Application of Differentiation

Differentiation is helpful in various application including

- Displacement, velocity and acceleration given as a function of time
- Rates of changes
- Small angles
- Tangents and normal
- Turning points and stationary points
- Maclaurin's theory
- Curve sketching

Displacement, velocity and acceleration

Displacement

Displacement is the distance covered by a particle/body in a specified direction.

The displacement ® of a particle is said to be maximum or minimum when $\frac{d}{dt}(r) = 0$ this enables us to obtain the time when r is maximum or minimum. Hence

$r_{\max\ or\ r_{min}}$ is the value $|r|$

Velocity

This is the rate of change of displacement or $v = \frac{d}{dt}(r)$ where r is displacement.

The velocity of a partice is maximum or minimum when $\frac{d}{dt}(v) = 0$, this enables us to obtain the time when v is maximum or minimum. Hence

$v_{\max\ or\ v_{min}}$ is the value $|v|$

Acceleration, a

This is the rate of change of velocity or $a = \frac{dv}{dt}$. The acceleration of a particle is minimum or maximum when $\frac{d}{dt}(a) = 0$

Example 1

(a) The distance, s meters of a particle from a fixed point is given by $s = t^2(t^2 + 6) - 4t(t - 1)(t + 1)$, where t is the time in seconds.

Find the velocity and acceleration of the particle when t 1s.

Solution

$$s = t^2(t^2 + 6) - 4t(t - 1)(t + 1)$$

$$= t^4 + 6t^2 - 4t(t^2 - 1)$$

$$= t^4 + 6t^2 - 4t^3 + 4t$$

Velocity = $\frac{ds}{tt} = 4t^3 + 12t - 12t^2 + 4$

Whet t = 1

$$v = 4 + 12 - 12 + 4 = 8ms^{-1}$$

Acceleration =$\frac{dv}{dt} = 12t^2 + 12 - 24t$

When t= 1

$$a = 12 + 12 - 24 = 0ms^{-2}$$

(b) A particle moves along a straight line OX so that its displacement x meters from the origin, O at time t second is given by

$$x = 4t^3 - 18t^2 + 24t$$

Find

(i) when and where the velocity of the particle is zero

$x = 4t^3 - 18t^2 + 24t$

$v = \frac{dx}{dt} = 12t^2 - 36t + 24$

For v = 0

$12t^2 - 36t + 24 = 0$

$t^2 - 3t + 2 = 0$

$(t-1)(t-2) = 0$

Either t = 1 or t= 2

$\therefore velocity = 0\ when$

$t = 1s\ or\ t = 2s$

When t = 1s

$x = 4(1)^3 - 18(1)^2 + 24(1)$

$x = 4 - 18 + 24 = 10m$

When t= 2

$x = 4(2)^3 - 18(12)^2 + 24(2)$

$x = 32 - 72 + 48 = 8m$

(ii) its acceleration at these instants

$a = \frac{dv}{dt} = \frac{d}{dt}(12t^2 - 36t + 24)$

$= 24t - 36$

When $t = 1s$,

$a = 24 - 36 = -12ms^2$

When $t = 2s$,

$a = 48 - 36 = 12ms^2$

(iii) its velocity when its acceleration is zero.

Acceleration is zero when $\frac{dv}{dt} = 0$

$24t - 36 = 0$

$t = \frac{36}{24} = \frac{3}{2}s$

Velocity v

$= 12\left(\frac{3}{2}\right)^2 - 36\left(\frac{3}{2}\right) = 24$

$= -3ms^{-1}$

i.e. the particle is moving in opposite direction.

(c) A particle of mass 5kg moves such that

$s = \begin{pmatrix} 2 - cos3t \\ 6sin2t \end{pmatrix}$

(i) Show that the particle never crosses the y-axis

For any point on the y-axis , x= 0

$2 - cos3t = 0$

cos3t = 2

$3t = cos^{-1}(2)$

$t = \frac{1}{3}cos^{-1}(2)$

Since $cos^{-1}(2)$ has no value, the particle does not cross y-axis

(ii) Find the velocity of the particle when $t = \frac{\pi}{6}$

$v = \frac{dx}{dt} = \frac{d}{dx}\begin{pmatrix} 2 - cos3t \\ 6sin2t \end{pmatrix}$

$= \begin{pmatrix} 3sin3t \\ 12cos2t \end{pmatrix}$

$At\ t = \frac{\pi}{6}$

$v = \begin{pmatrix} 3sin\frac{3\pi}{6} \\ 12cos\frac{2\pi}{6} \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} ms^{-1}$

(d) The acceleration of a car t s after starting from rest is $\frac{75-10t-t^2}{20}ms^{-2}$ until the instant when this expression vanishes. After this instant, the speed of this car remains constant. Find the maximum acceleration.

Solution

A is maximum when $\frac{d(a)}{dt} = 0$

$\frac{d}{dt}\left(\frac{75+10t-t^2}{20}\right) = \frac{10-2t}{20}$

A is maximum when $\frac{10-2t}{20} = 0$

t = 5s

$a_{max} = \frac{75+10(5)-(5)^2}{20} = \frac{100}{20} = 5ms^{-2}$

(e) The distance s m of a particle from a fixed point is given by $s = t^2(t^2+6))$ where t is the time. Find the velocity and acceleration of the particles when t = 1s

$Solution$

$s = t^2(t^2 + 6)$

$= t^4 + 6t^2$

$v = \frac{d(s)}{dt} = \frac{d}{dt}(t^4 + 6t^2)$

$= 4t^3 + 12t$

$at\ t = 1s$

$v = 4(1)^3 + 12(1) = 16ms^{-1}$

$a = \frac{d(v)}{dt} = \frac{d}{dt}(4t^3 + 12t\ )$

$= 12t^2 + 12$

$at\ t = 1s$

$a = 12(1)^2 + 12 = 24ms^{-2}$

Revision exercise 1

1. A ball is thrown vertically upwards and its height after t seconds is h m where
   $h = 25.2t - 4.9t^2$
   Find
   (a) its height and velocity after 3s
   (b) when it is momentarily at rest
   (c) the greatest height reached
   (d) the distance moved in the 3rd second
   (e) the acceleration when $t = 2\frac{4}{7}$
   $$\begin{bmatrix} (a)\quad 31.5m, -4.2ms^{-1}; \\ (b)\ t = 2\frac{4}{7}; (c)32.4m; (d)2.5m; \\ (e) - 9.8ms^2(constant) \end{bmatrix}$$
2. A particle moves along a straight line in such a way that its distance s m from the origin after t s is given by $s = 7t + 12t^2$.
   (a) What does it travel in the 9th second?
   (b) What are its velocity and acceleration at the end of 9th second?
   $[(a)211s; (b)223cms^{-1}\ (c)24ms^{-2}]$
3. A point moves along a straight line OX so that its distance x from the point O at t s is given by $s = t^3 - 6t^2 + 9t$.Find
   (a) at what times and in what position the point will have zero velocity.
   (b) its acceleration at those instants
   (c) its velocity when its acceleration is zero.
   $$\begin{bmatrix} (a)1s, 3s, 4cm, 0; (b) - 6, 6cms^{-2}; \\ (c) - 3cms^{-1} \end{bmatrix}$$
4. A particle moves in a straight line so that after t s it is 5m from a fixed point O on the line where$s\ = t^4 + 3t^2$. Find
   (a) The acceleration when t = 1, t = 2 and t = 3s.
   (b) The average acceleration between t = 1 and t =3s
   $[(a)18, 54, 114ms^{-1}; (b)58ms^{-2}]$
5. A particle moves a long a straight line so that after t s, its distance from a fixed point O on the line is 5m where $s = t^3 - 3t^2 + 2t$
   (a) When is the particle is at O?
   (b) What is the velocity and acceleration at these times?
   (c) What is the average acceleration between t = 0 and t = 2s.
   $$\begin{bmatrix} (a)after\ 0, 1, 2s; (b)2, -1, 2ms^{-1}; \\ -6, 0, 6ms^{-2}; (c)0ms^{-1}; (d)0ms^{-1} \end{bmatrix}$$

Rates of change/measurement

This deals with aspects that vary with others.

Example 2

(a) The side of a cube is increasing at the rate of 0.3ms^{-1}.
Find the rate of volume when the length is 5m.
Solution
Let L = length of each side of the cube.
$v = L^3$
$\frac{dv}{dL} = 3L^2$
$\frac{dL}{dt} = 0.3$
$\frac{dv}{dt} = \frac{dv}{dL}.\frac{dL}{dt} = 3L^2.0.3 = 0.9L^2$
When L= 5m
$$\frac{dv}{dt} = 0.9(5)^2 = 22.5m^3s^{-1}$$

(b) The volume of a cube is increasing at the rate of 2m^{3}s^{-1}. Find the rate of change of the side when its side is 10m.
$v = L^3$
$\frac{dv}{dL} = 3L^2$
$\frac{dv}{dt} = \frac{dv}{dL}.\frac{dL}{dt}$
But $\frac{dv}{dt} = 2$
$2 = 3L^2.\frac{dL}{dt}$
$\frac{dL}{dt} = \frac{2}{3L^2}$
When L= 10m
$\frac{dL}{dt} = \frac{2}{3(10)^2} = 0.007ms^{-1}$

(c) The volume of a cube increases uniformly at $a^3m^3s^{-1}$. Find an expression for the rate of increase of the surface area when the area of a face is b^2m^2.
Solution
Let L = side of the cube
A = surface area
V = volume of the cube
$v = L^3$
$\frac{dv}{dL} = 3L^2$

$\frac{dv}{dt} = \frac{dv}{dL}.\frac{dL}{dt}$

But $\frac{dv}{dt} = a^2$

$2 = 3L^2.\frac{dL}{dt}$

$\frac{dL}{dt} = \frac{a^3}{3L^2}$

For face area = $b^2 = L^2$ since L=b

Surface area of a cube = $6b^2$

$A = 6L^2$

$\frac{dA}{dL} = 12L$

$But\ \frac{dA}{dt} = \frac{dA}{dL}.\frac{dL}{dt}$

$= 12L.\frac{a^3}{3L^2} = \frac{4a^3}{L} = \frac{4a^3}{b}$

(d) A spherical balloon is inflated at a rate of $5m^3s^{-1}$. Find the rate of increase of radius when the radius is 3m.

Solution

If v and r are the volume and radius of the sphere at time t, then

$v = \frac{4}{3}\pi r^3$

$\frac{dv}{dr} = 4\pi r^2$

$\frac{dv}{dr} = 5$

But $\frac{dv}{dt} = \frac{dv}{dr}.\frac{dr}{dt}$

$5 = 4\pi r^2.\frac{dr}{dt}$

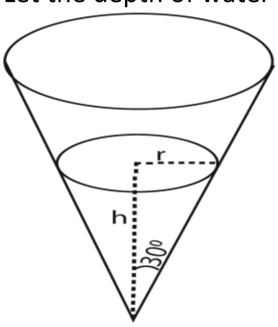
$\frac{dr}{dt} = \frac{5}{4\pi r^2}$

When r=3

$\frac{dr}{dt} = \frac{5}{4\pi(3)^2} = \frac{5}{36\pi} ms^{-1}$

(e) A hollow can of semi-vertical angle 30^0 is held with its vertex downwards. Water is poured into the cone at the rate of $3m^3s^{-1}$. Find the rate at which the depth of water in the cone is increasing when the depth is 5m.

Let the depth of water in the cone be h m



From the diagram above

$\tan 30 = \frac{r}{h}$

$\frac{1}{\sqrt{3}} = \frac{r}{h}$

$r = \frac{h}{\sqrt{3}}$

The volume v m^3 of water in the cone is given by $v = \frac{1}{3}\pi r^2 h$

Substituting for r

$v = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h = \frac{\pi h^3}{9}$

$\frac{dv}{dh} = \frac{\pi h^2}{3}$

But $\frac{dv}{dt} = \frac{dv}{dh}.\frac{dh}{dt}$

$3 = \frac{\pi h^2}{3}.\frac{dh}{dt}$

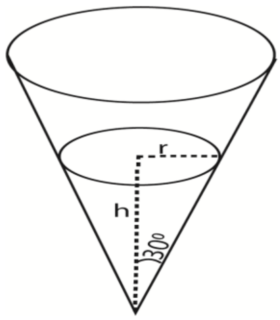
$\frac{dh}{dt} = \frac{9}{\pi h^2}$

When h = 5

$\frac{dh}{dt} = \frac{9}{\pi(5)^2} = \frac{9}{25\pi} ms^{-1}$

∴the rate of change of height is $\frac{9}{25\pi} ms^{-1}$

(f) An inverted cone with a vertical angle of 60^0 is collecting water leaking from a tap at a rate of $2m^3s^{-1}$. If the height of water collected in the cone is 10m, find the rate at which the surface are of eater is increasing.



$\tan 30 = \frac{r}{h}$

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$r = \frac{h}{\sqrt{3}}$

The volume v m^3 of water in the cone is given by $v = \frac{1}{3}\pi r^2 h$

Substituting for r

$v = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h = \frac{\pi h^3}{9}$

$\frac{dv}{dh} = \frac{\pi h^2}{3}$

Let A = surface area

$A = \pi r^2$

Substituting for r

$A = \pi\left(\frac{h}{\sqrt{3}}\right)^2 = \frac{\pi h^2}{3}$

$\frac{dA}{dh} = \frac{2}{3}\pi h$

But $\frac{dv}{dt} = \frac{dv}{dh}.\frac{dh}{dt}$

$2 = \frac{\pi h^2}{3}.\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{6}{\pi h^2}$

Also, $\frac{dA}{dt} = \frac{dA}{dh}.\frac{dh}{dt} = \frac{2}{3}\pi h . \frac{6}{\pi h^2} = \frac{4}{h}$

Substituting for h =10

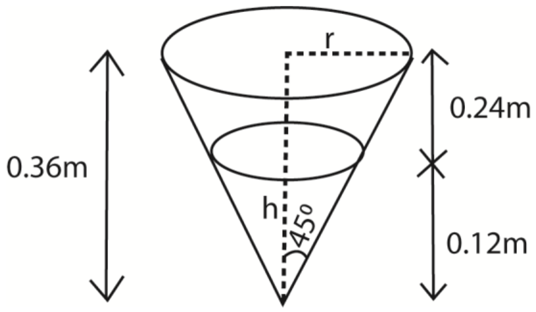
$\frac{dA}{dt} = \frac{4}{10} = 0.4m^2s^{-1}$

∴the rate at which the surface area is changing is $0.4m^2s^{-1}$.

(g) A hollow circular cone with vertical angle 90^0 and height 0.36m is inverted and filled with water. This water begins to leak away through a small hole in the vertex. If the level of the water begins to sink at a rate of 0.01m in 120s, and the water continues to leak a way at the same rate, at what rate is the level sinking when the water is 0.24m from the top?

Solution

When water is full



$\tan 45^0 = \frac{r}{h}$

$1 = \frac{r}{h}$

$r = h$

The volume v m^3 of water in the cone is given by $v = \frac{1}{3}\pi r^2 h$

Substituting for r

$v = \frac{1}{3}\pi(1)^2 h = \frac{\pi h^3}{3}$

$\frac{dv}{dh} = \pi h^2$

$\frac{dh}{dt} = \frac{0.01}{120}$

But $\frac{dv}{dt} = \frac{dv}{dh}.\frac{dh}{dt}$

$= \pi h^2.\frac{0.01}{120}$

Substituting for h

$\frac{dv}{dt} = \pi(0.36)^2.\frac{0.01}{120}$……………….(i)

When water level is 0.36 - 0.24 = 0.12m

$\frac{dv}{dt} = \pi(0.12)^2.\frac{dh}{dt}$……………….(ii)

$Equating\ (i) and\ (ii)$

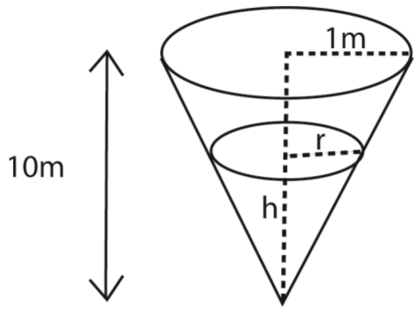
$\pi(0.36)^2.\frac{0.01}{120} = \pi(0.12)^2.\frac{dh}{dt}$

$$\frac{dh}{dt} = \frac{(0.36)^2}{(0.12)^2}.\frac{0.01}{120} = 7.5x10^{-4}ms^{-1}$$

(h) A hollow right circular cone of height 10m and base radius 1m is catching the drips from a tap leaking at a rate $0.002m^3s^{-1}$. Find the rate at which the surface area of water is increasing when water is half way up the cone

Solution

Let h and r be the height and radius of water level at time t



Expressing r in term of h, from similarity of figures,

$\frac{h}{10} = \frac{r}{1}$

$r = \frac{h}{10}$

Volume, v of a cone $= \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi\left(\frac{h}{10}\right)^2 h$$

$$= \frac{\pi h^3}{300}$$

$\frac{dv}{dh} = \frac{\pi h^2}{100}$

But $\frac{dv}{dt} = \frac{dv}{dh}.\frac{dh}{dt}$

$0.002 = \frac{\pi h^2}{100}.\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{0.2}{\pi h^2}$

Surface area, A = πr^2

Substituting for r

$A = \pi\left(\frac{h}{10}\right)^2 = \frac{\pi h^2}{100}$

$\frac{dA}{dh} = \frac{\pi h}{50}$

Now $\frac{dA}{dt} = \frac{dA}{dh}.\frac{dh}{dt} = \frac{\pi h}{50}.\frac{0.2}{\pi h^2} = \frac{0.004}{h}$

When water is half way up, h=5m

$\frac{dA}{dt} = \frac{0.004}{(5)} = 0.0008m^2s^{-1}$

Revision exercise 2

1. The side of a square is increasing at the rate of 5cms^{-1}. Find the rate of increase of the area when the length of the side is 10cm. [100cm^{2}s^{-1}]
2. The volume of a cube is increasing at the rate of 18cm^{3}s^{-1}. Find the rate of increase of the length of a side when the volume is 125cm^3. $\left[\frac{6}{25}cms^{-1}\right]$
3. The radius of a circle is increasing at the rate of $\frac{1}{3}cms^{-1}$. Find the rate of increase of the area when the radius is 5cm.$\left[\frac{10\pi}{3}cm^2s^{-1}\right]$
4. The volume of a sphere is increasing at a rate of (12π)cm^{3}s^{-1}. Find the rate of increase of the radius when the radius is 6cm. $\left[\frac{1}{12}cms^{-1}\right]$
5. The area of a square is increasing at the rate of 7cm^{2}s^{-1}. Find the rate of increase of the length of a side when this area is 100cm^2.$\left[\frac{7}{10}cms^{-1}\right]$
6. The area of a circle is increasing at the rate of (4π)cm^{2}s^{-1}. Find the rate of increase of the radius when this radius is $\frac{1}{2}$cm. [4cms^{-1}]
7. The surface area of a sphere is increasing at a rate of 2cm^{2}s^{-1}. Find the rate of increase of the radius when the surface area is (100π) cm^2? $\left[\frac{1}{20\pi}cms^{-1}\right]$
8. A boy is inflating a spherical balloon at the rat of 10cm^{3}s^{-1}. Find the rate of increase of the surface area of the balloon when the radius is 5m.[4cm^{2}s^{-1}]
9. A hollow cone of semi-vertical angle 45^0 is held with its vertex pointing downwards. It receives water at a rate of 3cm^3 per minute. Find the rate at which the depth of water in the cone is increasing when the depth is 2cm. $\left[\frac{3}{4\pi}cmmin^{-1}\right]$

Small changes

Suppose a function y = f(x) and δy and δx are increments in y and x respectively

Then as δx → 0, $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$

⇨ $\delta y = \frac{dy}{dx}.\delta x$

The above expression is used to find small changes in the variable x.

Example 3

(a) If $y = x^5$, find the approximate percentage increase in y due to increase of 0.1 percent in x.

$y = x^5$

$\frac{dy}{dx} = 5x^4$

But $\delta y = \frac{dy}{dx}.\delta x = 5x^4.\delta x$

$\frac{\delta y}{y} = \frac{5x^4.\delta x}{x^5} = 5\frac{\delta x}{x}$

But $\frac{\delta x}{x} = 0.1\%$

$\frac{\delta y}{y} = 5x0.1\%$ =0.5%

(b) An error of 2 ½ % is made in the measurement of the area of a circle. What percentage error results in

(i) The radius

A= πr^2

$\frac{dA}{dr} = 2\pi r$

$\delta A = \frac{dA}{dr}.\delta r$

$= 2\pi r\ .\delta r$

$\frac{\delta A}{A} = \frac{2\pi r}{\pi r^2}.\delta r = 2\frac{\delta r}{r}$

$\frac{1}{2}.\frac{\delta A}{A} = \frac{\delta r}{r}$

$\frac{\delta r}{r} = \frac{1}{2}.\frac{5}{2} = \frac{5}{4} = 1\frac{1}{4}\%$

(ii) The circumference

$c = 2\pi r$

$\frac{dc}{dr} = 2\pi$

$\delta c = \frac{dc}{dr}.\delta r = 2\pi\delta r$

$\frac{\delta c}{c} = \frac{2\pi\delta r}{2\pi r} = \frac{\delta r}{r}$

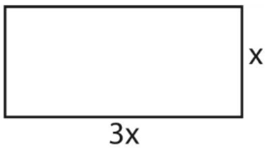
$\frac{\delta c}{c} = 1\frac{1}{4}\%$

(c) One side of a rectangle is three times the other. If the perimeter increases by 2%. What is the percentage increase in area?

Solution

Let the width of the rectangle = x

The length of the rectangle = 3x



P = 2(3x +x) = 8x

$\frac{dP}{dx} = 8$

$\delta P = \frac{dP}{dx}.\delta x = 8\delta x$

$\frac{\delta P}{P} = \frac{8\delta x}{8x} = \frac{\delta x}{x} = 2\%$

$A = 3x^2$

$\frac{dA}{dx} = 6x$

$\delta A = \frac{dA}{dx}\delta x = 6x\delta x$

$\frac{\delta A}{A} = \frac{6x\delta x}{3x^2} = \frac{2\delta x}{x}$

$\frac{\delta A}{A} = 2x2\% = 4\%$

(d) Find an approximate for $\sqrt{25.01}$

Let y = $\sqrt{x} = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$\delta y = \frac{dy}{dx}\delta x = \frac{1}{2\sqrt{x}}\delta x$

Taking x= 25 and δx=0.01

$\delta y = \frac{1}{2\sqrt{25}}x0.01 = \frac{1}{10}x0.01 = 0.001$

$Now\ (x + \delta x)^{\frac{1}{2}} = y + \delta y$

$\sqrt{25.01} = 5 + 0.001 = 5.001$

(e) Find an approximate of $\sqrt{101}$

Solution

Let y = $\sqrt{x} = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$\delta y = \frac{dy}{dx}\delta x = \frac{1}{2\sqrt{x}}\delta x$

Taking x= 100 and δx=1

$\delta y = \frac{1}{2\sqrt{100}}x0.01 = \frac{1}{20}x1 = 0.05$

$Now\ (x + \delta x)^{\frac{1}{2}} = y + \delta y$

$\sqrt{101} = 10 + 0.05 = 10.05$

(f) Find $\sqrt[3]{30}$

Let y = $\sqrt[3]{x} = x^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{1}{3(\sqrt[3]{x})^2}$

$\delta y = \frac{dy}{dx}\delta x = \frac{1}{3(\sqrt[3]{x})^2}\delta x$

Taking x= 27 and δx= 3

$\delta y = \frac{1}{3\left(\sqrt[3]{27}\right)^2}x3 = \frac{1}{9} = 0.11$

$Now\ (x + \delta x)^{\frac{1}{2}} = y + \delta y$

$\sqrt[3]{30} = 2 + 0.11 = 3.11$

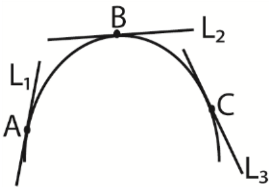
Revision exercise 3

1. If the side of a square can be measured accurately to 0.1cm, what is the possible error in the area of the square whose side measured to be 200cm?[40cm^2]
2. Find the approximate percentage change on the square of a quantity when the quantity itself changes by 0.1 percent. Hence calculate an approximate value of $(10.01)^2$. [0.2%, 100.2]
3. An error of 3% is made in measuring the radius of the sphere. Find the percentage error in volume. [9%]
4. The radius of a closed cylinder is equal to its height. Find the percentage increase in total surface area corresponding to unit percentage increase in height. [2%]
5. The volume of a sphere increases by 2%. Find the corresponding percentage increase surface area. $\left[1\frac{1}{3}\%\right]$
6. If $y = x + \frac{1}{x}$. Find the approximate increase in y when x increases from 2 to 2.4. [0.03]
7. Find the percentage increase in the volume of a cube when all the edges of the cube are increased in length of 2%. [6%]
8. The time period, T, of a pendulum of length L is given by $T = 2\pi\sqrt{\frac{L}{g}}$; where π and g are constants. Find the approximate percentage increase in T when the length of the pendulum increases by 4% [2%]

9. Find an approximate value for $\sqrt[3]{64.96}$. [4.02]
10. Find an approximate value for $(5.02)^3$. [126.5]

Tangents and Normals to curves

A tangent is a straight line drawn that touches a curve at only one point.



Lines 1, 2 and 3 above touch y= f(x) at point A, B and C respectively, hence they are tangents to the curve.

Gradient of a curve

A curve has varying gradients. A gradient at a point on a curve is obtained by finding a gradient of a tangent to the curve.

For a curve y = f(x), the gradient of a curve at any particular point is given by

$\frac{dy}{dx} = f'(x)$

Example 4

(a) Find the gradient of a curve $f(x) = x^2 + \frac{1}{x}$ at the point (1, 2)

Solution

$f(x) = x^2 + \frac{1}{x} = x^2 + x^{-1}$

$f'(x) = 2x - x^{-2} = 2x - \frac{1}{x^2}$

At point (1, 2), x = 1

$f'^{(x)} = 2(1) - \frac{1}{(1)^2} = 2 - 1 = 1$

∴the gradient of the curve at point (1,) is 1

(b) Find the coordinates of points for the curve y = x^3 whose gradient is 12

y = x^3

$\frac{dy}{dx} = 3x^2$

⇨ $12 = 3x^2$

$x^2 = 4$

$x = \pm 2$

When x = 2, y = 2^3 = 8

Hence the point is (2,8)

When x = -2, y = -2^3 = -8

Hence the point is (2,-8)

(c) The curve is defined by y = ax^2 + b where a and b are constants. Given that the gradient of the curve at the point (2, -2) is 3. Find the values of a and b.

Solution

y = ax^2 + b

$\frac{dy}{dx} = 2ax$

⇨ $2ax = 3$

At point (2, -2), x = 2 and y = -2

⇨ 4a = 3

a = $\frac{3}{4}$

Also $2x\frac{3}{4}x(2)^2 + b = -2$

6 + b =-2

b = -8

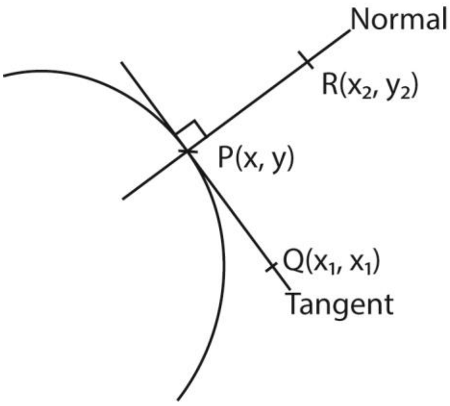
Hence a = $\frac{3}{4}$ and b = -8

Revision exercise 4

1. Find the gradient at the given point of the following curves.
   (a) $y = 2x^3 + 4$ at(3, 58) [54]
   (b) $y = \frac{x+5}{x}$ at (-1, -4) [-5]
   (c) $y = 6\sqrt{x} + \frac{1}{2\sqrt{x}}$ at $\left(\frac{1}{9}, \frac{7}{2}\right)$ $\left[\frac{9}{4}\right]$
   (d) $y = \frac{4-x^3}{x^2}$ at (-2, 3) [0]
2. Find the coordinates of the points on each of the following curves where the gradient is as stated
   (a) $y = 3x^2$, gradient -6 (-1, 3)
   (b) $y = x^3 - x^2 + 3$, grad 0 $\left(\frac{2}{3}, \frac{77}{27}\right)$
   (c) $y = \frac{x^2+3}{2x^2}$, grad 3 (-1, 2)
   (d) $y = 4\sqrt{x} - x$, grad5 $\left(\frac{1}{9}, \frac{11}{9}\right)$
3. A curve is given by $y = ax^2 + b\sqrt{x}$, where a and b are constants. If the gradient of the curve at (1, 1) is 5; find a and b.
   [a= 3 and b =-2]
4. Given that the curve $y = px^2 + qx$ has a gradient 7 at the point (6,8), find the values of the constants A and B. $\left[p = \frac{17}{18}; q = \frac{-13}{3}\right]$

5. A curve $y = \frac{p}{x} + q$ passes through the point (3, 9) with gradient 5. Find the value of constant p and q. [p= -45; q=24]

Equation of a tangent and the normal to the curve



Equation of the tangent

The gradient of the curve at any point P(x, y) is $\frac{dy}{dx} = m.$

If Q (x_1, y_1) is another point on the tangent then;

Gradient of the tangent $\overline{PQ} = m$

$\frac{y_1 - y}{x_1 - x} = m$

By cross multiplication, the equation of the tangent is obtained

Equation of the normal

The normal to a curve at any point say P is a straight line through P which is perpendicular to the tangent at P

When the gradient of the tangent to the curve is m, then the gradient of the normal is $\frac{-1}{m}$.

If R (x_2, y_2) is another point on the normal then

The gradient of the normal $\overline{PR}$ = $\frac{-1}{m}$.

Then$\frac{y_2 - y}{x_2 - x} = \frac{-1}{m}$

By cross multiplication, the equation of the normal is obtained

Example 5

(a) Find the equations of the equation of the tangent and the normal to the curve $y = x^3$ at P(2,8)

Solution

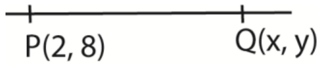
$y = x^3$

$\frac{dy}{dx} = 3x^2$

At x = 2

The gradient of the tangent = $3(2)^2$ = 12

Let Q (x, y) lie on the tangent



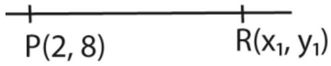
Grad $\overline{PQ} = 12$

$\frac{y-8}{x-2} = 12$

$y - 8 = 12(x - 2)$

$y = 12x{-}16$

Let R(x1, y1) lie on the normal



Grad $\overline{PR} = \frac{-1}{12}$

$\frac{y-8}{x-2} = \frac{-1}{12}$

$12(y - 8) = -(x - 2)$

$y = \frac{2-x}{12} + 8$

$y = \frac{98-x}{12}$

(b) The equation of the curve at the point$P\left(-2, \frac{1}{4}\right)$is given by $f(x) = \frac{1}{x^2}$. Find

(i) equation of the tangent

let y = f(x)

$y = \frac{1}{x^2} = x^{-2}$

$\frac{dy}{dx} = -2x^{-3} = \frac{-2}{x^3}$

At x= -2, gradient =$\frac{-2}{(-2)^2} = \frac{1}{4}$

Let Q(x, y) lie on the tangent, then

$\frac{y - \frac{1}{4}}{x-(-2)} = \frac{1}{4}$

$4\left(y - \frac{1}{4}\right) = x + 2$

4y – 1 = x+2

$y = \frac{x+3}{4}$

(ii) the equation of the normal at P

Gradient of the normal = $-1 \div \frac{1}{4} = -4$

Let R(x, y) lie on the normal, then

$\frac{y-\frac{1}{4}}{x-(-2)} = -4$

$y - \frac{1}{4} = -4(x+2)$

$4y - 1 = -16(x+2)$

$y = \frac{-16(x+2)+1}{4} = \frac{-16x-15}{4}$

(c) Find the equations of the normal to the curve $y = x^3 - 3x^2 + 4$ which are perpendicular to the line y – 24x = 1

Solution

For y – 24x = 1

y = 24x+1

$\frac{dy}{dx} = 24$

$For\ y = x^3 - 3x^2 + 4$

$\frac{dy}{dx} = 3x^2 - 6x$

Gradient of the normal =$\frac{-1}{3x^2-6x}$

Since the normal and the line and the line y = 24x + 1 are perpendicular, this means that the product of their gradient is -1

⇨ $\frac{-1}{3x^2-6x} . 24 = -1$

$3x^2 - 6x = 24$

$x^2 - 2x - 8 = 0$

(x – 4)(x + 2)= 0

Either x – 4 = 0 and x = 4

Or(x + 2) = 0 and x = -2

When x = 4

$y = (4)^3 - 3(4)^2 + 4 = 20$

(x, y) = (4, 20)

Gradient of the normal =$\frac{-1}{3(4)^2-6(4)} - \frac{-1}{24}$

Let (x, y) lie on the normal,

Then $\frac{y-20}{x-4} = -\frac{1}{24}$

Simplifying

$y = \frac{-x}{24} + \frac{121}{6}$

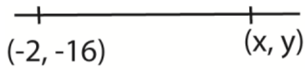
When x = -2

$y = (-2)^3 - 3(-2)^2 + 4 = -16$

(x, y) = (-2, -16)

Gradient of the normal =$\frac{-1}{3(-2)^2-6(-2)} - \frac{-1}{24}$ or since the two normal are parallel, they have the same gradient

Let (x, y) lie on the normal,



Then, $\frac{y-(-16)}{x-(-2)} = -\frac{1}{24}$

$\frac{y+16}{x+2)} = -\frac{1}{24}$

After simplifying

$y = \frac{-x}{24} + \frac{193}{12}$

(d) The tangent to the curve $y = 2x^2 + ax + b$ at the point (-2, 11) is perpendicular to the line 2y = x+ 7. Find a and b.

Solution

The point (-2, 11) satisfies the equation

$y = 2x^2 + ax + b$

$11 = 2(-2)^2 + a(-2) + b$

-2a + b = 3 ………………………………….(i)

For line 2y = x + 7

$y = \frac{x}{2} + \frac{7}{2}$

Gradient = $\frac{1}{2}$

For line $y = 2x^2 + ax + b$

$\frac{dy}{dx} = 4x + a$

Since the tangent to the curve and given line are perpendicular.

$\frac{1}{2}.(4x + a) = -1$

$4x + a = -2$

At x = -2

$4(-2) + a = -2$

$a = 6$

From (i)

-2(6) + a = 3
b = 15
Hence a = 6 and b = 15

(e) The curve is given by parametrically by $x = \frac{2}{t}$ and $y = 3t^2 - 1$, at the point (2, 2). Find

(i) Equation of the tangent

Solution

$x = \frac{2}{t} = 2t^{-1}$

$\frac{dx}{dt} = -2t^{-1} = \frac{-2}{t^2}$

$y = 3t^2 - 1,$

$\frac{dx}{dt} = 6t$

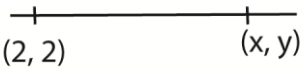
$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 6t \cdot \left(-\frac{t^2}{2}\right) = -3t^3$

At point (2, 2), x = 2

$2 = \frac{2}{t}$; t = 1

⇨ $\left\{\frac{dx}{dt}\middle| t = 1\right\} = -3(1)^3 = -3$

Let (x, y) lie on the tangent,



$\frac{y-2}{x-2} = -3$

$y = -3x + 8$

(ii) The equation of the normal

The gradient of the normal = $\frac{-1}{-3} = \frac{1}{3}$

Let (x, y) lie on the normal,

$\frac{y-2}{x-2} = \frac{1}{3}$

3(y – 2) = x – 2

3y = x + 4

$y = \frac{x+4}{3}$

(f) Find the equation of tangent and normal of hyperbola with parametric equations

x = asecθ and y = btanθ

Solution

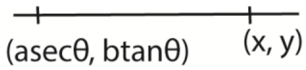
$\frac{dx}{d\theta} = asec\theta tan\theta$

$\frac{dy}{d\theta} = bsec^2\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{bsec^2\theta}{asec\theta tan\theta} = \frac{bsec\theta}{atan\theta}$

Gradient of the tangent =$\frac{bsec\theta}{atan\theta}$

Let (x, y) lie on the tangent,



$\frac{y-btan\theta}{x-asec\theta} = \frac{bsec\theta}{atan\theta}$

Simplifying

$\frac{x}{a}sec\theta - \frac{y}{b}tan\theta - 1 = 0$

Gradient of the normal =$-\frac{atan\theta}{bsec\theta}$

Let (x, y) lie on the normal,

$\frac{y-btan\theta}{x-asec\theta} = -\frac{atan\theta}{bsec\theta}$

Simplifying

$by + xasin\theta - (a^2 + b^2)tan\theta = 0$

(g) Find the equation of the tangent and normal to the rectangular hyperbola at a $P\left(ct, \frac{c}{t}\right)$

Solution

x = ct

$\frac{dx}{dt} = c$

$y = \frac{c}{t} = ct^{-1}$

$\frac{dx}{dt} = \frac{-c}{t^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{-c}{t^2} \cdot \frac{1}{c} = -\frac{1}{t^2}$

Gradient of the tangent =$-\frac{1}{t^2}$

Let (x, y) lie on the tangent,

$\frac{y-\frac{c}{t}}{x-ct} = -\frac{1}{t^2}$

Simplifying

$yt^2 + x = 2ct$

Gradient of the normal =t^2

Let (x, y) lie on the normal,

$\frac{y-\frac{c}{t}}{x-ct} = t^2$

Simplifying

$yt - t^3x = (1 - t^4)$c

Revision exercise 5

1. Find the equation of the normal to the curve $y = 3x^2 + 7x - 2$ at a point P where x= -1 $[y = x -]$

2. The normal to the curve $y = x^2 - 4x$ at point (3, -3) cuts x-axis at A and y-axis at B. find the equation of the normal and the coordinates of A and B. $\left[y = -\frac{1}{2}(x-3); A(-3,0); B\left(0, \frac{-3}{2}\right)\right]$
3. Find the equation of the tangent to the curve $y = x^2 - 3x + 1$ at a point where the cuts the y-axis [y = -3x+1]
4. Find the equation of the tangent and the normal to the curve x = 6t^2 and y = t^3 – 4t, at the point where t = -1. $\left[y = \frac{1}{12}x + \frac{5}{2}; y = -12x + 75\right]$
5. Find the equation of the normal to the curve $y = x^3 - 8$ at the point where the curve cuts the x-axis [12y +x =3]
6. The two tangents to the curve y = x2 at the point where y=9, intersect at point P, find the coordinates of P. [(0.-9)]
7. Find the coordinates of the point of intersection of two normal to the curve $y = x^2 + 3x + 5$ which make an angle of 45^0 with the x-axis. $\left[\frac{-3}{2}, \frac{7}{2}\right]$
8. (a) Find the equation of the normal at the point (2, 3) on the curve $y = 2x^3 - 12x^2 + 23x - 11$ [y=x+1]
   (b) Find also the coordinate of points where the normal meets the curve again
   [(1, 2),(3, 4)]
9. The tangent of the curve $y = ax^2 + 1$ at the point (1, b) has gradient 6. Find the values of a and b. [a = 3, b = 4]
10. Find the equation of the tangents to the curve $x = t^2 and\ y = 6t -$ at the points where x = 1 [y = 3x – 4; y + 3x + 10 = 0]
11. Find the equations of the tangents to the curve $x = \frac{4}{t} and\ y = t^2 - 3t + 2$ at the point where the curve crosses the x-axis. [y + x = 2, 4y – x + 4 =0]
12. A curve is given by $x = t^3, y = 4t$. The tangent at the point t =2meets the tangent at the point t =-1 at point Q. find the coordinates of Q. [(-2, 2)]

Turning/ stationary points

A point on a curve such that its gradient is zero, $\frac{dy}{dx} = 0$, is called a stationary point.

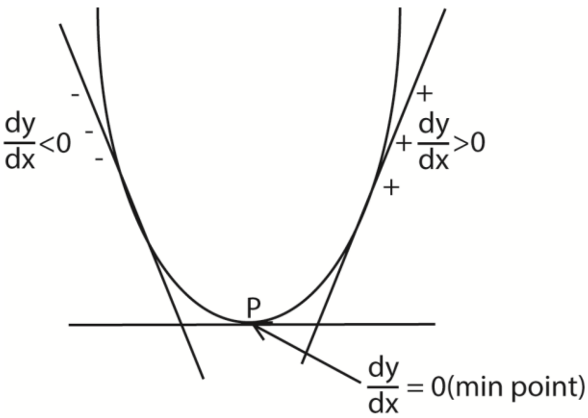
At this, the tangent to the curve is horizontal and the curve is 'flat'

There are three types of stationary points;

- Minimum point
- Maximum point
- Point of inflexion

Minimum point

This is obtained at the lowest point of the curve (valley like). In this, the gradient of the curve is negative to the left of the left of turning point and positive to the right



In summary we have

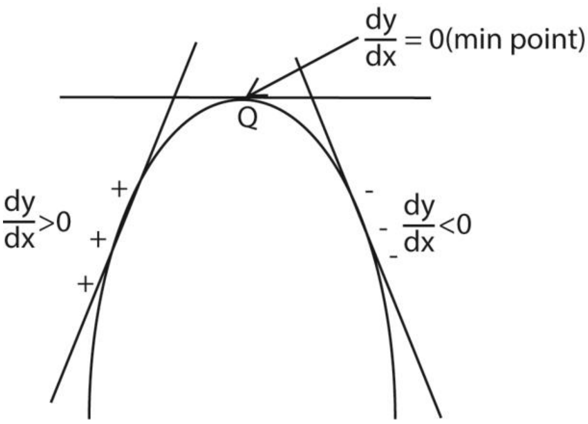
To the left of P	At point P	To the right of P
$\frac{dy}{dx} < 0(-)$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} > 0(+)$

When using second derivative to investigate where the turning point is minimum. In this case, considering point P, the gradient of the derived function is positive.

i.e. at point P, $\frac{d^2y}{dx^2} > 0$ (positive value)

Maximum point

This is obtained at the highest point of curve (mountain like). In this case, the gradient of the curve is positive to the left and negative to the right of the turning point



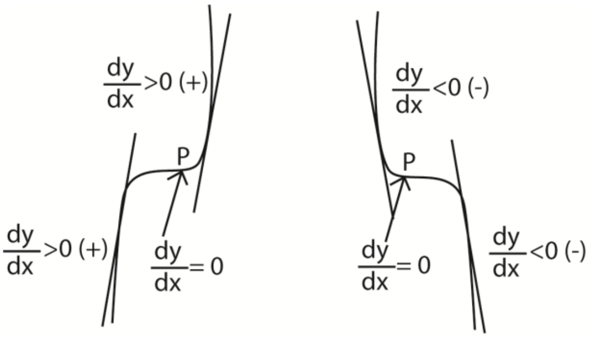
In summary, we have

To the left of Q	At point Q	To the right of Q
$\frac{dy}{dx} > 0(+)$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} < 0(-)$

When using second derivative to investigate where the turning point is minimum. In this case, considering point Q, the gradient of the derived function is negative.

Point of inflexion

In this case, the gradient has the same sign on each side of the stationary point



In summary

To the left of P	At point P	To the right of P
$\frac{dy}{dx} > 0(+)$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} > 0(+)$

Or

To the left of P	At point P	To the right of P
$\frac{dy}{dx} < 0(-)$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} < 0(-)$

When using second derivative to investigate whether the turning point is a point of inflexion. In this case, considering point P, the gradient of the derived function is zero. I.e. at point P, $\frac{d^2y}{dx^2} = 0$

However, the second derivatives can also be zero at a maximum or minimum point. For this reason, therefore, we must examine the sign of $\frac{dy}{dx}$ at each side of the point.

Example 6

(a) Find the coordinates of the stationary points on the curve $y = x^3 + 3x^2 + 1$ and determine their nature.

Solution

$y = x^3 + 3x^2 + 1$

$\frac{dy}{dx} = 3x^2 + 6x$

At stationary point $\frac{dy}{dx} = 0$

$\therefore 3x^2 + 6x = 0$

$3x(x + 2) = 0$

Either 3x = 0, x =0

Or (x + 2) = 0, x= -2

When x = 0

$y = (0)^3 + 3(0)^2 + 1$ =1

$(x, y) = (0,1)$

When x = -2

$y = (-2)^3 + 3(-2)^2 + 1$ =5

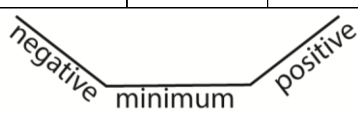
$(x, y) = (-2,5)$

Hence stationary points are (0, 1) and (-2, 5)

Determining the nature of stationary points.

For point (0, 1)

x	-1	0	1
$\frac{dy}{dx}$	-3	0	9



The stationary point (0, 1) is minimum

Alternatively

$\frac{dy}{dx} = 3x^2 + 6x$

$\frac{d^2y}{dx^2} = 6x + 6$

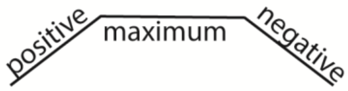
At point (0, 1)

$\frac{d^2y}{dx^2} = 6(0) + 6 = 6$

Hence the stationary point (0, 1) is minimum

For point (-2, 5)

x	-2.5	-2	1
$\frac{dy}{dx}$	3.75	0	-2.25



The stationary point (-2, 5) is maximum

Alternatively

$\frac{dy}{dx} = 3x^2 + 6x$

$\frac{d^2y}{dx^2} = 6x + 6$

At point (-2, 5)

$\frac{d^2y}{dx^2} = 6(-2) + 6 = -6$

Hence the stationary point (-2, 5) is maximum

(b) Find and distinguish between the nature of the turning points of the curves

(i) $y = x^3 - x^2 - 5x + 6$

Solution

$y = x^3 - x^2 - 5x + 6$

$\frac{dy}{dx} = 3x^2 - 2x - 5$

At stationary point $\frac{dy}{dx} = 0$

$\therefore 3x^2 - 2x - 5 = 0$

$(x + 1)(3x - 5) = 0$

Either $(x + 1)$ = 0, x = -1

Or $(3x - 5) =$ = 0, x= $\frac{5}{3}$

When x = -1

$y = (-1)^3 - (-1)^2 - 5(-1) + 6 = 9$

$(x, y) = (-1,9)$

When x = $\frac{5}{3}$

$y = \left(\frac{5}{3}\right)^3 - \left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right) + 6 = \frac{-13}{27}$

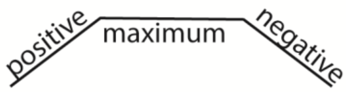
$(x, y) = (\frac{5}{3}, \frac{-13}{27})$

Hence stationary points are $(-1,9)$and $(\frac{5}{3}, \frac{-13}{27})$

Determining the nature of stationary points.

For point (-1, 9)

x	-2	-1	0
$\frac{dy}{dx}$	11	0	-1



The stationary point (-1, 9) is maximum

Alternatively

$\frac{dy}{dx} = 3x^2 - 2x - 5$

$\frac{d^2y}{dx^2} = 6x - 2$

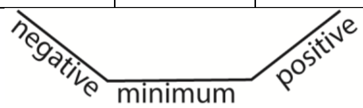
At point (-1, 9)

$\frac{d^2y}{dx^2} = 6(-1) - 2 = -8$ (< 0)

Hence the stationary point (-1, 9) is maximum

For point $(\frac{5}{3}, \frac{-13}{27})$

x	1	$\frac{5}{3}$	2
$\frac{dy}{dx}$	-4	0	3



The stationary point $(\frac{5}{3}, \frac{-13}{27})$is a minimum.

Alternatively

$\frac{dy}{dx} = 3x^2 - 2x - 5$

$\frac{d^2y}{dx^2} = 6x - 2$

At point $(\frac{5}{3}, \frac{-13}{27})$

$\frac{d^2y}{dx^2} = 6\left(\frac{5}{3}\right) - 2 = 8$ (> 0)

Hence the stationary point $(\frac{5}{3}, \frac{-13}{27})$ is maximum

(ii) $y = x^4 + 2x^3$

Solution

$y = x^4 + 2x^3$

$\frac{dy}{dx} = 4x^3 + 6x^2$

At stationary point $\frac{dy}{dx} = 0$

$\therefore 4x^3 + 6x^2 = 0$

$2x^2(2x + 3) = 0$

Either $2x^2$ = 0, x = 0

Or $(2x + 3) =$ = 0, x= $-\frac{3}{2}$

When x = 0

$y = (0)^4 + 2(0)^3 = 0$

$(x, y) = (0,0)$

When x = $-\frac{3}{2}$

$y = \left(-\frac{3}{2}\right)^4 + 2\left(-\frac{3}{2}\right)^3 = \frac{27}{16}$

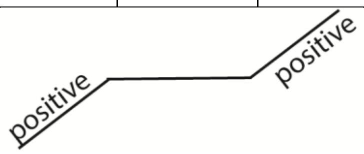
$(x, y) = \left(-\frac{3}{2}, \frac{27}{16}\right)$

Hence stationary points are (0,0)and $\left(-\frac{3}{2}, \frac{27}{16}\right)$

Determining the nature of stationary points.

For point (0, 0)

x	-1	0	1
$\frac{dy}{dx}$	2	0	10



The stationary point (0, 0) is inflexion

Alternatively

$\frac{dy}{dx} = 4x^3 + 6x^2$

$\frac{d^2y}{dx^2} = 12x^2 + 12x$

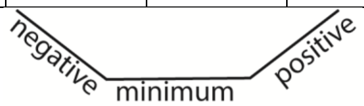
For (0, 0)

$\frac{d^2y}{dx^2} = 12(0)^2 + 12(0) = 0$

Hence the stationary point (0, 0) is inflexion

For point $\left(-\frac{3}{2}, \frac{27}{16}\right)$

x	-2	$-\frac{3}{2}$	-1
$\frac{dy}{dx}$	-8	0	2



The stationary point $\left(-\frac{3}{2}, \frac{27}{16}\right)$is minimum

Alternatively

$\frac{dy}{dx} = 4x^3 + 6x^2$

$\frac{d^2y}{dx^2} = 12x^2 + 12x$

At point $(\frac{5}{3}, \frac{-13}{27})$

$\frac{d^2y}{dx^2} = 12\left(\frac{5}{3}\right)^2 + 12\left(\frac{5}{3}\right)$ =9 (> 0)

Hence the stationary point $(\frac{5}{3}, \frac{-13}{27})$ is maximum

Revision exercise 6

1. Find the coordinates and the nature of the stationary points of the curves
   (a) $y = x^2 + \frac{16}{x}$ [(2, 12), minimum]
   (b) $x = 4 - t^3$ and $y = t^2 - 2t$ [(3, -1), minimum]
   (c) $y = \frac{2-x^3}{x^4}\left[\left(2, \frac{-3}{8}\right), mim\right]$
   (d) $y = \frac{2}{x^3} - \frac{1}{x^2}\left[\left(3, \frac{-1}{27}\right), mim\right]$
   (e) $y = \frac{1}{x} - \frac{3}{x^2}\left[\left(6, \frac{1}{12}\right), max\right]$
2. Find the maximum and minimum values of the function 2sint+ cos2t $\left[\left(\frac{\pi}{2}, 1\right), min; \left(\frac{\pi}{6}, \frac{3}{2}\right), max; \left(\frac{5\pi}{6}, \frac{3}{2}\right), min\right]$
3. If $p = 4x^2 - 10x + 7$, find the minimum of p and the corresponding value of x at which it occurs. $\left[\frac{3}{4}, \frac{5}{4}\right]$
4. If $v = 30x - 6x^2$, find the maximum of v and the corresponding value of x at which it occurs.$\left[37\frac{1}{2}, 2\frac{1}{2}\right]$

Application of maxima and minima to problem solving

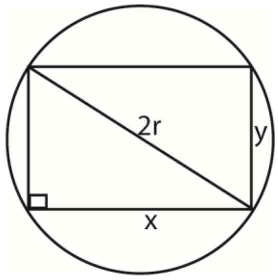
In all case of maximum or minimum values of functions, their derivatives are equal to zero.

Example 7

(a) Find the dimensions of a rectangle with maximum area that can be inscribed in a circle of radius r.

Solution

Let x and y be the dimensions of the rectangle



Area of a rectangle $A = xy$

From the figure, $x^2 + y^2 = (2r)^2$

$x^2 + y^2 = 4r^2$

$y = \sqrt{4r^2 - x^2}$

$\Rightarrow A = x\sqrt{4r^2 - x^2} = x(4r^2 - x^2)^{\frac{1}{2}}$

$$\frac{dA}{dx} = \left(\sqrt{4r^2 - x^2}\right)(1) + \frac{x(-2x)}{2\sqrt{4r^2 - x^2}}$$

$$= \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}$$

Area is maximum when$\frac{dA}{dx} = 0$

$\Rightarrow \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}} = 0$

$4r^2 - 2x^2 = 0$

$4r^2 = 2x^2$

$x = r\sqrt{2}$

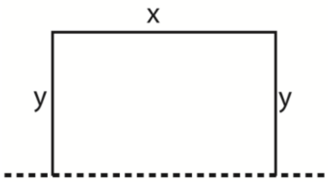
$y = \sqrt{2r^2 = r\sqrt{2}}$

Hence the figure is a square of side $r\sqrt{2}$

(b) A figure wishes to enclose a rectangular piece of land of area 1250cm2 whose one side is bound by a straight bank of river. Find the least possible length of barbed wire required to fence the other three sides of land

Solution

Let x and y be the dimensions of the rectangular land



Type equation here.

Area = xy

1250 = xy

y= $\frac{1250}{x}$

Perimeter, P(length of the wire) = 2y + x

$P = 2\left(\frac{1250}{x}\right) + x = \frac{2500}{x} + x$

$\frac{dP}{dx} = 1 - \frac{2500}{x^2}$

P is minimum when $\frac{dP}{dx} = 0$

$\Rightarrow 1 - \frac{2500}{x^2} = 0$

$x^2 = 2500$

$x = 50m$

$y = \frac{1250}{50} = 25m$

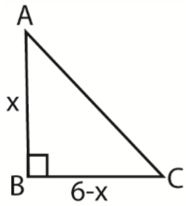
∴the minimum possible length

P = 25 x 2 + 50 = 100m

(c) In a right angled triangle ABC where <ABC = 90^0, the length AB and BC vary such that their sum is 6. Find the maximum area of the rectangle

Solution

Let AB = x , then BC = 6-x



$A = \frac{1}{2}(x)(6-x) = 3x - \frac{x^2}{2}$

$$\frac{dA}{dx} = 3 - x$$

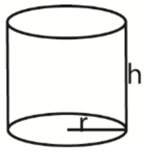
Area is maximum when $\frac{dA}{dx} = 0$

⇨ 3 – x =0

x = 3

Hence maximum area =$\frac{1}{2}(3)(6-3) = 4.5$

(d) A company that manufactures animal feed wishes to pack the feed in enclosed cylindrical tins. What should be the dimensions of each tin if each is to have a volume of 250πcm^3 and the minimum possible surface area?



Volume, $v = \pi r^2 h$

$\pi r^2 h = 250\pi$

$h = \frac{250}{r^2}$

Surface area A = 2 $\pi r^2 + 2\pi rh$

By substitution

A = 2 $\pi r^2 + 2\pi r\left(\frac{250}{r^2}\right) = 2\,\pi r^2 + \frac{500\pi}{r}$

$\frac{dA}{dr} = 4\pi r - \frac{500\pi}{r^2}$

Surface area is minimum when $\frac{dA}{dr} = 0$

$\Rightarrow 4\pi r - \frac{500\pi}{r^2} = 0$

$r^3 = 125$

$r = 5cm$

$h = \frac{250}{(5)^2} = 10cm$

minimum surface area

= 2 $\pi(5)^2 + 2\pi(5)(10)$

=150πcm^2

(e) An enclosed, right circular base radius rcm and height hm has a volume 54πcm^3, show that the total surface area $A = \frac{108\pi}{r} + 2\pi r^2$. Hence find the radius and height corresponding to the minimum surface area.

Volume, $v = \pi r^2 h$

$\pi r^2 h = 54\pi$

$h = \frac{54}{r^2}$

Surface area A = 2 $\pi r^2 + 2\pi rh$

By substitution

A = 2 $\pi r^2 + 2\pi r\left(\frac{54}{r^2}\right)$

$= \frac{108\pi}{r} + 2\pi r^2$(as required)

$\frac{dA}{dr} = 4\pi r - \frac{108\pi}{r^2}$

Surface area is minimum when$\frac{dA}{dr} = 0$

$\Rightarrow 4\pi r - \frac{108\pi}{r^2} = 0$

$r^3 = 27$

$r = 3cm$

$h = \frac{54}{(3)^2} = 6cm$

Hence the surface area is minimum when radius = 3cm and height = 6cm

(f) Write down an expression of the volume v and surface area s of a closed cylinder of radius r and height h. if the surface area is kept constant, show that the volume of the cylinder will be maximum when h = 2r.

Solution

$v = \pi r^2 h$

$s = 2\pi rh + 2\pi r^2$

From $s = 2\pi rh + 2\pi r^2$

$h = \frac{s-2\pi r^2}{2\pi r}$

Substituting h in $v = \pi r^2 h$

$v = \pi r^2 \left(\frac{s-2\pi r^2}{2\pi r}\right)$

$= r\left(\frac{s-2\pi r^2}{2}\right)$

$= \frac{sr}{2} - \pi r^3$

$\frac{dv}{dr} = \frac{s}{2} - 3\pi r^2$

v is maximum when$\frac{dv}{dr} = 0$

$\Rightarrow \frac{s}{2} - 3\pi r^2 = 0$

$\frac{s}{2} = 3\pi r^2$

$s = 6\pi r^2$

Substituting s in h

$h = \frac{6\pi r^2 - 2\pi r^2}{2\pi r} = \frac{4\pi r^2}{2\pi r} = 2r$

$\therefore h = 2r$

Revision exercise 7

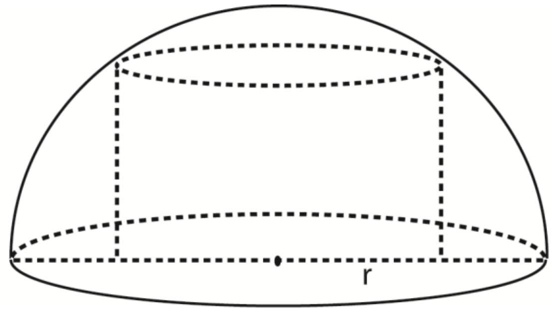
1. A rectangular enclosure is formed by using 1200m fencing. Find the greatest possible area that can be enclosed in this way and the corresponding dimensions of the rectangular enclosure.
[90,000m^2, 300m square]
2. An open tank with a square base is made from 12m^2 of metal sheet. Find the length of the side of the base for the volume of the tank to be maximum and find this volume.
[2m, 4m^2]
3. A cylindrical tin without a lid is made of sheet of metal. If s the area of the sheet used, without waste v, the volume of the tin and r the radius of the cross-section, prove that 2v = sr – πr^3. If s is given, prove that the volume of the tin is greatest when the ratio of the height to the diameter is 1:2.
4. A strip of wire of length 150cm is cut into two pieces. One piece is bent to form a square of x cm and the other piece is bend to form a rectangle which is twice long as wide.
Find the expression, interms of x, for the
(i) Width of the rectangle $\left[25 - \frac{2}{3}x\right]$
(ii) Length of rectangle$\left[50 - \frac{4}{3}x\right]$
(iii) Area of rectangle $\left[1250 - \frac{200}{3}x + \frac{8}{9}x^2\right]$

(iv) Given also that the sum of the two areas enclosed is a minimum, calculate the value of x. $\left[\frac{300}{17}\right]$

5. A closed cuboid plastic box is to be made with an external surface area of 216cm^2. The base is to be such that its length is four times its breadth. Find the length of the base of the box if the volume of the box is to be maximum and find this maximum volume. [12m, 172.8cm^3]
6. A cylindrical can, with no lid, has a circular base of radius r cm, the total surface area is 300π cm^2.
   (a) $Show\ that\ the\ volume\ v\ cm^3$ of the can is given by $v = \frac{\pi r}{2}(300 - r^2)$
   (b) Given that r may vary, find the positive value of r for which $\frac{dv}{dr} = 0$ [10]
   (c) Show that this value of r gives a maximum value of v. [v''(10)=-30π <0]
7. A cylindrical tank, open at the top and of height hm and radius rm, has a capacity of 1m^3. Show that $h - \frac{1}{\pi r^2}$
   If its total internal surface area is sm^2. Show that $s = \frac{2}{r} + \pi r^2$
   Determine the value of r which makes surface area s as small as possible$\left[\sqrt[3]{\frac{1}{11}}\right]$

Revision exercise 8 (topical)

1. The distance s m of a particle from a fixed $s = t^2(t^2 + 6) - 4t(t - 1)(t + 1)$ where t is time. Find the velocity and acceleration of the particle when t = 1s [8ms^{-1}, 0ms^{-2}]
2. Find the equation of the tangent to the curve $x^2 + y^2 - 2xy = 4$ at (1, -1)[y =-1]
3. A curve is defined by parametric equation $x = t^2 - t, y = 3t + 4$, find the equation of the tangent at (2, 10) [y =x+8]
4. A hemisphere bowl of radius a cm is initially full of water. The water runs out through a small hole at the bottom of the bowl at a rate$a^2\{36x(2a - x\}^{-1}cms^{-1}$. Find how long it will take for the depth of the water to be $\frac{1}{3}a\ cm$ and the rate at which the depth is decreasing at this instant.$\left[20.4s, \frac{a}{20}cms^{-1}\right]$
5. An inverted cone with vertical angle of 60^0 is collecting water leaking from a tap at a rate of 0.2cm^{3}s^{-1}. If the height of water in the cone is 10cm, find the rate the surface are of water is increasing [12cm^{2}s^{-1}]
6. A cylinder is inscribed in a semi-hemisphere of radius r as shown in the figure below



   Find the maximum volume of the cylinder in terms of r.$\left[\frac{2\pi r^3}{3\sqrt{3}}\right]$
7. Using calculus of small increments or otherwise find $\sqrt{98}$ correct to 1 decimal place. [9.9]
8. If $y = \sqrt{(5x^2 + 3)}$, show that
   $$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 5$$
9. A right circular cone of radius r has a minimum volume, the sum of vertical height h, and the circumference is 15cm. if the radius varies, show that the maximum volume of the cone is $\frac{625}{\pi}cm^3$.
10. The distance of the particle moving in a straight line from a fixed point after time t is given by $x = e^{-1}sint$. Show that the particle is instantaneously at rest at $t = \frac{\pi}{4}$. Find the acceleration at $t = \frac{\pi}{4}$s.[-0.6447]
11. A spherical balloon is inflated such that the rate at which its radius is increasing is 0.5cms-1. Find the rate at which
    (a) The volume is increasing at this instant [157.08cm^{3}s^{-1}]
    (b) The surface area is increasing when r= 8.5cm [106.814cm^{2}s-1]
12. A curve is represented by parametric equations $x = 3t\ and\ y = \frac{4}{t^2+1}$. Find the general equation for the tangent to the curve in terms of x, y and t. hence determine the equation of the tangent at the point (3,2).

$[3y(t^2+1)^2 + 8tx = 24t^2 + 12(t^2+1; 12]$

13. Given that $R = q\sqrt{(1000 - q^2)}$, find
    (a) $\frac{dR}{dq}$
    (b) The value of q when R is maximum$\left[\frac{1000-2q^2}{\sqrt{(1000-q^2)}}, (b)\sqrt{500}\right]$
14. The base radius of a right circular cone increase and the volume changes by 2%. If the height of the cone remains constant, find the percentage increase in the circumference of the base [1%]
15. Find the equation of the normal to the curve $x^2y + 3y^2 - 4x - 12 = 0$ at the point (0,2) [y= -3x +2]
16. A curve has the equation$y = \frac{2}{1+x^2}$. Determine the nature of the turning point on the curve.$[(0,2), \max]$
17. A cylinder has radius r and height 8r. The radius increases from 4cm to 44.1cm. Find the approximate increase in the volume. (use π= 3.14) (05marks) [120.576cm^3]
18. Given that x = $\frac{t^2}{1+t^3}$ and y =$\frac{t^3}{1+t^3}$, find$\frac{d^2y}{dx^2}$.
    $\left[\frac{6}{t}\left(\frac{1+t^3}{2-t^3}\right)^3\right]$
19. A container is in form of an inverted right angled circular cone. Its height is 100cm and base radius is 40cm. the container is full of water and has a small hole at its vertex. Water is flowing through the hole at a rate of 10cm^{3}s^{-1}. Find the rate at which the water level in the container is falling when the height of water in the container is halved. (05marks) [0.00796]
20. A curve whose equation is x^{2}y + y^2 -3x =3 passes through points A(1, 2)and B(-1, 0). The tangent at A and the normal at the curve at B intersect at point C. Determine;
    (a) equation of the tangent. (06marks)
    $\left[y = \frac{1}{5}x + \frac{11}{5}\right]$
    (b) coordinates of C. (06marks)$[\mathrm{C}(-19, 6)]$

    Determine the equation of the tangent to the curve y^3 + y^2 –x^4 = 1 at the point (1, 1) (05marks) [5y = 4x + 1]
21. Find the equation of the tangent to the curve y = $\frac{a^3}{x^2}$ at the point $(\frac{a}{t}, at^2)$. (05marks)
22. given that y = $In\left\{e^x\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}$, show that $\frac{dy}{dx} = \frac{x^2-1}{x^2-4}$ (05marks)
23. A rectangular sheet is 50cm long and 40cm wide. A square of x cm is cut off from each corner. The remaining sheet is folded to form an open box. Find the maximum volume of the box (08marks) [6564.22554cm^3]

Thank you

Dr. Bbosa Science