13}{5}$
And
$(x - 4y) = -1$ …………………… (i)
3x = 8y = 11…………………… (ii)
2(eqn (i)) + eqn. (ii)
5x = 9
$x = \frac{9}{5}$
From equation (i)
$4y = \frac{9}{5} + 1$
$y = \frac{7}{10}$
$\therefore$ (x,y) = $\left(\frac{13}{5}, \frac{2}{5}\right), \left(\frac{9}{5}, \frac{7}{10}\right)$

Three linear simultaneous equations n three unknown.

When solving for three unknowns, there must be three equation that will be solved simultaneously. The methods that will be used to such equation are

- Elimination and substitution
- Row reduction to echelon

Elimination and substitution

This involves elimination of one unknown variable so as to remain two unknowns which can easily be solved

Example 7

(a) Solve the simultaneous equations

x – 2y - 2z = 0
2x + 3y + z = 1
3x –y – 3z = 3
X – 2y -2z = 0 ……………………… (i)
2x + 3y + z = 1…………………… (ii)
3x –y – 3z = 3 ………………………(iii)
Eqn. (i) + 2eqn. (ii)
4x + 4y = 2 ……………………………(iv)
3Eqn. (ii) + eqn. (iii)
9x + 8y = 6 ………………………… (v)
2eqn. (iv) – eqn. (v)
x = -2
Substituting x = -2 into eqn. (iv)
5(-2) + 4y = 2
y = 3
Substituting x = -2 and y = 3 into eqn. (ii)
2(-2) + 3(3) + z =2
z = -4
∴ x = -2, y = 3 and z = -4

(b) 2x = 3y = 4z

$x^2 - 9y^2 – 4z + 8 = 0$

2x = 3y = 4z, substituting 4z = 2x and y = $\frac{2x}{3}$ into the equation $x^2 – 9y^2 – 4z + 8 = 0$
$x^2 – (2x)^2 – 2x + 8 = 0$
$-3x^2 – 2x + 8= 0$
x = $\frac{2\pm\sqrt{(-2)^2-4(-3)(8)}}{2(-3)}$; x = -2 or x = $\frac{4}{3}$
When x = -2; y = $\frac{2\ x\ (-2)}{3}$ = $\frac{-4}{3}$; z= $\frac{2\ x\ (-2)}{4}$ = -1
(x, y, z) = (-2, $\frac{-4}{3}, -1$)
When x = $\frac{4}{3}$; y = $\frac{2\ x\ (\frac{4}{3})}{3}$ = $\frac{8}{9}$; z= $\frac{2\ x\ (\frac{4}{3})}{4}$ = $\frac{2}{3}$
(x, y, z) = ($\frac{4}{3}$, $\frac{8}{9}$, $\frac{2}{3}$)

Alternatively
2x = 3y = 4z, substituting 4z = 3y and x = $\frac{3y}{2}$ into the equation $x^2 – 9y^2 – 4z + 8 = 0$
$\left(\frac{3}{2}y\right)^2 - 9y^2 – 3y + 8 = 0$
$9y^2 – 36y^2 – 12y + 32 = 0$
$-27y^2 – 12y + 32 = 0$
y = $\frac{12\pm\sqrt{(-12)^2-4(-27)(32)}}{2(-27)}$; y = $\frac{-4}{3}$ or x = $\frac{8}{9}$
When y = $\frac{-4}{3}$; x = $\frac{3}{2}\ x\ \frac{-4}{3} = -2$; z= $\frac{3}{4}\ x\ \frac{-4}{3}$ = -1
(x, y, z) = (-2, $\frac{-4}{3}, -1$)
When y =$\frac{8}{9}$; x = $\frac{3}{2}\ x\ \frac{8}{9} = \frac{4}{3}$; z= $\frac{3}{4}\ x\ \frac{8}{9}$ = $\frac{2}{3}$
(x, y, z) = ($\frac{4}{3}$, $\frac{8}{9}$, $\frac{2}{3}$)

Row reduction to Echelon matrix form

This method involves the expression of the three equations into matrix form known as augmented matrix and thereafter transforming the augmented matrix to a unity triangular matrix (a matrix whose elements in the major diagonal are unity and zero below)

Example

Solve the simultaneous equations

(a) x – 2y - 2z = 0
2x + 3y + z = 1
3x –y – 3z = 3

Solution

Expressing the equation in matrix form

$$\begin{pmatrix}1 & -2 & -2\\ 2 & 3 & 1\\ 3 & -1 & -3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 1\\ 3\end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix}1 & -2 & -2:0\\ 2 & 3 & 1\ :1\\ 3 & -1 & -3:3\end{pmatrix}$$

Transforming augmented matrix to a unity triangular matrix

$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\begin{pmatrix}1 & -2 & -2:0\\ 2 & 3 & 1\ :1\\ 3 & -1 & -3:3\end{pmatrix}\begin{matrix}\to \quad R_1 = R_1\\ \to 2R_1 - R_2 = R_2\\ \to 3R_1 - R_3 = R_3\end{matrix}\begin{pmatrix}1 & -2 & -2: & 0\\ 0 & -7 & -5: & -1\\ 0 & -5 & -3: & -3\end{pmatrix}$$

$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\begin{pmatrix}1 & -2 & -2: & 0\\ 0 & -7 & -5: & -1\\ 0 & -5 & -3: & -3\end{pmatrix}\begin{matrix}\to \quad R_1 = R_1\\ \to 2R_1 - R_2 = R_2\\ \to \frac{3R_1-R_3}{-4} = R_3\end{matrix}\begin{pmatrix}1 & -2 & -2: & 0\\ 0 & 1 & \frac{5}{7}: & {}^1/_7\\ 0 & 0 & 1: & -4\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1 & -2 & -2\\ 0 & 1 & \frac{5}{7}\\ 0 & 0 & -4\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ {}^1/_7\\ 16\end{pmatrix}$$

z = -4

y + $\frac{5}{7}z$ = ${}^1/_7$
y + $\frac{5}{7}x - 4$ = ${}^1/_7$
y = 3

x – 2y – 2z = 0
x -2(3) – 2(-4) = 0
x = -2
∴ x = -2, y = 3 and z = -4

(b) 3x – y – 2z = 0
x +3y – z = 5
2x – y + 4z = 26

Expressing the equation into matrix form

$$\begin{pmatrix}3 & -1 & -2\\ 1 & 3 & -1\\ 2 & -1 & 4\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 5\\ 26\end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix}3 & -1 & -2: & 0\\ 1 & 3 & -1: & 5\\ 2 & -1 & 4\ : & 26\end{pmatrix}$$

Transforming the augmented matrix into unity

$$\begin{pmatrix}3 & -1 & -2: & 0\\ 1 & 3 & -1: & 5\\ 2 & -1 & 4\ : & 26\end{pmatrix}\begin{matrix}\to \quad R_1 = R_1\\ \to R_1 - 3R_2 = R_2\\ \to 2R_1 - 3R_3 = R_3\end{matrix}\begin{pmatrix}3 & -1 & -2: & 0\\ 0 & -10 & 1\ : & -15\\ 0 & 1 & -16: & -78\end{pmatrix}$$

$$\begin{pmatrix}3 & -1 & -2: & 0\\ 0 & -10 & 1\ : & -15\\ 0 & 1 & -16: & -78\end{pmatrix}\begin{matrix}\to \quad R_1 \div 3 = R_1\\ \to 10R_1 \div -10 = R_2\\ \to \frac{R_2+10R_3}{159}\end{matrix}\begin{pmatrix}1 & \frac{-1}{3} & \frac{-2}{3} & : 0\\ 0 & 1 & -\frac{1}{10} & : \frac{15}{10}\\ 0 & 0 & 1 & : \ 5\end{pmatrix}$$

Z = 5

$y - \frac{1}{10}\ x5 = \frac{15}{10}$
10y - 5 =15
y = 2

$x - \frac{1}{3}y - \frac{2}{3}z = 0$
$x - \frac{1}{3}(2) - \frac{2}{3}(5) = 0$
x = 4
∴ x = 4, y = 2 and z = 5

(c) 3x – 2y – z = 5
x +3y – z = 4
2x – y + 4z = 13 [3, 1, 2]

Expressing the equation into matrix form

$$\begin{pmatrix} 3 & -2 & -1 \\ 1 & 3 & -1 \\ 2 & -1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 13 \end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix} 3 & -2 & -1 & : 5 \\ 1 & 3 & -1 & : 4 \\ 2 & -1 & 4 & : 13 \end{pmatrix}$$

Transforming the augmented matrix into unity

$$\begin{pmatrix} 3 & -2 & -1 & : 5 \\ 1 & 3 & -1 & : 4 \\ 2 & -1 & 4 & : 13 \end{pmatrix} \begin{matrix} \rightarrow R_1 = R_1 \\ \rightarrow R_1 - 3R_2 = R_2 \\ \rightarrow 2R_1 - 3R_3 = R_3 \end{matrix} \begin{pmatrix} 3 & -2 & -1 & : 5 \\ 0 & -11 & 2 & : -7 \\ 0 & -1 & -14 & : -29 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -2 & -1 & : 5 \\ 0 & -11 & 2 & : -7 \\ 0 & -1 & -14 & : -29 \end{pmatrix} \begin{matrix} \rightarrow R_1 \div 3 \\ \rightarrow R_2 \div -11 \\ \rightarrow \frac{R_2 - 11R_3}{156} \end{matrix} \begin{pmatrix} 1 & \frac{-2}{3} & \frac{-1}{3} & : \frac{5}{3} \\ 0 & 1 & \frac{-2}{11} & : \frac{7}{11} \\ 0 & 0 & 1 & : 2 \end{pmatrix}$$

Z = 2

$y - \frac{2}{11}\ x\ z = \frac{7}{11}$
11y - 4 =7
y = 1

$x - \frac{2}{3}y - \frac{1}{3}z = \frac{5}{3}$
$x - \frac{2}{3}(1) - \frac{1}{3}(2) = \frac{5}{3}$
x = 3
∴ x = 3, y = 1 and z = 2

Revision exercise 1

1. Using elimination method solve the following pairs of simultaneous equation
   (a) -3x + 2y =-16
   x + 5y = 11 [x = 6, y = 1]
   (b) 3y – 2x = -18
   2y + 3x = -6 [x = 0, y = -3]
   (c) 2x - 3y = 7
   X + 4y = -2 [x = 2, y = -1]
   (d) 5x + 3y = 8
   3x + 2y = 6 [x = -2, y = 6)
2. Using substitution method solve the following pairs of simultaneous equation
   (a) -3x + 2y = 16
   5x + 3y = 33 [x = 6, y = 1]
   (b) y – 2x = -3
   7y + 3x = -21 [x = 0, y = -3]
   (c) 2x + 5y = 26
   3x + 2y = 6 [x = -2, y = 6]
3. Solve the following pairs of simultaneous equation using the matrix method
   (a) 2x – 3y = 7
   2x + 3y = 1 [x =2, y = -1]
   (b) 2x – 7y = 1
   3x + 3y = 15
   (c) 3x – 4y = 5
   6x – 3y = 0 [x = -1, y =-2]
   (d) 3x + 2y = 3
   x -6y = 1[x =1, y = 0]
   (e) 2x + 3y= 1

3x + y = 5 [x =2, y = -1]

4. Solve the following simultaneous equations elimination and substitution method
   (a) 3x – 2y – 2z = -2
   x + 3y – 3z = -5
   2x – y + 4z = 26 [(x, y, z)=(4, 2, 5)]
   (b) 2x + 2y – 3z = 1
   3x + 3y – z = 5
   4x – 2y + 2z = 4 [(x, y, z)=(1, 1, 1)]
   (c) 4x –y + 2z = 7
   x + y + 6z = 2
   8x + 3y – 10z = -3 [(x, y, z)=(1, -2, $\frac{1}{2}$)]
5. By row reducing the appropriate matric to echelon form solve the systems of equations below
   (a) x +2y – 2z =0
   2x + y – 4z = -1
   4x – 3y + z = 11 [(x, y, z), (3, 1, 2]
   (b) x – 2y + 3z = 6
   3x + 4y – z = 3
   4x + 6y – 5z = 0 [(x, y, z)=(2, $\frac{-1}{2}$, 1)]
   (c) 2x – y + 3z = 10
   x +2y -5z = 9
   5x + y + 4z [x =2, y = -2, z= 1]
   (d) p + 2q – r = -1
   3p –q + 2r = 16
   2p + 3q + r =3 [p= 4, q = -2, r = 1]

Thank you

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