Surds

These are irrational numbers which cannot be expressed in terms of $\frac{a}{b}$ where a and b are rational. Irrational numbers may be defined as square roots of prime numbers.

Examples are $\sqrt{2}, \sqrt{3}, \sqrt{5}$ ….

Expression of root of numbers in surd form

Example 1

Write the following as the simplest surds

(i) $\sqrt{32}$ (ii) $\sqrt{50}$ $(iii)\sqrt{8}$ $(iv)\sqrt{27}$

Solution

(i) $\sqrt{32} = \sqrt{16\,x\,2} = \sqrt{16}\,x\,\sqrt{2} = 4\sqrt{2}$
(ii) $\sqrt{50} = \sqrt{25\,x\,2} = \sqrt{25}\,x\,\sqrt{2} = 5\sqrt{2}$
(iii) $\sqrt{8} = \sqrt{4\,x\,2} = \sqrt{4}\,x\,\sqrt{2} = 2\sqrt{2}$
(iv) $\sqrt{27} = \sqrt{9\,x\,3} = \sqrt{9}\,x\,\sqrt{3} = 3\sqrt{3}$

Addition and subtraction of surds

This is done by expressing the surds in their simplest form

Example 2

(i) $\sqrt{75} - 3\sqrt{27} + 2\sqrt{12}$

= $\sqrt{25\,x\,3} - 3\sqrt{9\,x\,3} + 2\sqrt{4\,x\,3}$

= $\sqrt{25}\,x\,\sqrt{3} - 3x\sqrt{9}x\,\sqrt{3} + 2\,x\,\sqrt{4}x\sqrt{3}$

= $5\,x\,\sqrt{3} - 3\,x\,3\,x\,\sqrt{3} + 2\,x\,2\,x\sqrt{3}$

$(5 - 9 + 4)\sqrt{3} = 0$

(ii) $\sqrt{50} + \sqrt{2} - 3\sqrt{18} + 2\sqrt{8}$
= $\sqrt{25\,x\,2} + \sqrt{2} - 3\sqrt{9x\,2} + 2\sqrt{4\,x\,2}$
=$5\sqrt{2} + \sqrt{2} - 3\,x\,3\sqrt{2} + 2x\,2\sqrt{2}$

$= (5 + 1 - 9 + 4)\sqrt{2} = \sqrt{2}$

Multiplication of surds

Finding the product of two surd numbers is the same as finding the root of the product of two numbers.

i.e. $\sqrt{a}\,x\,\sqrt{b} = \sqrt{ab}$

Example 3

Find the value of the following and give your answers in the simplest form

(i) $\sqrt{2}\,x\,\sqrt{2}$
(ii) $\sqrt{2}\,x\,\sqrt{20}$
(iii) $(3\sqrt{2} - 2\sqrt{3})^3$

Solution

(i) $\sqrt{2}\,x\,\sqrt{2} = \sqrt{2x2} = \sqrt{4} = 2$
(ii) $\sqrt{2}\,x\,\sqrt{30} = \sqrt{2x30} = 2\sqrt{15}$
(iii) $(3\sqrt{2} - 2\sqrt{3})^3$

Using Pascal's triangle; the coefficients of the terms in the expansion $(a + b)^3$ are 1 3 3 1

$(3\sqrt{2} - 2\sqrt{3})^3 =$

$(3\sqrt{2})^3 + 3(3\sqrt{2})^2(-2\sqrt{3}) + 3(2\sqrt{3})(-2\sqrt{3})^2 + (-2\sqrt{3})^3$

$=(54\sqrt{2}) + 3(18)(-2\sqrt{3}) + 3(12)(3\sqrt{2}) - 24\sqrt{3}$

$= 162\sqrt{2} - 132\sqrt{3}$

Division of surds

There are two types of division of surds;

- A fraction whose denominators has a single term such as $\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{3}}$, etc.

- A fraction whose denominator has double terms, such as $\frac{1}{\sqrt{2}+\sqrt{3}}, \frac{2+\sqrt{3}}{2-\sqrt{3}}, \frac{1}{1+\sqrt{5}}$, etc.
- In both cases, first eliminate the surds from the denominator. The process of eliminating surds from the denominator is called rationalization.
- In the first case rationalize the fraction by multiplying the numerator and denominator by the surd term of the denominator.

Example 4

Rationalize the following

(i) $\frac{2}{3\sqrt{5}}$
(ii) $\frac{1}{\sqrt{2}}$
(iii) $\frac{5}{\sqrt{3}}$

Solution

(i) $\frac{2}{3\sqrt{5}} = \frac{2\sqrt{5}}{3\sqrt{5}\,x\,\sqrt{5}} = \frac{2\sqrt{5}}{3\,x\,5} = \frac{2\sqrt{5}}{15}$
(ii) $\frac{1}{\sqrt{2}} = \frac{1\,x\sqrt{2}}{\sqrt{2}\,x\,\sqrt{2}} = \frac{\sqrt{2}}{2}$
(iii) $\frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{\sqrt{3}\,x\,\sqrt{3}} = \frac{5\sqrt{3}}{3}$

In the second case rationalize by multiplying the numerator and denominator by the conjugate of the denominator.

Note

(i) The conjugate a + $\sqrt{b}$ is a - $\sqrt{b}$ and that a - $\sqrt{b}$ is a + $\sqrt{b}$
(ii) The product of a surd function and its conjugate is equal to the difference of two squares. i.e. (a + $\sqrt{b}$)(a - $\sqrt{b}$) = $\left(a^2 - \left(\sqrt{b}\right)^2\right)$

Example 5

Rationalize the following

(i) $\frac{2}{2-\sqrt{2}}$
(ii) $\frac{1}{3-\sqrt{5}}$
(iii) $\frac{3-\sqrt{5}}{\sqrt{5-3}}$

solution

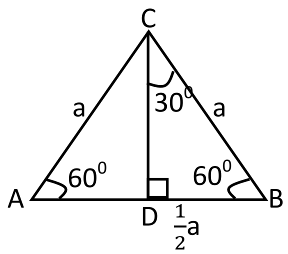
(i) $\frac{2}{2-\sqrt{2}} = \frac{2(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})} = \frac{2(2+\sqrt{2})}{4-2} = (2+\sqrt{2})$

(ii) $\frac{1}{3-\sqrt{5}} = \frac{1(3+\sqrt{5})}{(3-\sqrt{5})((3+\sqrt{5}))} = \frac{(3+\sqrt{5})}{9-5} = \frac{(3+\sqrt{5})}{4}$

(iv) $\frac{3-\sqrt{5}}{\sqrt{5-3}} x \frac{\sqrt{5+3}}{\sqrt{5+3}} = \frac{(3)^2-(\sqrt{5})^2}{(\sqrt{5})^2-(3)^2} = \frac{9-5}{5-9} = -1$

Set square angle (30^0, 45^0 and 60^0)

(a) Consider an equilateral triangle ABC of each side = a units



$ED^2 = a^2 - \left(\frac{1}{2}a\right)^2 = \frac{3a^2}{4}$

EC=$\frac{a\sqrt{3}}{2}$

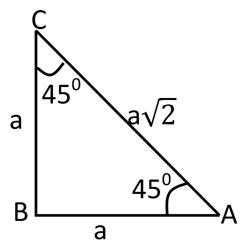
cos 60^0 = $\frac{a}{2a} = \frac{1}{2}$; sin60^0 =$\frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$

tan 60^0= $\frac{\sqrt{3}}{2}\ x\frac{2}{1} = \sqrt{3}$

cos 30^0 = $\frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$; sin60^0 =$\frac{a}{2a} = \frac{1}{2}$

tan 60^0= $\frac{1}{2}\ x\frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

(b) Given a right angled isosceles triangle ABC with equal perpendicular sides each of length a units



$AC^2 = a^2 + a^2 = 2a^2$
$AC = a\sqrt{2}$
cos 45^0 = sin45^0 = $\frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$
tan 45^0 = $\frac{a}{a} = 1$

Example 6

Express without a surd in the denominator each of the following

(a) $\frac{1+\tan 30^0}{1-\tan 30^0}$

(b) $\left(\frac{1+\cos 45^0}{2-\sin 60^0}\right)^2$

Solution

(a) $\frac{1+\tan 30^0}{1-\tan 30^0} = \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)\left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$

$= \frac{3+2\sqrt{3}+1}{3-1} = \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3}$

(b) $\left(\frac{1+\cos 45^0}{2-\sin 60^0}\right)^2 = \left(\frac{1+\frac{\sqrt{2}}{2}}{2-\frac{\sqrt{3}}{2}}\right)^2 = \left(\frac{2+\sqrt{2}}{4-\sqrt{3}}\right)^2$

$= \frac{4+4\sqrt{2}+2}{16-8\sqrt{3}+3}$

$= \left(\frac{6+4\sqrt{2}}{19-8\sqrt{3}}\right)\left(\frac{19+8\sqrt{3}}{19+8\sqrt{3}}\right)$

$= \frac{114+48\sqrt{3}+76\sqrt{2}+32\sqrt{6}}{169}$

Revision exercise

1. Simplify
   (a) $\sqrt{48}$ [$4\sqrt{3}$]
   (b) $\sqrt{162}$ [$9\sqrt{2}$]
   (c) $\sqrt{28}$ [$2\sqrt{7}$]
   (d) $\sqrt{45}$ [$3\sqrt{5}$]
   (e) $\sqrt{125}$ [$5\sqrt{5}$]
   (f) $\sqrt{147}$ [$7\sqrt{3}$]
2. Simplify the following suds
   (a) $\sqrt{8}+\sqrt{200}-4\sqrt{18}$ [$2\sqrt{2}$]
   (b) $5\sqrt{20}+2\sqrt{45}+2\sqrt{5}$ [$18\sqrt{5}$]
   (c) $3\sqrt{50}+2\sqrt{32}-2\sqrt{75}+2\sqrt{12}-\sqrt{27}$ [$23\sqrt{2}-7\sqrt{3}$]
3. Given that $4\sqrt{20}+3\sqrt{5}-5\sqrt{125} = x\sqrt{5}$, find the value of x [-14]
4. Find the value of the following simplifying the answer as much as possible
   (a) $(5\sqrt{2}-\sqrt{5})(3\sqrt{5}-2\sqrt{2})$[$5+13\sqrt{10}$]
5. Express each of the following in the form $\frac{a\sqrt{b}}{c}$ where a, b and c are integers
   (a) $\frac{2}{\sqrt{7}}$ $\left[\frac{2\sqrt{7}}{7}\right]$
   (b) $\frac{3}{\sqrt{2}}$ $\left[\frac{3\sqrt{2}}{2}\right]$
   (c) $\frac{14\sqrt{5}}{\sqrt{7}}$ [$2\sqrt{35}$]
   (d) $\frac{3\sqrt{50}}{5\sqrt{27}}$ $\left[\frac{\sqrt{6}}{3}\right]$
   (e) $\frac{4\sqrt{45}}{5\sqrt{8}}$ $\left[\frac{3\sqrt{10}}{5}\right]$
6. Express the following in the form a +b$\sqrt{c}$
   (a) $\frac{2\sqrt{3}-3\sqrt{2}}{2\sqrt{3}+3\sqrt{2}}$ [$-5+2\sqrt{6}$]
   (b) $\frac{4+3\sqrt{2}}{4-3\sqrt{2}}$ [$-17-12\sqrt{2}$]
   (c) $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt{3}+1}$ $\left[\frac{1}{2}-\frac{1}{4}\sqrt{2}+\frac{1}{4}\sqrt{6}\right]$
7. Show that $\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)^2+\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)^2$=194
8. Express without a surd in the denominator each of the following
   (a) $\frac{1}{1-\sin 30^0}$ [2]
   (b) $\frac{1+\tan 60^0}{1-\tan 60^0}$ [-2-$\sqrt{3}$]
9. Express each of the following in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c and d are integers
   (a) $\frac{3}{6+\sqrt{3}}$ $\left[\frac{6-\sqrt{3}}{11}\right]$
   (b) $\frac{3+\sqrt{2}}{5-\sqrt{2}}$ $\left[\frac{17+8\sqrt{2}}{23}\right]$
   (c) $\frac{3+\sqrt{24}}{2+\sqrt{6}}$ $\left[\frac{6-\sqrt{6}}{2}\right]$
10. Solve the equation
    $\sqrt{2x+3}-\sqrt{x+1} = \sqrt{x-2}$ [3]
11. Without using mathematical tables or calculators, find the value of
    $\frac{(\sqrt{5}+2)^2-(\sqrt{5}-2)^2}{8\sqrt{5}}$ [1]

Thank you

Dr. Bbosa Science