## Approximations

### The Maclaurin's Theorem

The polynomial of Maclaurin's series of any infinitely differentiable function, f(x)m whose value and all values of all its derivatives, exist at x = 0

$$f(x) = f(0)x + \frac{f\prime(0)}{2!}x^2 + \frac{f\prime\prime(0)}{3!}x^3 + \cdots$$

### Maclaurin's series of sinx

Let $f(x) = \ sinx \ => \ f(0) = \sin(0) = \ 0$

$f'(x) \ = cosx \ => \ f'(0) \ = \ cos(0) \ = \ 1$

$f''(x) = -sinx \ => \ f''(0) - sin(0) \ = 0$

$f'''(x) = -cosx => \ f'''(0) \ = -cos(0) \ = -1$

$f^{iv}(x) = sinx => \ f'''(0) = sin(0) = -1$

Note that the fourth derivative takes us back to the starting point. So these values repeat in a cycle of four as 0, 1, 0, -1; 0, 1, 0, -1; etc.

By substitution, we have

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The Maclaurin's series of sin x is valid for all values of x.

### Maclaurin series of cos x

$$cosx = \frac{d}{dx}(sinx)$$

$$= \frac{d}{dx}(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{3!} - \frac{x^6}{6!} + \cdots$$

The Maclaurin's series of cos x is valid for all values of x.

### Maclaurin's series of $e^x$

Let f(x) = $e^x$ => f(0) = $e^0$=1

f'(x) = $e^x$ => f'(0) = $e^0$=1

f''(x) = $e^x$ => f''(0) = $e^0$=1

f'''(x) = $e^x$ => f'''(0) = $e^0$=1 etc.

Here we see that the function and all its derivatives are the same, so these values repeat themselves indefinitely at 1, 1, 1,1, etc. by substitution we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots$$

The Maclaurin's series of $e^x$ is valid for all values of x

### Maclaurin series of Inx

Let f(x) = Inx => f(0) =In(0) = ?

f'(x) =$\frac{1}{x}$ => f'(0) = $\frac{1}{0}$=?

f''(x) = $-\frac{1}{x^2}$ => f''(0) = $\frac{1}{0^2}$=?

Here we notice that neither the function nor any of the derivatives exist as x= 0, so there is no polynomial Maclaurin's expansion of natural logarithm, Inx.

### Maclaurin series of In(1+x)

Let f(x) = In(1+x) => f(0) =In(1+0) = 0

f'(x) =$-\frac{1}{1+x}$ => f'(0) = $-\frac{1}{1+0}$= 1

f''(x) = $\frac{-1}{(1+x)^2}$ => f''(0) = $\frac{1}{(1+0)^2}$ =-1

f'''(x) = $\frac{2}{(1+x)^3}$ => f'''(0) = $\frac{2}{(1+0)^3}$ =2

$f^{iv}$(x) = $-\frac{3x2}{(1+x)^4}$ => $f^{iv}$ (0) = $\frac{-3x2}{(1+0)^4}$ =-3x2 etc by substitution we have

$$In(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

The Maclaurin's series of In(1+x) is valid for
-1< x ≤ 1

Summary

| f(x) | Expansion | Validity |
|---|---|---|
| $e^x$ | $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ | for all x |
| $e^{-x}$ | $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ | for all x |
| sinx | $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ | for all x |
| cosx | $1 - \frac{x^2}{3!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ | for all x |
| $tan^{-1}x$ | $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ | for -1< x ≤ 1 |
| In(1+x) | $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ | for -1< x ≤ 1 |
| $(1+x)^k$ | $1 + kx - \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$ | for -1< x ≤ 1 |
| $\frac{1}{1-x}$ | $1 + x + x^2 + x^3 + \ldots\ldots$ | for -1< x ≤ 1 |
| $\frac{1}{(1-x)^2}$ | $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$ | for -1< x ≤ 1 |

## Answering questions

The questions usually require to produce Maclaurin's series of a function to a specifies nth term and then its application.

Examples

1. Find the Maclaurin series for $(1+x)^{-1}$ as far as $x^4$.
   Deduce the Maclaurin series for
   (i) In(1+x)
   (ii) In(1-x)
   (iii) $In\left(\frac{1-x}{1+x}\right)$

   Solution

   Maclaurin expansion series is given by
   $f(x) = f(0) + f'^{(0)}x + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!}$+...

   Let f(x) = $(1+x)^{-1}$ => f(0)= $(1+0)^{-1}$ = 0

   f'(x) = $-1(1+x)^{-2}$ => f'(0) = $-1(1+0)^{-2}$=-1

   f''(x) = $2(1+x)^{-3}$ => f'(0) = $2(1+0)^{-3}$= 2

   f'''(x) = $-6(1+x)^{-4}$ => f'''(0) = $-6(1+0)^{-4}$= -6

   $f^{iv}$(x) = $14(1+x)^{-5}$ => $f^{iv}$(0) = $24(1+0)^{-5}$= 24

   By substitution we have

   $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4$

   (i) We know the $\int \frac{dx}{1+x}$=In(1+x)
   ⇨ In(1+x)=$\int(1 - x + x^2 - x^3 + x^4)dx$
   =$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$

   This valid for-1 < x ≤ 1

   (ii) Replacing x by –x in (i)

   In(1-x) =$-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5}$

   This valid for-1 < x ≤ 1

   (iii) Subtracting (ii) from (i)
   In(1+x) – In(1-x)= $2x + \frac{2x^3}{3} + \frac{2x^5}{5}$

2. Find the Maclaurin series for (1+x)-1 as far as $x^4$.
   Deduce the Maclaurin series for

Note the validity of **Maclaurin series** is arrived at by using ratio test theorem whose derivation is outside the scope of our coverage

(i) $\frac{1}{1+x^2}$ as far as $x^6$.

(ii) $tan^{-1}x$ as far as $x^7$.

Show that $tan^{-1}\left(\frac{1}{2}\right) + tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

Solution

Maclaurin expansion series is given by

$f(x) = f(0) + f'^{(0)}x + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!}+...$

Let $f(x) = (1+x)^{-1}$ => $f(0)= (1+0)^{-1} = 0$

$f'(x) = -1(1+x)^{-2}$ => $f'(0) = -1(1+0)^{-2}=-1$

$f''(x) = 2(1+x)^{-3}$ => $f'(0) = 2(1+0)^{-3}= 2$

$f'''(x) = -6(1+x)^{-4}$ => $f'''(0) = -6(1+0)^{-4}= -6$

$f^{iv}(x) = 14(1+x)^{-5}$ => $f^{iv}(0) = 24(1+0)^{-5}= 24$

By substitution we have

$(1+x)^{-1} = 1 – x + x^2 – x^3 + x^4$

(i) Replacing x by $x^2$ gives

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6$$

(ii) We know that $\int \frac{dx}{1+x^2} = tan^{-1}x$

⇨ $tan^{-1}x = \int(1 - x^2 + x^4 - x^6)\,dx$

$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$

We also know that

$tan^{-1}A + tan^{-1}B = \frac{A+B}{1-A.B}$

⇨ $tan^{-1}\left(\frac{1}{2}\right) + tan^{-1}\left(\frac{1}{3}\right) = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}$

$= tan^{-1}(1) = \frac{\pi}{4}$

3. Use Maclaurin theorem to expand $e^x$ up to the term $x^4$, use your expansion to evaluate e correct to 4 decimal places.

Let $f(x) = e^x$ => $f(0) = e^0=1$

$f'(x) = e^x$ => $f'(0) = e^0=1$

$f''(x) = e^x$ => $f''(0) = e^0=1$

$f'''(x) = e^x$ => $f'''(0) = e^0=1$ etc.

by substitution we have

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

Evaluating e

$e = e^1$, substituting for x = 1

$e^1 = 1 + (1) + \frac{(1)^2}{2!} + \frac{(1)^3}{3!} + \frac{(1)^4}{4!}$

$= 2 + \frac{1}{6} + \frac{1}{24} = \frac{65}{24} = 2.7083\ (4d.p)$

4. Expand $\sqrt{\left(\frac{1+2x}{1-x}\right)}$ up to the term $x^2$. Hence find the value of $\sqrt{\left(\frac{1.04}{0.98}\right)}$ to four significant figures. (12marks)

$\sqrt{\left(\frac{1+2x}{1-x}\right)} = (1+2x)^{\frac{1}{2}}(1-x)^{\frac{-1}{2}}$

Using $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots$

$\sqrt{\left(\frac{1+2x}{1-x}\right)} = \left(1 + x - \frac{1}{2}x^2\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right)$

$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + x + \frac{1}{2}x^2 - \frac{1}{2}x^2$

$= 1 + \frac{3}{2}x + \frac{3}{8}x^2$

$\therefore \sqrt{\left(\frac{1+2x}{1-x}\right)} \approx 1 + \frac{3}{2}x + \frac{3}{8}x^2$

Substituting for x = 0.02

$\sqrt{\left(\frac{1.04}{0.98}\right)} = \sqrt{\frac{1+2(0.02)}{1-0.02}}$

$= 1 + \frac{3}{2}(0.02) + \frac{3}{8}(0.02)^2$

$= 1.030$

5. Obtain the first two non-zero terms of Maclaurin's series for sec x

$f(x) = f(0) + f'^{(0)}x + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!}+...$

$f(x) = \sec x => f(0) = sec0 = 1$

f'(x) =secxtanx => f'(0)= sec0tan0 =0

$f''(x) = \sec x\sec^2 x + \tan x\sec x\tan x$

⇨ $f''(0)=\sec 0\sec^2 0+\tan 0\sec 0\tan 0= 1+0 = 1$

Hence the first two non-zero terms of Maclaurin series of $sec\ x = 1 + \frac{x^2}{2}$

**Revision exercise**

1. Use Maclaurin theorem to expand the following up to
   (i) $In\left(\frac{1+x}{1-x}\right)$ up to $x^3$. Hence, find the approximation of In2 correct to 3 significant figure

   $\left[In\left(\frac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3; 0.691\right]$

   (ii) $e^{-x}sinx$
   $\left[x - x^2 + \frac{1}{3}x^3\right]$

   (iii) $In\sqrt{\left(\frac{1+sinx}{1-sinx}\right)}$ $\left[2x + \frac{x^3}{6}\right]$

   (iv) $In(1 + sinx)$ $\left[x - \frac{x^2}{2} + \frac{x^3}{6}\right]$

   (v) $In(1 + x)^2$ $\left[2x - x^2 + \frac{2x^3}{3} - \frac{x^4}{2}\right]$

   (vi) $\frac{1}{\sqrt{(1+x)}}$ $\left[1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16}\right]$

2. Given $y = tan^{-1}\sqrt{1-x}$ show that
   (i) $(2 - x)\frac{dy}{dx} + \frac{1}{2\sqrt{(1-x)}} = 0$
   (ii) $(2 - x)\frac{d^2y}{dx^2} - \frac{dy}{dx} + \frac{1}{4}(1 - x)^{-\frac{1}{2}} = 0$

3. Use Maclaurin theorem to show that
   (i) $\frac{cosx}{1-x^2} = 1 + \frac{1}{2}x^2 + \frac{11}{24}x^4$
   (ii) $e^{-x}sinx = \frac{x}{3}(x^2 - 3x + 3)$. Hence evaluate$e^{-x}sin\frac{\pi}{3}$ to 4d.p [0.3334]

4. Given that $y = e^{tan^{-1}x}$, show that $(1 + x^2)\frac{d^2y}{dx^2} + (2x - 1)\frac{dy}{dx} = 0$. Hence or otherwise, determine the first four non-zero terms of the Maclaurin expansion of y
   $\left[1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right]$

5. Given that y = $In\left\{e^x\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}$, show that
   $\frac{dy}{dx} = \frac{x^2-1}{x^2-4}$

6. Use Maclaurin's theorem to express In(sinx+cosx) as a power series up to the term $x^2$.[$x - x^2$]

Thank you

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