Binomial theorem

Pascal's triangle

The Pascal's triangle below and its further extension is used to determine the coefficients of the expansion of $(p + q)^n$

1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Observations

- The coefficients are symmetrical; they are the same irrespective of which side they are read from, for instance the coefficients of $(p + q)^6$ are
  1 6 15 20 15 6 1
- The coefficient of the 2nd term in the expansion is the index of a given expansion. E.g. in the expansion of $(p + q)^6$, the coefficient of the 2nd term is 6.
- The number of terms in the expansion exceeds the index by one, e.g. $(p + q)^4$ with index 4, has 5 terms
- The index of the first term of the expansion decreases by one, from the index given till zero, whereas, the index of the second term increases by one from zero to the given index
  For $(p + q)^3 = 1p^3q^0 + 3p^2q^1 + 3p^1q^2 + 1p^0q^3$
  $=p^3 + 3p^2q + 3pq^2 + q^3$
- The sum of indices in each term s constant and equal to the index of the expansion.

Example 1

Use Pascal's triangle to expand

(a) $(p + q)^4$

The coefficients are 1 4 6 4 1

Terms are $1p^4q^0 + 4p^3q^1 + 6p^2q^2 + 4p^1q^3 + 1q^4$

$= p^4 +4p^3q + 6p^2q^2 + 4pq^3 + q^4$

(b) $(2x + 3y)^3$

The coefficients are 1 3 3 1

Terms are

$1(2x)^3(3y)^0 + 3(2x)^2(3y)^1 + 3(2x)^1(3y)^2 + 1(2x)^0(3y)^3$

$= 8x^3+ 36x^2y + 54xy^2 + 27y^3$

(c) $(a + b + c)^2$

The terms can be grouped into two ways:

Either a and (b + c) or (a + b) and c

The coefficients are 1 2 1

Either

Terms are: $1a^2(b + c)^0 + 2a^1(b + c)^1 + 1a^0(b + c)^2$

$= a^2 + 2a(b + c) + (b + c)^2$

$= a^2 + 2ab + 2ac + b^2 + 2bc + c^2$

Or

$(a + b + c)^2$

$= 1(a + b)^2c^0 + 2(a + b)_1c^1 + 1c^2$

$= (a + b)^2 + 2c(a + b) + c^2$

$=a^2 + 2ab + b^2 + 2ac + 2bc + c^2$

Example 2

Expand $(a + b)^4$ using Pascal's triangle. Hence find $(1.996)^4$ correct to 3 decimal places

Solution

From Pascal's triangle the coefficients are
1 4 6 4 1

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(1.996)^4 = (2- 0.004)^4$

Substituting a = 2 and b = -0.004

$24 + 4(2)^3(-0.004) + 6(2)^2(-0.004)^2 + 4(2)(-0.004)^3 + (-0.004)^4$

16 – 0.128 + 0.000384 = 15.872 (3D)

Example 3

(a) Expand $(2 – 3x)^4$ using Pascal's triangle. Hence evaluate $(1.97)^4$ correct to 3 decimal places.

Solution

Coefficients: 1 4 6 4 1

$(2 – 3x)^4 = 1 (2)^4(-3x)^0 + 4(2)^3(-3x)^1 + 6(2)^2(-3x)^2 + 4(2)^1(-3x)^3 + 1(2)^0(-3x)^4$

$= 16 – 96x + 216x^2 – 216x^3 + 81x^4$

Now $(1.97)^4 = (2 – 0.03)^4 = (2 – 3(0.01))^4$

⇨ x = 0.01

$(1.97)^4 = 16 – 96 (0.01) + 216(0.01)^2 - 216(0.01)^3 + 81(0.01)^4 = 15.0613848$

= 15.061 (3d.p)

(b) Use Pascal's triangle to evaluate $(1.02)^3$ correct 5 significant figures

Solutions

$(1.02)^3 = (1 + 02)^3$

Coefficients are : 1 3 3 1

$(1 + 02)^3 = 1^3 + 3(1)^2(0.02) + 3(1)(0.02)^2 + (0.02)^3 = 1.061208$

= 1.0612 (5 significant figures)

The idea of factorial and combination can also be used to determine the coefficients of the expansions

Example 3

Expand

(a) $(p + 3q)^3$

Solution

The coefficients are ${}^3C_0$ ${}^3C_1$ ${}^3C_2$ ${}^3C_3$

Terms are p^3 $p^2(3q)$ $p(3q)^2$ $(3q)^3$

$(p + 3q)^3 = {}^3C_0p^3 + {}^3C_1p^2(3q) + {}^3C_2p(3q)^2 + {}^3C_3(3q)^3$

$= P^3 + 9P^23q + 27pq^2 + 27q^3$

(b) $(1 – x)^4$. Hence evaluate $(0.99)^4$ correct to four decimal places

Solution

Coefficients are ${}^4C_0$ ${}^4C_1$ ${}^4C_2$ ${}^4C_3$ ${}^4C_4$

Or simply $\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

$1 – x)^4 = {}^4C_01^4(-x)^0 + {}^4C_11^3(-x)^1 + {}^4C_21^2(-x)^2 + {}^4C_31^1(-x)^3 + {}^4C_41^0(-x)4^0$

$= 1 – 4x + 6x^2 – 4x^3 + x^4$

Now 0.99 = 1 – 0.01) = x = 0.01

$(0.99)^4$

$= 1 – 4(0.01) + 6(0.01)^2 – 4(0.01)^3 + (0.01)^4$

= 0.96059601

= 0.9606 (4d.p)

The binomial theorem for positive integral index

Consider the expansion of $(1 + x)^4$, the result is

$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

Observations

(i) The indices of x increase by 1 from term to term
(ii) The index of the last term being the same as the power to which (1 + x) s raised
(iii) The coefficients of the terms of expansion are $^4C_0$ $^4C_1$ $^4C_2$ $^4C_3$ $^4C_4$

Hence the expansion of $(1 + x)^n$ is

$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots x^n$

The expansion of $(a + x)^n$ is

$$(a + x)^n = a^n\left(1 + \frac{x}{a}\right)^n$$

$$= a^n\left[1 + \binom{n}{1}\left(\frac{x}{a}\right) + \binom{n}{2}\left(\frac{x}{a}\right)^2 + \cdots x\left(\frac{x}{a}\right)^n\right]$$

$$= a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \cdots x^n$$

Example 4

Expand $(1 + 4x)^{14}$ in ascending power of x up to and include the 4^{th} term. Hence evaluate $(1.0004)^{14}$ correct to four decimal places

Solution

$(1 + 4x)^{14}$

$= 1 + \binom{14}{1}(4x) + \binom{14}{2}(4x)^2 + \binom{14}{3}(4x)^3$

$= 1 + 56x + 1456x^2 + 23296x^3$

Now $(1.0004)^{14} = (1 + 0004)^{14} \Rightarrow x = 0.0001$

$(1 + 4x)^{14}$

$= 1 + 56(0.0001) + 1456(0.0001)^2 +$

$23296(0.0001)^3$

$= 1.005614583$

$= 1.0056$ (4d.p)

Note the next term in the expansion is

$\binom{14}{4}(4x)^4 = 256256x^4$

Its value $= 256256(0.0001)^4 = 2.56256 \times 10^{-11}$

Which when added to the above answer there will negligible change in value

Example 5

Expand $(3 – 2x)^{12}$ in ascending powers of x up to and including the term x^3. Hence evaluate $(2.998)^{12}$ correct to the nearest whole number.

Solution

$(3 – 2x)^{12} = 3^{12} + \binom{12}{1}(3)^{11}(-2x)^1 + \binom{12}{2}(3)^{10}(-2x)^2 + \ldots$

$= 531441 – 4251528x + 1588936x^2$

Now $(2.998)^{12} = (3 – 0.002)^{12} \Rightarrow x = 0.001$

$(2.998)^{12} = 531441 – 4251528(0.001) + 1588936(0.001)^2$

$= 527205.0609$

$= 527205$ (nearest whole number)

Particular terms of binomial expansion

As earlier seen, the expansion of

$(a + x)^5 = a^5x^0 + 5a^4x^1 + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$

In general if r is the power of x in the expansion $(a + x)^n$, then

the U_{r+1} term of x = $^nC_ra^{n-r}x^r$

Example 6

Find the term of x^4 in the expression of

(i) $(1 + x)^9$

By using U_{n+1}; term of x = $^nC_ra^{n-r}x^r$

n = 9, r = 4, a = 1

$U_5 = {}^9C_4 1^{9-4}x^4 = {}^9C_4x^4 = 126x^4$

(ii) (3 + x)7

n = 7, a = 3, r = 4

$U_5 = {}^7C_4 3^{7-4}x^4 = {}^7C_4(3)^3x^4 = 945x^4$

(iii) $\left(2 - \frac{x}{2}\right)^{12}$

$n = 12, a = 2$

$U_5 = {}^{12}C_4 2^{12-4}\left(-\frac{x}{2}\right)^4 = {}^{12}C_4(2)^8\left(-\frac{x}{2}\right)^4 = 7920x^4$

Example 7

Find the term indicated in expansion of the following expression

(i) $\left(3x - \frac{2}{x}\right)^5$ [x^3]

$$\left(3x - \frac{2}{x}\right)^5 = 3^5x^5\left(1 - \frac{2}{3x^2}\right)^5$$

The term in $x^3 = 3^5x^5.{}^5C_1\left[1^4\left(-\frac{2}{3x^2}\right)^1\right]$

$= 3^5x^5.{}^5C_1\left(-\frac{2}{3x^2}\right) = -810x^3$

(ii) $\left(2x + \frac{5}{x}\right)^6$ [x^4]

$$\left(2x + \frac{5}{x}\right)^5 = 2^6x^6\left(1 + \frac{5}{2x^2}\right)^6$$

The term in $x^3 = 2^6x^6.{}^6C_1\left[1^5\left(\frac{5}{2x^2}\right)^1\right]$

$= 2^6x^6.{}^6C_1\left(\frac{5}{2x^2}\right)$ $960x^4$

Finding terms independent of x

The term is said to be independent of x if the power of x is zero

Example 8

Find the term independent of x in the following expansion

(a) $\left(2x + \frac{1}{x^2}\right)^{12}$

Solution

$$\left(2x + \frac{1}{x^2}\right)^{12} = 2^{12}x^{12}\left(1 + \frac{1}{2x^3}\right)^{12}$$

The term independent of x = $2^{12}x^{12}$ multiplied by the term in x^{-12} in the expansion $\left(2x + \frac{1}{x^2}\right)^{12}$

⇨ The term independent of x

$= 2^{12}x^{12}.{}^{12}C_4\left(\frac{1}{2x^3}\right)^4 = 2^{12}x^{12} \times {}^{12}C_4 \times \frac{1}{2^4x^{12}}^4$

$= 2^8 \times 495 = 126720$

Alternatively

By using $U_{r+1} = {}^nC_r a^{n-r} x^r$

The term independent of x is got by equating the index of x to zero

$U_{r+1} = {}^{12}C_r(2x)^{12-r}\left(\frac{1}{x^2}\right)^r$

$= {}^{12}C_r(2x)^{12-r}x^{-2r}$

$= {}^{12}C_r.\ 2^{12-r}.x^{12-3r}$

Equating the $x^{12-3r} = 1$

⇨ $12 - 3r = 0$
$r = 4$

Term independent of x = ${}^{12}C_4.\ 2^{12-4}$

$= {}^{12}C_4.\ 2^8$

$= 126720$

(b) $\left(2x^2 + \frac{3}{x}\right)^{12}$

Solution

$$\left(2x + \frac{3}{x}\right)^{12} = 2^{12}x^{12}\left(1 + \frac{3}{2x^2}\right)^{12}$$

Un+ r = ${}^{12}C_r(2x^2)^{12-r}(3x^{-1})^r$

$= {}^{12}C_r(2x^2)^{12-r}(3^r)x^{-r}$

$= {}^{12}C_r.\ 2^{12-r}(3^r)x^{24-3r}$

Equating the $x^{12-3r} = 1$

⇨ $24 - 3r = 0$
$r = 8$

Term independent of x = ${}^{12}C_8.\ 2^{12-4}.3^8$

$= {}^{12}C_8.\ 2^4.3^8$

$= 51, 963,120$

Binomial expansion of terms with fractional or negative powers

As noted earlier

$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots.\ x^n$

The following is noted

(i) For positive integral value of n, i.e. n ≥ 1, the series above terminates at the term x^n and its sum is $(1 + x)^n$.

(ii) For fractional or negative values of n, the series above does not terminate but instead converges to $(1 + x)^n$ as the limit of its sum only -1 < x< 1 or $|x| < 1$

Example 9

Expand $\sqrt{(1-2x)}$ up to the term x^3. Hence evaluate $\sqrt{0.98}$ correct to four decimal places.

Solution

$\sqrt{(1-2x)} = (1-2x)^{\frac{1}{2}}$

Using

$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + ... x^n$

Comparing with $(1-2x)^{\frac{1}{2}}$ => x ≡ -2x and n = ½

$$\sqrt{(1-2x)} = 1 + \frac{1}{2}(-2x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)(-2x)^2}{2!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-2x)^3}{3!}$$

$$=1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3$$

Now $\sqrt{0.98} = \sqrt{(1-0.02)}$

Comparing with $\sqrt{(1-2x)}$; x = 0.01

Substituting

$\sqrt{0.98} = 1 - (0.01) - \frac{1}{2}(0.01)^2 - \frac{1}{2}(0.01)^3$

= 0.9899495

$\therefore \sqrt{0.98} = 0.9899$ (4d.p)

Example 10

Given that x is very small that its cube and higher powers can be neglected, show that

$$\sqrt{\left(\frac{1-x}{1+x}\right)} = 1 - x + \frac{1}{2}x^2$$

By putting x = $\frac{1}{8}$ show that $\sqrt{7} = \frac{339}{128}$

Solution

By rationalizing the denominator

$$\sqrt{\left(\frac{1-x}{1+x}\right)\left(\frac{1-x}{1-x}\right)} = \sqrt{\frac{(1-x)^2}{(1-x^2)}}$$

$$= (1-x)(1-x^2)^{\frac{1}{2}}$$

But $(1-x^2)^{\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-x^2) + ...$

$$= 1 + \frac{1}{2}x^2$$

$$\sqrt{\left(\frac{1-x}{1+x}\right)} = (1-x)\left(1 + \frac{1}{2}x^2\right)$$

$$= 1 - x + \frac{1}{2}x^2$$

Putting x = $\frac{1}{8}$

$$\sqrt{\left(\frac{1-\frac{1}{8}}{1+\frac{1}{8}}\right)} = 1 - \frac{1}{8} + \frac{1}{2}\left(\frac{1}{8}\right)^2$$

$$\sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{3} = 1 - \frac{1}{8} + \frac{1}{128} = \frac{113}{128}$$

$$\sqrt{7} = \frac{339}{128}$$

Example 11

Expand $\sqrt{\left(\frac{1+x}{1-x}\right)}$ up to the term x^3

Use your expansion to evaluate $\sqrt{23}$ correct to 3 decimal places taking x = $\frac{1}{24}$.

Solution

By rationalizing the denominator

$$\sqrt{\left(\frac{1+x}{1-x}\right)\left(\frac{1+x}{1+x}\right)} = (1+x)(1-x^2)^{\frac{1}{2}}$$

But$(1-x^2)^{\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \cdots$

$$\sqrt{\left(\frac{1+x}{1-x}\right)} = (1+x)\left(1 + \frac{1}{2}x^2\right)$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3$$

Hence putting x = $\frac{1}{24}$

$$\sqrt{\left(\frac{1+\frac{1}{24}}{1-\frac{1}{24}}\right)} = 1 + \frac{1}{24} + \frac{1}{2}\left(\frac{1}{24}\right)^2 + \frac{1}{2}\left(\frac{1}{24}\right)^3$$

$\sqrt{\frac{25}{23}} = \frac{5}{\sqrt{23}} = \frac{28825}{27648}$

$\sqrt{23} = \frac{5 \ x\ 27648}{28825} = 4.7958$

$\therefore \sqrt{23} = 4.7958$ (4d.p)

Example 12

Use the binomial theorem to expand $\sqrt[3]{(1-x)}$ up to x^3. Use your expansion to evaluate $\sqrt[3]{7}$ correct to four decimal places

Solution

$\sqrt[3]{(1-x)} = (1-x)^{\frac{1}{3}}$

$= 1 + \frac{1}{3}(-x) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)(-x)^2}{2!} + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(-x)^3}{3!} + \ldots$

$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{1}{81}x^3$

Hence $\sqrt[3]{7} = \sqrt[3]{(8-1)} = \sqrt[3]{8\left(1-\frac{1}{8}\right)}$

$= 2\left(1-\frac{1}{8}\right)^{\frac{1}{3}}$ => x = $\frac{1}{8}$

$\sqrt[3]{7} = 2\left[1 + \frac{1}{3}\left(\frac{1}{8}\right) - \frac{1}{9}\left(\frac{1}{8}\right)^2 + \frac{1}{81}\left(\frac{1}{8}\right)^3\right]$

=1.9130(4d.p)

Example 13

Write down the expansion of $\sqrt{(1-x)}$ in ascending powers of x as far as the term x^4. Use your expansion to find $\sqrt{80}$ correct to four significant figures.

Solution

$\sqrt{(1-x)} = (1-x)^{\frac{1}{2}}$

For binomial expansion

$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots. x^n$

Comparing $(1-x)^{\frac{1}{2}}$ with$(1 + x)^n$

n = $\frac{1}{2}$ and x = -x

$(1-x)^{\frac{1}{2}} = 1 + \frac{1}{2}(-x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)(-x)^2}{2!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^3}{3!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-x)^4}{4!}$

$(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x - \frac{x^2}{8} - \frac{x^3}{6} - \frac{x^4}{24}$

Now $\sqrt{80} = \sqrt{81-1} = \sqrt{81\left(1-\frac{1}{81}\right)}$

$= 9\left(1-\frac{1}{81}\right)^{\frac{1}{2}}$

Comparing $\left(1-\frac{1}{81}\right)^{\frac{1}{2}}$ with $(1-x)^{\frac{1}{2}}$

⇨ x = $\frac{1}{81}$

Substituting for x

$\left(1-\frac{1}{81}\right)^{\frac{1}{2}}$

$= 1 - \frac{1}{2}\left(\frac{1}{81}\right) - \frac{\left(\frac{1}{81}\right)^2}{8} - \frac{\left(\frac{1}{81}\right)^3}{6} - \frac{\left(\frac{1}{81}\right)^4}{24}$

$= \frac{52163}{52488}$

$\sqrt{80} = 8.944$ (4 S.F)

Not the termx3 has been neglected as it does not affect the answer to 4 significant figured

Example 14

John operates an account with a certain bank which pays a compound interest rate of 13.5% per annum. He opened the account at the beginning of the year with sh. 500,000 and deposits the same amount of money at the beginning of every year. Calculate how much he will accumulate at the end of 9 years. After how long will the money have accumulated to sh. 3.32 millions?

Solution

The 1st deposit will grow to

$500000\left(1+\frac{13.5}{100}\right)$= 500000 x 1.135

2^{nd} deposit will grow to 500000 x 1.135^2

n^{th} deposit will grow to 500000 x 1.135^n

9^{th} deposit will grow to 500000 x 1.135^9

The total = 500,000[1.135 +1.135^2 + … + 1.135^9]

$=500000\left[a\left(\frac{r^n-1}{r-1}\right)\right]$

$=500000\ \left[1.135\left(\frac{1.135^9-1}{1.135-1}\right)\right]$

=8,936,381

Finding how long it will take the money to accumulate to sh. 3,320,000

$500000\ \left[1.135\left(\frac{1.135^n-1}{1.135-1}\right)\right]=3320000$

n = 4.6years

Example 15

Expand $(1 + x)^{-2}$ in descending powers of x including the term x^{-4}. If x = 9 find the percentage error in using the first two terms of the expression.

Solution

From

$$(1+x)^n=1+nx+\frac{2(n-1)x^2}{2!}+\cdots+x^n$$

Now $(1 + x)^{-2}$ = $x^{-2}\left(1+\frac{1}{x}\right)^{-2}$

$x^{-2}\left(1+\frac{1}{x}\right)^{-2}$

$=x^{-2}\left[1+(-2)\frac{1}{x}+\frac{(-2)(-3)}{2!}\left(\frac{1}{x}\right)^2\right]$

$=x^{-2}\left[1-\frac{2}{x}+\frac{3}{x^2}\right]$

$=x^{-2}[1-2x^{-1}+3x^{-2}]$

$=x^{-2}-2x^{-3}+3x^{-4}$

If x = 9

$(1 + x)^{-2}=9^{-2}-2(9)^{-3}+3(9)^{-4}$

$=\frac{1}{81}-\frac{2}{729}$ (using the first 2 terms)

$=\frac{7}{729}$

The exact value is $(1 + 9)^{-2}$ = $\frac{1}{100}$

Error = $\frac{1}{100}-\frac{7}{729}=\frac{29}{72900}$

%error = $\frac{29}{72900}\ x100\ x100$= 3.978%(3d.p)

Example 16

(a) Find the three terms of the expansion $(2 – x)^6$ and use it to find $(1.998)^6$ correct to two decimal places (06 marks)

$(2 – x)^6$ = $2^6+\binom{6}{1}2^5(-x)^1+\binom{6}{2}4(-x)^2$

= 64 -192x + $240x^2$

$(1.998)^2=(2-0.002)^2$

$$\begin{aligned}&=64-192(0.002)+240(0.002)^2\\&=64-0.384+0.00096\\&=63.61696\\&=63.62\ (2D)\end{aligned}$$

Example 17

(a) Prove by induction

1.3 + 2.4 + … + n(n + 2) = $\frac{1}{6}n(n+1)(2n+7)$ for all values of n.
Suppose n = 1
L.H.S = 1 x 3 = 3
R.H.S = $\frac{1}{6}x1(1+1)(2+7)=3$
L.H.S = R.H.S, hence the series holds for n =1

Suppose n = 2

L.H.S= 1 x 3 + 2 x 4 = 11

R.H.S = $\frac{1}{6}\ x\ 2(2+1)(4+7) = 11$

L.H.S = R.H.S, hence the series holds for n =2

Suppose n = k

1.3 + 2.4 + … + k(k + 2) = $\frac{1}{6}k(k+1)(2k+7)$

For n = k + 1

1.3 + 2.4 + … + k(k + 2), (k + 1)(k + 3) = $\frac{1}{6}k(k+1)(2k+7)$+(k +1)(k + 3)

=(k + 1)$\left[\frac{1}{6}k(2k+7)+(k+3\right]$

= $\frac{1}{6}(k+1)(2k^2+13k+18)$

= $\frac{1}{6}(k+1)(2k^2+4k+9k+18)$

= $\frac{1}{6}(k+1(k+2)(2k+9)$

= $\frac{1}{6}(k+1)(k+2)[2(k+1)+7]$

Which is equal to R.H.S when n = k + 1

It holds for n = 1, 2, 3 …, hence it holds for all integral values of n.

(b) A man deposits Shs. 150,000 at the beginning of every year in a micro finance bank with the understanding that at the end of the seven years he is paid back his money with 5% per annum compound interest. How much does he receive?

Using amount, A = P$\left(1+\frac{r}{100}\right)^n$

=150000$\left(1+\frac{5}{100}\right)^7$=211,065.06

Alternatively

1st year

P = 150,000

He is paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 150,000$\left(1+\frac{5}{100}\right)$=157,500

2nd year

P = 157, 500

He is paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 157500$\left(1+\frac{5}{100}\right)$=165375

3rd year

P = 165375

Interest = $\frac{5}{10}\ x\ 165375 = 8268.75$

He is paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 165375 $\left(1+\frac{5}{100}\right)$=173643.75

4th year

P=173643.75

He is paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 173643.75$\left(1+\frac{5}{100}\right)$=182325.94

5th year

P =182325.94

His paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 182325.94$\left(1+\frac{5}{100}\right)$=191442.23

6th year

P = 191442.23

He is paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 191442.23$\left(1+\frac{5}{100}\right)$=201014.35

7^{th} year

P = 201014.35

He is paid back principal plus interest; $P\left(1 + \frac{5}{100}\right)$ = 201014.35$\left(1 + \frac{5}{100}\right)$=211,065.06

∴ by the 7^{th} year he has accumulated shs. **211,065.06**

Example 18

Expand $\sqrt{\left(\frac{1+2x}{1-x}\right)}$ up to the term x^2. Hence find the value of $\sqrt{\left(\frac{1.04}{0.98}\right)}$ to four significant figures. (12marks)

$$\sqrt{\left(\frac{1+2x}{1-x}\right)} = (1+2x)^{\frac{1}{2}}(1-x)^{\frac{-1}{2}}$$

$$using\ (1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots$$

$$\sqrt{\left(\frac{1+2x}{1-x}\right)} = \left(1 + x - \frac{1}{2}x^2\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right)$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + x + \frac{1}{2}x^2 - \frac{1}{2}x^2$$

$$= 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

$$\therefore \sqrt{\left(\frac{1+2x}{1-x}\right)} \approx 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

$$substituting\ for\ x = 0.02$$

$$\sqrt{\left(\frac{1.04}{0.98}\right)} = \sqrt{\frac{1+2(0.02)}{1-0.02}}$$

$$= 1 + \frac{3}{2}(0.02) + \frac{3}{8}(0.02)^2 = 1.030$$

Revision exercise

1. Given that the ratio of the 3rd to the 4th term of the expansion (2+3x)n is 5:14, find the value of n when x = $\frac{2}{5}$. [n= 16]
2. Expand $(3 – 4x)^5$ in ascending order of x up to and including the term x^3. Hence evaluate $(4.96)^5$ correct to 2 d.p. [3001.98]
3. (a) Find the coefficient of x2 in the expansion of $(1 – 2x)^n$ is 24 [4]
   (b) Find the term independent of x in the expansion of $\left(2x^3 - \frac{1}{x}\right)^{20}$ [-496128]
   (c) Use the binomial expansion to expand $\sqrt[4]{(1+2x)}$ up to the term x^3. Hence evaluate $\sqrt[4]{83}$ correct to three decimal places [3.018]
4. Expand $\sqrt{\left(\frac{1+2x}{1-2x}\right)}$ up to and including the term x3. Hence find the value of $\sqrt{\frac{1.02}{0.98}}$ to four significant figures. Deduce the value of $\sqrt{51}$ to 3 significant figures [1.0202, 7.14]
5. Five millions shillings are invested each year at a rate of 15%. In how many years will it accumulate to more than 50millions? [6years]
6. Expand $\left(1 - \frac{x}{3}\right)^{\frac{1}{2}}$ as far as x3. Hence evaluate $\sqrt{8}$ correct to 3 decimal places [2.829]
7. A man deposits sh. 800,000 into his saving account on which interest is 15% per annum. If he makes no withdraws, after how many years will his balance exceed sh. 8millions? [16.5 years]
8. Determine the binomial expansion of $\left(1 + \frac{x}{2}\right)^4$. Hence evaluate $(2.1)^4$ correct to 2 decimal places.[19.45]
9. Determine the binomial expansion of $\left(1 - \frac{x}{2}\right)^5$. Hence evaluate $(0.875)^5$ correct to four decimal places [0.5129]

10. A financial credit society give a compound interest of 2% per annum to its members. If Bbosa deposits sh. 10000 at the beginning of every year. How much would he accumulate at the end of the fifth year if no withdraws within this period [sh. 530812]

11. Expand $\sqrt{\left(\frac{1+x}{1-x}\right)}$ in ascending powers of x up to a term x^2. $\left[1 + x + \frac{x^2}{x} + \cdots\right]$

12. (a) Using the expansion $(1 + x)^{\frac{1}{2}}$ up to the term x^3, find the value of $\sqrt{1.08}$ to four decimal places [1.0392]

(b) Express $\sqrt{1.08}$ in the form $\frac{a}{b}\sqrt{c}$. Hence evaluate $\sqrt{3}$ correct to 3 significant figure $\left[\frac{3}{5}\sqrt{3}, 1.73\right]$

13. (a) obtain the first four non – zero terms of the binomial expansion in ascending powers of x of $(1 - x^2)^{-\frac{1}{2}}$ give that$|x| < 1$ $\left[1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16}\right]$

(b) show that, when x = $\frac{1}{3}$;

$(1 - x^2)^{-\frac{1}{2}} = \frac{3\sqrt{2}}{4}$

(c) substitute x = $\frac{1}{3}$ into your expansion and hence obtain an approximation of $\sqrt{2}$, give your answer to five decimal places [1.41415]

14. (a) show that $\frac{1}{\sqrt{4-x}} = \frac{1}{2}\left(1 - \frac{x}{4}\right)^{-\frac{1}{2}}$

(c) Write the first three terms in the binomial expansion of $\left(1 - \frac{x}{4}\right)^{-\frac{1}{2}}$ in ascending power of x stating the range for which this expansion is valid. $\left[1 + \frac{x}{8} + \frac{3x^2}{128};\ |x| < 4\right]$

(d) Find the first three terms in the expansion of $\frac{2(1+x)}{\sqrt{4-x}}$ in ascending power of x for small values of x.$\left[1 + \frac{9x}{8} + \frac{19x^2}{128};\right]$

Thank you

Dr. Bbosa Science