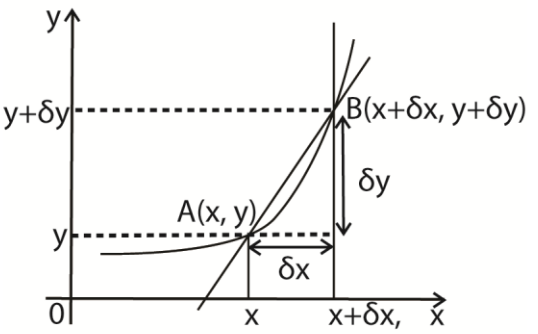
Differentiation

Consider point A(x, y) lying on a curve drawn below, if another point B(x + δx, y + δy) lies in the same curve, where δx and δy are small increments in x and y respectively, the straight line AB, drawn through the curve is called a chord of the curve.



As the distance δx becomes smaller and smaller, point B moves close to A and the chord AB approaches the position of the target at A

Now, Gradient, $M_{AB} = \frac{(y+\delta y-y}{x+\delta x-x}$

$$M_{AB} = \frac{\delta y}{\delta x}$$

As δx tends to zero, i.e.δx →0.

$\frac{\delta y}{\delta x}$ approaches the value of the gradient of the target line at A. This value is called limiting value of $\frac{\delta y}{\delta x}$ and is written as $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$.

The limiting value of $\frac{\delta y}{\delta x}$ is called a differential coefficient or first derivative of y with respect to x which is denoted by $\frac{dy}{dx}$.

Note: the process of finding this limiting value is called differentiation.

Direct differentiation of explicit functions

Explicit functions are functions where one variable is expressed in terms of the other variable. Examples $y = x^2$, $y = x^4 + 2x$ etc.

Given the function $x = x^n$, the derivative of y with respect to x, denoted by either y' or $\frac{dy}{dx}$ is given by $y' = \frac{dy}{dx} = nxn^{-1}$.

This result applies for all rational values of n. this means that multiply the term given by the give power index and then reduce the power by one.

Note: If

(i) y = f(x) + g(x) + h(x), then

$$\frac{dy}{dx} = \frac{d}{dx}\big(f(x)\big) + \frac{d}{dx}\big(g(x)\big) + \frac{d}{dx}\big(h(x)\big)$$

(ii) If y = a, this is written as $y = 0a^0$,

$$\frac{dy}{dx} = 0(ax^{-1}) = 0$$

Example 1

Find the derivatives of the following with respect to x

(a) $y = x^3$

solution

$$\frac{dy}{dx} = 3x^{3-2} = 3x^2$$

(b) $y = 2x^2 + 3$

Solution

$y = 2x^2 + 3x^0$

$$\frac{dy}{dx} = \frac{d}{dx}(2x^2) + \frac{d}{dx}(3x^0)$$
$$= 2(\,2x^{2-1}) + 0(3x^{0-1})$$
$$= 4x + 0 = 4x$$

(c) $y = \frac{1}{x}$

Solution

y = x^{-1}

$\frac{dy}{dx} = -1x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$

(d) y = $\sqrt{x}$

Solution

$y = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

(e) y = $\frac{-2}{x}$

Solution

y = $-2x^{-1}$

$\frac{dy}{dx} = -2(-1x^{-1-1}) = 2x^{-2} = \frac{2}{x^2}$

(f) y = $x^4 + 3x^2 + 2$

Solution

y = $x^4 + 3x^2 + 2x^0$

$\frac{dy}{dx} = 4x^{4-1} + 2(3x^{2-1}) + 0(2x^{0-1})$

= $4x^3 + 6x + 0$

=$4x^3 + 6x$

(g) y = $\frac{3}{\sqrt{x}} - 2\sqrt{x}$

Solution

$y = 3x^{-\frac{1}{2}} - 2x^{\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{1}{2}\left(3x^{-\frac{1}{2}-1} - \frac{1}{2}\left(2x^{\frac{1}{2}-1}\right)\right)$

$-\frac{3}{2}x^{\frac{-3}{2}} - x^{-\frac{1}{2}} = -\frac{3}{2x^{\frac{2}{3}}} + \frac{1}{x^{\frac{1}{2}}}$

(h) y = x4(x + 1)

solution

y= $x^5 + x^4$

$\frac{dy}{dx} = 5x^{5-1} + 4x^{4-1} = 5x^4 + 4x^3$

(i) y =$6\sqrt{x}(x^2 - 2x)$

Solution

$y = 6x^{\frac{5}{2}} - 12x^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{5}{2}\left(6x^{\frac{5}{2}-1}\right) - \frac{3}{2}\left(12x^{\frac{3}{2}-1}\right)$

$15x^{\frac{3}{2}} - 18x^{\frac{1}{2}}$

Revision exercise 1

Find the derivatives of the following with respect to x

(a) y = $3x^2$ [6x]

(b) y = $2x^4 + 2$ [$8x^3$]

(c) y = b [0]

(d) y = $\frac{9}{2x^3}$ $\left[-\frac{27}{2x^4}\right]$

(e) y = $2x^{-2}$ [$-4x^{-3}$]

(f) y = $\frac{-3}{4x^4}$ $\left[\frac{3}{x^5}\right]$

(g) y = $\sqrt[1]{x}$ $\left[\frac{1}{4x^{\frac{3}{4}}}\right]$

(h) y = $\frac{4}{5\sqrt{x}}$ $\left[\frac{2}{5x^{\frac{3}{2}}}\right]$

(i) y = $\frac{-6}{\sqrt[3]{x}}$ $\left[\frac{2}{x^{\frac{4}{3}}}\right]$

(j) $6\sqrt{x}(x^3 - 2x + 1)$

$\left[21x^{\frac{5}{2}} - 18x^{\frac{1}{2}} + \frac{3}{x^{\frac{1}{2}}}\right]$

Differentiation of functions from first principles

There are four basic steps followed when differentiating functions from first principles.

Given the function y = f(x), the steps are

(i) Add small changes in x and y to the function y = f(x) i.e. y + δy = f(x = δx)

(ii) Subtract y = f(x) from the established function in step one above i.e. δy = f(x+δx)- f(x)

(iii) Divide the function in step (ii) by δx i.e. $\frac{\delta y}{\delta x} = \frac{f(x+\delta x)-f(x)}{\delta x}$

(iv) Find the limit of the above quotient when δx→0. This is the derivative required

Differentiation of polynomial functions from first principles

These are functions in terms of y = ax^n where n is both rational and irrational numbers.

Example 2

Differentiated the following with respect to x from first principles

(a) y = x^2

Solution

y = x^2

y + δy = $(x + δx)^2$

δy = (x + δx) – x^2 …………………. (i)

Eqn. (i) is difference of two squares expression

δy = (x +δx + x)(x + δx – x)

$\delta y = (2x + \delta x)\delta x = 2x\delta x + (\delta x)^2$

$\frac{\delta y}{\delta x} = 2x + \delta x$

$\frac{\delta y}{\delta x} = \max_{\delta x \to 0} \frac{dy}{dx} = 2x$

$\therefore \frac{dy}{dx} = 2x$

(b) y = $\sqrt{x}$

Solution

y = $\sqrt{x}$

y + $\delta y = \sqrt{x + \delta x}$

$\delta y = \sqrt{x + \delta x} - y$

$\delta y = \sqrt{x + \delta x} - \sqrt{x}$

Dividing through by δx

$\frac{dy}{dx} = \frac{\sqrt{x+\delta x}-\sqrt{x}}{\delta x}$

Rationalizing the numerator on the RHS

$\frac{\delta y}{\delta x} = \frac{\sqrt{x+\delta x}-\sqrt{x}}{\delta x}\left(\frac{(\sqrt{x+\delta x}+\sqrt{x})}{(\sqrt{x+\delta x}+\sqrt{x})}\right)$

$\frac{\delta y}{\delta x} = \frac{x+\delta x - x}{\delta x(\sqrt{x+\delta x}+\sqrt{x})} = \frac{\delta x}{\delta x(\sqrt{x+\delta x}+\sqrt{x})}$

$\frac{\delta y}{\delta x} = \frac{1}{(\sqrt{x+\delta x}+\sqrt{x})}$

$\frac{\delta y}{\delta x} = \max_{\delta x \to 0} \frac{dy}{dx} = \frac{1}{(\sqrt{x}+\sqrt{x})} = \frac{1}{2\sqrt{x}}$

$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

(c) y = $\frac{1}{x^2}$

Solution

y = $\frac{1}{x^2}$

y + δy = $\frac{1}{(x + \delta x)^2}$

$\delta y = \frac{1}{(x + \delta x)^2} - y$

$\delta y = \frac{1}{(x + \delta x)^2} - \frac{1}{x^2}$

$\delta y = \frac{x^2 - (x + \delta x)^2}{x^2(x + \delta x)^2} = \frac{(x+x+dx)(x-x-\delta x}{x^2(x + \delta x)^2}$

$\delta y = \frac{(2x+\delta x)(-\delta x)}{x^2(x + \delta x)^2} = \frac{-2x\delta x - (\delta x)^2}{x^2(x + \delta x)^2}$

Dividing through by δx

$\frac{\delta y}{\delta x} = \frac{-2x - \delta x}{x^2(x + \delta x)^2}$

$\frac{\delta y}{\delta x} = \max_{\delta x \to 0} \frac{dy}{dx} = \frac{-2x}{x^4} = \frac{-2}{x^3}$

$\therefore \frac{\delta y}{\delta x} = \frac{-2}{x^3}$

(d) y = $2x^3$

Solution

y = $2x^3$

y + $\delta y = 2(x + \delta x)^3$

$\delta y = 2(x + \delta x)^3 - 2x^3$

$\delta y = 2x^3 + 6x^2\delta x + 6x(\delta x)^2 - 2x^3$

$\delta y = 6x^2\delta x + 6x(\delta x)^2$

$\frac{\delta y}{\delta x} = 6x^2 + 6x\delta x$

$\frac{\delta y}{\delta x} = \max_{\delta x \to 0} \frac{dy}{dx} = 6x^2$

$\therefore \frac{dy}{dx} = 6x^2$

(e) y = $\frac{x}{1+x^2}$

Solution

y = = $\frac{x}{1+x^2}$

y + δy = $\frac{x+\delta d}{1+(x+\delta x)^2}$

$\delta y = \frac{x+\delta d}{1+(x+\delta x)^2} - \frac{x}{1+x^2}$

$\delta y = \frac{(x+\delta d)(1+x^2) - x(1+(x+\delta x)^2)}{(1+x^2)(1+(x+\delta x)^2)}$

$\delta y = \frac{x + x^3 + \delta x + x^2\delta x - x - x^3 - 2x^2\delta x - x(\delta x)^2}{(1+x^2)(1+(x+\delta x)^2)}$

$\delta y = \frac{\delta x - x^2\delta x - x(\delta x)^2}{(1+x^2)(1+(x+\delta x)^2)}$

$\frac{\delta y}{\delta x} = \frac{1 - x^2 - x\delta x}{(1+x^2)(1+(x+\delta x)^2)}$

$\frac{\delta y}{\delta x} = \max_{\delta x \to 0} \frac{dy}{dx} = \frac{1-x^2}{(1+x^2)(1+x^2)} = \frac{1-x^2}{(1+x^2)^2}$

$\therefore \frac{dy}{dx} = \frac{1-x^2}{(1+x^2)^2}$

(f) y=x^n

Solution

y=x^n

$y + \delta y = (x + \delta x)^n$

$\delta y = (x + \delta x)^n - x^n$

Since n is assumed to be positive, we expand $(x+\delta x)^n$ using binomial expansion

$\delta y = x^n + \binom{n}{1} x^{n-1}\delta x + \binom{n}{2} x^{n-2}(\delta x)^2 + \cdots + -x^n$

$\delta y = nx^{n-1}\delta x + \binom{n}{2} x^{n-2}(\delta x)^2 + \cdots + (\delta x)^n$

$\frac{\delta y}{\delta x} = nx^{n-1} + \binom{n}{2} x^{n-2}\delta x + \cdots + (\delta x)^{n-1}$

$\frac{\delta y}{\delta x} = \max_{\delta x\to 0}\frac{dy}{dx} = nx^{n-1}$

$\therefore \frac{dy}{dx} = nx^{n-1}$

Revision exercise 2

Differentiated the following with respect to x from first principles

(a) $y = 3x^2$ [6x]
(b) $y = 2x^4 + 2$ [$8x^3$]
(c) y = b [0]
(d) $y = \frac{9}{2x^3}$ $\left[-\frac{27}{2x^4}\right]$
(e) $y = 2x^{-2}$ [$-4x^{-3}$]
(f) $y = \frac{-3}{4x^4}$ $\left[\frac{3}{x^5}\right]$
(g) $y = \sqrt[1]{x}$ $\left[\frac{1}{4x^{\frac{3}{4}}}\right]$
(h) $y = \frac{4}{5\sqrt{x}}$ $\left[\frac{2}{5x^{\frac{3}{2}}}\right]$
(i) $y = \frac{-6}{\sqrt[3]{x}}$ $\left[\frac{2}{x^{\frac{4}{3}}}\right]$
(j) $y = 6\sqrt{x}(x^3 - 2x + 1)$ $\left[21x^{\frac{5}{2}} - 18x^{\frac{1}{2}} + \frac{3}{x^{\frac{1}{2}}}\right]$
(k) $y = X^3 + x^2$ [$3x^2 + 2x$]
(l) $y = \frac{2}{\sqrt{(x+2)}}$ $\left[\frac{1}{\sqrt{(x+2)}}\right]$
(m) $y = 4x + 2x^2$ $[4 + 4x]$

Differentiation of trigonometric functions from first principles

These include trigonometric functions with single or multiple angles and those with higher or fractional powers.

Note the following formula as well

$cosA + cos B = 2cos\frac{A+B}{2}cos\frac{A-B}{2}$

$cosA - cos B = -2sin\frac{A+B}{2}sin\frac{A-B}{2}$

$sinA + sinB = 2sin\frac{A+B}{2}cos\frac{A-B}{2}$

$sinA - sinB = 2cos\frac{A+B}{2}sin\frac{A-B}{2}$

Example 3 single angle

Differentiate the following functions from first principle

(a) cosx

Solution

Let y = cos x

y + δy = cos(x + δx)

δy = cos(x + δx)- y

δy = cos(x + δx)- cosx

From $cosA - cos B = -2sin\frac{A+B}{2}sin\frac{A-B}{2}$

$\delta y = -sin\left(x + \frac{\delta x}{2}\right) sin\frac{1}{2}\delta x$

Dividing through by δx

$\frac{\delta y}{\delta x} = \frac{-2\sin\left(x+\frac{1}{2}\delta x\right)sin\frac{1}{2}\delta x}{\delta x}$

$\frac{\delta y}{\delta x} = -2\sin\left(x + \frac{1}{2}\delta x\right)\frac{sin\frac{1}{2}\delta x}{\delta x}$

$\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{dy}{dx} = -2sinx.\frac{\frac{1}{2}\delta x}{\delta x}$

$= \frac{-2sinx}{2}$

$= -sinx$

$\therefore \frac{d}{dx}cosx = -sinx$

(b) sinx

let y = sinx
$y + \delta y = \sin(x+\delta x)$
$\delta y = \sin(x+\delta x) - \sin x$

From $sinA - sinB = 2cos\frac{A+B}{2}sin\frac{A-B}{2}$

$\delta y = 2\cos\left(x + \frac{1}{2}\delta x\right) sin\frac{1}{2}\delta x$

Dividing through by δx

$\frac{\delta y}{\delta x} = \frac{2\cos\left(x+\frac{1}{2}\delta x\right)sin\frac{1}{2}\delta x}{\delta x}$

$\frac{\delta y}{\delta x} = 2\cos\left(x + \frac{1}{2}\delta x\right)\frac{\sin\frac{1}{2}\delta x}{\delta x}$

$\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{dy}{dx} = 2\cos x.\frac{\frac{1}{2}\delta x}{\delta x}$

$= \frac{2\cos x}{2} = cos\ x$

$\therefore \frac{d}{dx}sinx = cosx$

(c) tanx

$let\ y = tanx$

$y + \delta y = \tan(x + \delta x)$

$\delta y = \tan(x + \delta x) - \tan x$

$= \frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{sinx}{cosx}$

$= \frac{\sin(x+\delta x)cosx - \cos(x+\delta x)sinx}{\cos(x+\delta x)cosx}$

$= \frac{sin\delta x}{\cos(x+\delta x)cosx}$

Divide by δx

$\frac{\delta y}{\delta x} = \frac{sin\delta x}{\cos(x+\delta x)cosx}.\frac{1}{\delta x}$

$\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{dy}{dx} = \frac{s\delta x}{cosxcosx}.\frac{1}{\delta x}$

$= \frac{1}{cos^2 x} = sec^2 x$

$\therefore \frac{d}{dx}tanx = sec^2 x$

(d) sec x

Let y = sec x

$y = \frac{1}{cosx}$

$y + \delta y = \frac{1}{\cos(x+\delta x)}$

$\delta y = \frac{1}{\cos(x+\delta x)} - \frac{1}{cosx}$

$= \frac{cox - \cos(x+\delta x)}{\cos x \cos(x+\delta x)}$

$= \frac{-2\sin\left(x+\frac{1}{2}\delta x\right)\sin\frac{1}{2}\delta x}{\cos x\cos(x+\delta x)}$

Dividing by δx

$\frac{\delta y}{\delta x} = \frac{-2\sin\left(x+\frac{1}{2}\delta x\right)\sin\left(-\frac{1}{2}\delta x\right)}{\cos x\cos(x+\delta x)\delta x}$

$\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{dy}{dx} = \frac{+2sinx.\frac{1}{2}\delta x}{cosx.cosx.\delta x}$

$= \frac{sinx}{cosx}.\frac{1}{cosx} = tanxsecx$

$\therefore \frac{d}{dx}secx = tanxsecx$

(e) cot x

Let y = cot x

$y = \frac{\cos x}{\sin x}$

$y + \delta y = \frac{\cos(x+\delta x)}{\sin(x+\delta x)}$

$\delta y = \frac{\cos(x+\delta x)}{\sin(x+\delta x)} - \frac{\cos x}{\sin x}$

$= \frac{\text{sinxcos}(x+\delta x) - cosxsin(x+\delta x)}{\text{sinxsin}(x+\delta x)}$

$= \frac{sin\{x-(x+\delta x)\}}{\text{sinxsin}(x+\delta x)}$

$= \frac{sin(-\delta x)}{\text{sinxsin}(x+\delta x)} = \frac{-sin\delta x}{\text{sinxsin}(x+\delta x)}$

Dividing by δx

$\frac{\delta y}{\delta x} = \frac{-sin\delta x}{\text{sinxsin}(x+\delta x)}.\frac{1}{\delta x}$

$\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{dy}{dx} = \frac{-\delta x}{\text{sinxsin}\,x}.\frac{1}{\delta x} = \frac{-1}{sin^2 x}$

$= -cosec^2 x$

$\therefore \frac{d}{dx}cot = -cosec^2 x$

Example 4 double angle

Differentiate the following functions from first principle

(a) cos 2x

Let y = cos2x

$y + \delta x = \cos 2(x + \delta x)$

$\delta y = \cos 2(x + \delta x) - cos2x$

$= -2sin(2x + \delta x)sin\delta x$

Dividing through by δx

$\frac{\delta y}{\delta x} = -2sin(2x + \delta x)\frac{sin\delta x}{\delta x}$

$\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{dy}{dx} = -2sin2x.\frac{\delta x}{\delta x} = -2sin2x$

$\therefore \frac{d}{dx}cos2x = -2sin2x$

(b) sin2x

(c) Let y = sin2x

$y + \delta x = \sin 2(x + \delta x)$

$\delta y = \sin 2(x + \delta x) - sin2x$

$= 2cos2(x + \delta x)sin\delta x$

Divide by δx

$\frac{\delta y}{\delta x} = 2cos2(x + \delta x)\frac{sin\delta x}{\delta x}$

$\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{dy}{dx} = 2cos2x\frac{\delta x}{\delta x} = 2cos2x$

$\therefore \frac{d}{dx}sin2x = 2cos2x$

(d) $cos\frac{1}{2}x$

Let $y = cos\frac{1}{2}x$

$y + \delta y = cos\frac{1}{2}(x + \delta x)$

$\delta y = cos\frac{1}{2}(x + \delta x) - cos\frac{1}{2}x$

$= -2sin\left(\frac{x}{2} + \frac{\delta x}{4}\right) sin\left(\frac{\delta x}{4}\right)$

Dividing through by δx

$\frac{\delta y}{\delta x} = -2sin\left(\frac{x}{2} + \frac{\delta x}{4}\right)\frac{sin\left(\frac{\delta x}{4}\right)}{\delta x}$

$\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{dy}{dx} = -2sin\left(\frac{x}{2}\right)\frac{\frac{\delta x}{4}}{\delta x}$

$= -\frac{2}{4}sin\left(\frac{1}{2}x\right) = -\frac{1}{2}sin\left(\frac{1}{2}x\right)$

$\therefore \frac{d}{dx}cos\frac{1}{2}x = -\frac{1}{2}sin\left(\frac{1}{2}x\right)$

(e) tan2x

let y = tan2x

y + δy = tan2(x+ δx)

δy = tan2(x+ δx)-tan2x

$=\frac{sin2(x+\delta x)}{cos2(x+\delta x)} - \frac{sin2x}{cos2x}$

$= \frac{cos2xsin2(x+\delta x) - sinxcos2(x+\delta x)}{cos2xcos2(x+\delta x)}$

$= \frac{sin2\delta x}{cos2xcos2(x+\delta x)}$

Dividing through by δx

$\frac{\delta y}{\delta x} = \frac{sin2\delta x}{cos2xcos2(x+\delta x)}.\frac{1}{\delta x}$

$\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{dy}{dx} = \frac{2\delta x}{cos2xcos2x}.\frac{1}{\delta x}$

$= \frac{2}{cos^2 2x} = 2sec^2 2x$

$\therefore \frac{d}{dx}tan2x = 2cos^2 2x$

Example 5 higher and fractional power

Differentiate the following functions from first principle

(a) sin^2x

Let y = sin^2x

y + δy = $sin^2(x + \delta x)$

δy = $sin^2(x + \delta x) - sin^2x$

= $\{\sin(x + \delta x) + sinx\}\{\sin(x + \delta x) - sinx\}$

= $\left\{2\sin\left(x + \frac{\delta x}{2}\right) cos\frac{\delta x}{2}\right\}\left\{2\cos\left(x + \frac{\delta x}{2}\right) sin\frac{\delta x}{2}\right\}$

δy = $2\sin\left(x + \frac{\delta x}{2}\right) 2\cos\left(x + \frac{\delta x}{2}\right) cos\frac{\delta x}{2} sin\frac{\delta x}{2}$

=$4\sin\left(x + \frac{\delta x}{2}\right) \cos\left(x + \frac{\delta x}{2}\right) cos\frac{\delta x}{2} sin\frac{\delta x}{2}$

Dividing through by δx

$\frac{\delta y}{\delta x} = 4\sin\left(x + \frac{\delta x}{2}\right) \cos\left(x + \frac{\delta x}{2}\right) cos\frac{\delta x}{2}\frac{sin\frac{\delta x}{2}}{\delta x}$

$\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{dy}{dx} = 4sinxcosx.\frac{\frac{\delta x}{2}}{\delta x}$

$= \frac{4}{2}sinxcosx = 2sinxcosx$

$\therefore \frac{d}{dx}sin^2x = 2sinxcosx$

(b) cos^2x

Let y =cos^2x

y + δy = $cos^2(x + \delta x)$

δy = $cos^2(x + \delta x) = cos^2x$

= $\{\cos(x + \delta x) + \cos x\}\{\cos(x + \delta x) - cosx\}$

= $\left\{2cos\left(x + \frac{1}{2}\delta x\right)cos\frac{1}{2}\delta x\right\}\left\{-2sin\left(x + \frac{1}{2}\delta x\right)sin\frac{1}{2}\delta x\right\}$

= $-4cos\left(x + \frac{1}{2}\delta x\right)sin\left(x + \frac{1}{2}\delta x\right)sin\frac{1}{2}\delta xcos\frac{1}{2}\delta x$

Dividing through by δx

Dividing through by δx

$\frac{\delta y}{\delta x} = -4cos\left(x + \frac{1}{2}\delta x\right)sin\left(x + \frac{1}{2}\delta x\right)\frac{sin\frac{1}{2}\delta x}{\delta x}cos\frac{1}{2}\delta x$

$\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{dy}{dx} = -4cosxsinx\frac{\frac{1}{2}\delta x}{\delta x} = -\frac{4}{2}cosxsinx = -2cosxsinx$

$\therefore \frac{d}{dx}cos^2x = -2cosxsinx$

(c) cos^22x

Let y = cos^22x

y + δy = $cos^22(x + \delta x)$

δy = $cos^22(x + \delta x) - cos^22x$

= $\{cos2(x + \delta x) + cos2x\}\{cos2(x + \delta x) - cos2x\}$

=$\{2cos(2x + \delta x)cos\delta x\}\{-2sin(2x + \delta x)sin\delta x\}$

=$-4cos(2x + \delta x)sin(2x + \delta x)sin\delta xcos\delta x$

Dividing through by δx

$\frac{\delta y}{\delta x} = -4cos(2x + \delta x)sin(2x + \delta x)\frac{sin\delta x}{\delta x}cos\delta x$

$\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{dy}{dx} = -4cos2x\sin 2x.\frac{\delta x}{\delta x}$

$= -4cos2xsin2x$

$\therefore \frac{d}{dx}cos^22x = -4cos2xsin2x$

(d) $\sqrt{cosx}$

Let y = $\sqrt{cosx}$

y + δy=$\sqrt{cos(x + \delta x)}$

δy = $\sqrt{cos(x + \delta x)} - \sqrt{cosx}$

by rationalizing

δy = $\frac{\sqrt{cos(x+\delta x)} - \sqrt{cosx}}{1}.\frac{\sqrt{cos(x+\delta x)} + \sqrt{cosx}}{\sqrt{cos(x+\delta x)} + \sqrt{cosx}}$

$= \frac{\cos(x+\delta x)-cosx}{\sqrt{cos(x+\delta x)}+\sqrt{cosx}}$

$= \frac{-2\sin\left(x+\frac{1}{2}\delta x\right)sin\frac{1}{2}\delta x}{\sqrt{cos(x+\delta x)}+\sqrt{cosx}}$

Dividing through by δx

$\frac{\delta y}{\delta x} = \frac{-2\sin\left(x+\frac{1}{2}\delta x\right)sin\frac{1}{2}\delta x}{\left(\sqrt{cos(x+\delta x)}+\sqrt{cosx}\right)\delta x}$

$\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{dy}{dx} = \frac{-\sin x.\frac{1}{2}\delta x}{2\sqrt{cosx}.\delta x} = \frac{-sinx}{2\sqrt{cosx}}$

$\therefore \frac{d}{dx}\sqrt{cosx} = -\frac{sinx}{2\sqrt{cosx}}$

(e) $3x^2 + cos3x$

Let $y = 3x^2 + cos3x$

$y + \delta y = 3(x+\delta x)^2 + cos3(x+\delta x)$

$\delta y = 3(x+\delta x)^2 - 3x^2 + cos3(x+\delta x) - cos3x$

$\delta y = 3(x^2 + 2x\delta x + (\delta x)^2) - 3x^2 + \left\{-2sin\left(3x+\frac{3}{2}\delta x\right)sin\frac{3}{2}\delta x\right\}$

$\delta y = 6x\delta x + 3(\delta x)^2 + \left\{-2sin\left(3x+\frac{3}{2}\delta x\right)sin\frac{3}{2}\delta x\right\}$

Dividing through by δx

$\frac{\delta y}{\delta x} = 6x + 3\delta x + \left\{-2sin\left(3x+\frac{3}{2}\delta x\right)\frac{sin\frac{3}{2}\delta x}{\delta x}\right\}$

$\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{dy}{dx} = 6x - 2sin3x.\frac{3}{2}\delta x = 6x - 3sin3x$

$\therefore \frac{d}{dx}(3x^2 + cos3x) = 6x - 3sin3x$

(f) $2x^2 + sin2x$

let $y = 2x^2 + sin2x$

$y + \delta y = 2(x+\delta x)^2 + sin2(x+\delta x)$

$\delta y = 2(x+\delta x)^2 - 2x^2 + sin2(x+\delta x) - sin2x$

$\delta y = 2(x^2 + 2x\delta x + (\delta x)^2) - 2x^2 + \{2cos(2x+\delta x)sin\delta x\}$

$\delta y = 4x\delta x + 2(\delta x)^2 + \{2cos(2x+\delta x)sin\delta x\}$

Divide through by δx

$\frac{\delta y}{\delta x} = 4x + 2\delta x + \left\{2cos(2x+\delta x)\frac{sin\delta x}{\delta x}\right\}$

$\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{dy}{dx} = 4x + 2cos2x.\frac{\delta x}{\delta x} = 4x + 2cos2x$

$\therefore \frac{d}{dx}(2x^2 + sin2x) = 4x + 2cos2x$

(g) Given that x = θ – sin θ and y = 1- cosθ. Show that $\frac{\delta y}{\delta x} = cot\frac{\theta}{2}$

Solution

x = θ - sinθ

$\frac{\delta x}{\delta \theta} = 1 - cos\theta$

y= $1 - cos\theta$

$\frac{\delta y}{\delta \theta} = sin\theta$

$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta \theta}.\frac{\delta \theta}{\delta x}$

$= sin\theta.\frac{1}{1-cos\theta}$

$= \frac{sin\theta(1+cos\theta)}{(1-cos\theta)(1+cos\theta)}$

$= \frac{sin\theta(1+cos\theta)}{(1-cos^2\theta)}$

$= \frac{sin\theta(1+cos\theta)}{sin^2\theta}$

$= \frac{(1+cos\theta)}{sin\theta}$

$= \frac{1+2cos^2\frac{\theta}{2}-1}{2sin\frac{\theta}{2}cos\frac{\theta}{2}}$

$= \frac{cos\frac{\theta}{2}}{sin\frac{\theta}{2}}$

$= cot\frac{\theta}{2}$

Example 6

If y = $\sqrt{x}$, show that $\frac{\delta y}{\delta x} = \frac{1}{\sqrt{(x+\delta x)}+\sqrt{x}}$. Hence deduce $\frac{dy}{dx}$.

y+ δy = $\sqrt{(x + \delta x)}$

$\delta y = \sqrt{(x + \delta x)} - \sqrt{x}$

Dividing through by δx

$\frac{\delta y}{\delta x} = \frac{\sqrt{(x+\delta x)}-\sqrt{x}}{\delta x}$

By rationalizing the numerator

$\frac{\delta y}{\delta x} = \frac{\sqrt{(x+\delta x)}-\sqrt{x}}{\delta x}.\frac{\sqrt{(x+\delta x)}+\sqrt{x}}{\sqrt{(x+\delta x)}+\sqrt{x}}$

$\frac{\delta y}{\delta x} = \frac{\left(\sqrt{(x+\delta x)}\right)^2-\left(\sqrt{x}\right)^2}{\delta x\left(\sqrt{(x+\delta x)}+\sqrt{x}\right)} = \frac{\delta x}{\delta x\left(\sqrt{(x+\delta x)}+\sqrt{x}\right)}$

$\therefore \frac{\delta y}{\delta x} = \frac{1}{\left(\sqrt{(x+\delta x)}+\sqrt{x}\right)}$

As δx →0; $\frac{\delta y}{\delta x} \rightarrow \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{(\sqrt{x}+\sqrt{x})} = \frac{1}{2\sqrt{x}}$

Example 7

Differentiate y= $\sqrt{\frac{1+sinx}{1-sinx}}$

y= $\sqrt{\frac{(1+sinx)(1+sinx)}{(1-sinx)(1+sinx)}} = \sqrt{\frac{(1+sinx)^2}{1-sin^2x}} = \sqrt{\frac{(1+sinx)^2}{cos^2x}}$

⇨ y =$\frac{1+x}{cosx}$

$\frac{dy}{dx} = \frac{cosx(cosx)-(1+sinx)(1-sinx)}{cos^2x}$

$= \frac{cos^2x+sinx+sin^2x}{cos^2x} = \frac{1+sinx}{cos^2x} = \frac{1+sinx}{1-sin^2x}$

$= \frac{1+sinx}{(1+sinx)(1-sinx)}$

$= \frac{1}{1-sinx}$

Hence $\frac{d}{dx}\sqrt{\frac{1+sinx}{1-sinx}} = \frac{1}{1-sinx}$

Revision exercise 3

1. Differentiate with respect to x

(a)	$5x^3$	[$15x^2$]
(b)	$1-x^2$	[-2x]
(c)	x - $\frac{3}{x}$	$\left[1 + \frac{3}{x^2}\right]$
(d)	$\sqrt{x}$	$\left[\frac{1}{2}\frac{1}{\sqrt{x}}\right]$
(e)	cos3x	[3sin3x]
(f)	cot2x	[-2cosec^{2}2x]
(g)	x + sinx	[1 + cosx]
(h)	cos^{2}x	[-cosxsinx]
(i)	$\sqrt{sinx}$	$\left[\frac{cosx}{2\sqrt{sinx}}\right]$
(j)	$sin^2 5x$	[10sin5xcos5x]
(k)	Sin^{3}2x	[6sin^{2}2xcosx]
(l)	$6sin\sqrt{x}$	$\left[\frac{3cos\sqrt{x}}{\sqrt{x}}\right]$
(m)	$(1 + sinx)^2$	[2cosx(1+sinx)]
(n)	$(sinx + cos2x)^3$	[3(cosx-2sin2x)(sinx + cos2x)^2]
(o)	$\frac{1}{1+cosx}$	$\left[\frac{sinx}{(1+cosx)^2}\right]$
(p)	$\sqrt{1 - 6sinx}$	$\left[\frac{-3cosx}{\sqrt{1-6sinx}}\right]$
(q)	$\frac{3x+4}{\sqrt{2x^2+3x-2)}}$	$\left[\frac{-7x-24}{2\sqrt{2x^2+3x-2)}}\right]$
(r)	$\frac{3x-1}{\sqrt{x^2+1}}$	$\left[\frac{x+3}{\sqrt{x^2+1}}\right]$
(s)	$\left(\frac{1+2x}{1+x}\right)^2$	$\left[\frac{2(1+2x)}{(1+x)^3}\right]$
(t)	$\frac{x^3}{\sqrt{(1-2x^2)}}$	$\left[\frac{3x^2-4x^4}{(1-2x^2)^{\frac{3}{2}}}\right]$

2. Given that $y = \sqrt{\frac{1+sinx}{1-sinx}}$ show that $\frac{dy}{dx} = \frac{1}{1-sinx}$
3. Show from first principles that $\frac{d}{dx}(tanx) = sec^2x$

Differentiation of product and quotient of a function

Given the function y = uv and that u and v are functions of x, the derivatives of y with respect to x is done from first principles.

Let δx be a small increment in x and let δu, δv and δy be the resulting small increment in u, v and y

y = uv

$y + \delta y = (u + \delta u)(v + \delta v)$

$\delta y = (u + \delta u)(v + \delta v) - uv$

$\quad = u\delta v + v\delta u + \delta u\delta v$

Dividing through by δx

$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u\delta v}{\delta x}$

As δx → 0; δu → 0; δv → 0 and δy → 0

⇨ $\frac{\delta y}{\delta x} \to \frac{dy}{dx}, \frac{\delta u}{\delta x} \to \frac{du}{dx}; \frac{\delta v}{\delta x} \to \frac{dv}{dx}$ and $\frac{\delta u\delta v}{\delta x} \to 0$

$\therefore \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

This can also be expressed as (uv)' = u'v + uv'

Example 8

Differentiate the following functions with respect to x.

(a) $x^2(x+2)^3$

Here u = x^2 and v = $(x+2)^3$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 3(x+2)^2$

But $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{\delta y}{\delta x} = 2x(x+2)^3 + 3x^2(x+2)^2$

$\quad = (x+2)^2(2x^2 + 4x + 3x^2)$

$\quad = (x+2)^2(5x^2 + 4x)$

$\quad = x(x+2)^2(5x+4)$

$\therefore \frac{\delta}{\delta x}(x^2(x+2)^3 = x(x+2)^2(5x+4)$

(b) $(x+2)^3(1-x^2)^4$
u = $(x+2)^3$ and $v =$ $(1-x^2)^4$

$\frac{du}{dx} = 3(x+2)^2$ and

$\frac{dv}{dx} = 4(1-x^2)^3(-2x) = -8x(1-x^2)^3$

But $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = -8x(x+2)^3(1-x^2)^3 + 3(1-x^2)^4(x+2)^2$

$= (1-x^2)^3(x+2)^2\ [-8x(x+2) + 3(1-x^2)]$

$= (1-x^2)^3(x+2)^2\ [-8x^2 - 16x + 3 - 3x^2]$
$= (1-x^2)^3(x+2)^2\ (3 - 16x - 11x^2)$

$\therefore \frac{\delta}{\delta x}\{(x+2)^3(1-x^2)^4\} = (1-x^2)^3(x+2)^2\ (3 - 16x - 11x^2)$

(c) $7x^2\sqrt{x^2-1}$

u = $7x^2$ and v = $(x^2-1)^{\frac{1}{2}}$

$\frac{du}{dx} = 14x\ and\ \frac{dv}{dx} = \frac{1}{2}(x^2-1)^{-\frac{1}{2}}(2x) = x(x^2-1)^{-\frac{1}{2}}$

But $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = 7x^2\left[x(x^2-1)^{-\frac{1}{2}}\right] + 14x(x^2-1)^{\frac{1}{2}}$

$= 7x^2\left[\frac{x}{(x^2-1)^{\frac{1}{2}}}\right] + 14x(x^2-1)^{\frac{1}{2}}$
$= \frac{7x^3+14x(x^2-1)}{(x^2-1)^{\frac{1}{2}}} = \frac{21x^3-14x)}{(x^2-1)^{\frac{1}{2}}} = \frac{7x(3x^2-2)}{(x^2-1)^{\frac{1}{2}}}$
$\therefore \frac{\delta}{\delta x}\left(7x^2\sqrt{x^2-1}\right) = \frac{7x(3x^2-2}{\sqrt{x^2-1}}$

(d) $2x^4(3x^2-6x+2)^3$
$u = 2x^4$ and $v = (3x^2-6x+2)^1$

$\frac{du}{dx} = 8x^3\ and\ \frac{dv}{dx} = 3(3x^2-6x+2)^3(6x-6)$

But $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = 2x^4[3(3x^2-6x+2)^2(6x-6)] + 8x^3(3x^2-6x+2)^3$
$=\ 4x^{\ 3}\ (3x^2-6x+2)^2\ \{9x^2 - 9x + 6x^2 - 12x + 4\}$
$=\ 4x^{\ 3}\ (3x^2-6x+2)^2\ \{15x^2 - 21x + 4\}$
$\therefore \frac{\delta}{\delta x}(2x^4(3x^2-6x+2)^3) =\ 4x^{\ 3}\ (3x^2-6x+2)^2\ (15x^2-21x+4)$

(e) $\sqrt{(6+x)}\sqrt{(3-2x)}$

$u = (6+x)^{\frac{1}{2}}$ and $v = (3-2x)^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2}(6+x)^{-\frac{1}{2}}\ and\ \frac{dv}{dx} = \frac{1}{2}(3-2x)^{-\frac{1}{2}}(-2) = -(3-2x)^{-\frac{1}{2}}$

But $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = \frac{-\sqrt{(6+x)}}{\sqrt{3-2x}} + \frac{\sqrt{(3-2x)}}{2\sqrt{(6+x)}} = \frac{-12-2x+3-2x}{2\sqrt{3-2x}\sqrt{(6+x)}} = \frac{-(9x+4)}{2\sqrt{3-2x}\sqrt{(6+x)}}$

$\therefore \frac{d}{dx}\left(\sqrt{(6+x)}\sqrt{(3-2x)}\right) = \frac{-(9x+4)}{2\sqrt{3-2x}\sqrt{(6+x)}}$

(f) $sin^2xcos2x$

$u = sin^2x \; and \; v = cos2x$

$u = sin^2x$ and $v = cos2x$

$\frac{du}{dx} = 2sinxcosx \; and \; \frac{dv}{dx} = -2sin2x$

But $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = -2sin2xsin^2x + cos2x(2cosxsinx)$

$= -2sin2xsin^2x + cos2xsin2x$

$= sin2x(cos2x - 2sin^2x)$

$\therefore \frac{d}{dx}sin^2xcos2x = sin2x(cos2x - 2sin^2x)$

Quotient rule

This is an extension of the product rule

Given the function y = $\frac{u}{v}$

Then

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \; or \; \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$

Example 9

Differentiate the following with respect to x

(a) $\frac{x^2+6}{2x-7}$

$u = x^2 + 6$ and $v = 2x - 7$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 2$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{(2x-7).x - (x^2+6).2}{(2x-7)^2}$

$= \frac{2(2x^2-7x-x^2-6)}{(2x-7)^2} = \frac{2(x^2-7x-6)}{(2x-7)^2}$

$\therefore \frac{d}{dx}\left(\frac{x^2+6}{2x-7}\right) = \frac{2(x^2-7x-6)}{(2x-7)^2}$

(b) tanx

From tan x =$\frac{sinx}{cosx}$

$sinx$ and $v = cosx$

$\frac{du}{dx} = cosx$ and $\frac{dv}{dx} = -sinx$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{sinxcosx - (-sinxcosx)}{cos^2x}$

$= \frac{sin^2x + cos^2x}{cos^2x}$

$= \frac{1}{cos^2x} = sec^2x$

$\therefore \frac{d}{dx}tanx = sec^2x$

(c) sec x

From sec x = $\frac{1}{cosx}$

u = 1 and v = cosx

$\frac{du}{dx} = 0$ and $\frac{dv}{dx} = -sinx$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{cosx.0 - 1(-sinx)}{cos^2x} = \frac{sinx}{cos^2x} = secxtanx$

$\therefore \frac{d}{dx}secx = secxtanx$

(d) $\frac{x}{(x^2+4)^3}$

u = u and v = $(x^2+4)^3$

$\frac{du}{dx} = 1$

$\frac{dv}{dx} = 2(x^2+4)^2.2x = 6x(x^2+4)^2$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{(x^2+4)^3 - 6x^2(x^2+4)^2}{((x^2+4)^3)^2}$

$= \frac{(4-5x^2)}{(x^2+4)^6}$

Revision exercise 4

1. Find the derivatives of each of the following

a. $\frac{sinx}{x}$ $\left[\frac{xcosx - sinx}{x^2}\right]$

b. $\frac{cosx}{x^2}$ $\left[\frac{-(xsinx + 2cosx}{x^3}\right]$

c. $\frac{2x+1}{3x-4}$ $\left[\frac{-11}{(3x-4)^2}\right]$

d. $\frac{3x-4}{2x+1}$ $\left[\frac{11}{(2x+1)^2}\right]$

e. $\frac{x^2-3}{2x+1}$ $\left[\frac{-2(x^2+1+3)}{(2x+1)^2}\right]$

f. $\frac{2x+1}{x^2-3}$ $\left[\frac{-2(x^2+1+3)}{(x^2-3)^2}\right]$

g. $\sqrt{\frac{x^3}{x^2-1}}$ $\left[\frac{\sqrt{x}(x^2-3)}{2\sqrt{x^2-1}}\right]$

h. $\sqrt{\frac{3+x}{2-3x}}$ $\left[\frac{11}{2\sqrt{(3+x)}\sqrt{(2-3x)}}\right]$

i. $\frac{\sqrt{x}+1}{\sqrt{x}-1}$ $\left[-\frac{1}{\sqrt{x}(\sqrt{x}-1)^2}\right]$

j. $\frac{2x}{\sqrt{x}+1}$ $\left[\frac{\sqrt{x}+2}{(\sqrt{x}+1)^2}\right]$

k. $\frac{x^2+1}{3x-1}$ $\left[\frac{3x \quad -2x-3}{(3x-1)^2}\right]$

l. $\frac{x(x-1)^3}{x-3}$ $\left[\frac{3(x^2-4x+1)(x-1)^2}{(x-3)^2}\right]$

m. $\frac{cos2x}{x+1}$ $\left[\frac{2(x+1)sin2x+cos2x}{(x+1)^2}\right]$

n. $\frac{1=sin2x}{cos2x}$ $\left[\frac{2(1+sin2x)}{cos^2 2x}\right]$

o. $\frac{x}{1+cos^2x}$ $\left[\frac{1+2xsinxcosx+cos^2x}{(1+cos^2x)^2}\right]$

p. $\frac{1+sinx}{1+cosx}$ $\left[\frac{1+sinx+cosx}{(1+cosx)^2}\right]$

2. Show that

(a) $\frac{d}{dx}\left(\frac{x(x-3)^3}{(x+3)(x+5)^2}\right)^2 = \frac{2x(x-3)^5(x^3+27x^2+69x-45)}{(x+3)^3(x+5)^5}$

(b) $\frac{d}{dx}\left(\frac{cosx+sinx}{cosx-sinx}\right) = \frac{1}{1-sun2x}$

Differentiation of functions by use of chain rule

Chain rule is a rule used to differentiate a function of a function i.e. if y is a function of u and u is a function of x, then

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

Example 10

Find $\frac{dy}{dx}$ of each of the following using chain rule

(a) $(x+5)^3$

Let u = (x + 5); thus y = u^3

$\frac{dy}{du} = 3u^2 \text{ and } \frac{du}{dx} = 1$

Using chain rule; $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = 3u^2.1 = 3u^2$

Substituting for u

$\frac{dy}{dx} = 3(x+5)^2$

$\therefore \frac{d}{dx}(x+5)^3 = 3(x+5)^2$

(b) $(2x-5)^{10}$

Let $u = 2x - 5$ so that y = u^{10}

$\frac{du}{dx} = 2 \text{ and } \frac{dy}{du} = 10u^9$

But, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = 10u^9.2 = 20u^9$

Substituting for u

$\frac{dy}{dx} = 20(2x-5)^9$

$\therefore \frac{d}{dx}(2x-5)^{10} = 20(2x-5)^9$

(c) $\cos x^2$

Let u = x^2 so that y = cos u

$\frac{du}{dx} = 2x \text{ and } \frac{dy}{du} = -\sin u$

But, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = -\sin u .2x = -2xsinu$

Substuting for u

$\frac{dy}{dx} = -2xsin\, x^2$

$\therefore \frac{d}{dx}\cos x^2 = -2xsin\, x^2$

(d) cos^2x

Since cos^2x = $(cosx)^2$

Let u = cosx so that y =u^2

$\frac{du}{dx} = -sinx \text{ and } \frac{dy}{du} = 2u$

But, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = 2u.-sinx = -2usinx$

Substituting for u

$\frac{dy}{dx} = -2cosxsinx$

$\therefore \frac{d}{dx}cos^2x$=$-2cosxsinx$

(e) sin 5x

Let u = 5x so that y = sinu

$\frac{du}{dx}5\ and\ \frac{dy}{du} = cosu$

But, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = cosu.5 = 5cosu$

Substituting for u

$\frac{dy}{dx} = 5cos5x$

$\therefore \frac{d}{dx}sin5x = 5cos5x$

(f) $(x^2 + x - 1)^4$

Let u = $x^2 + x - 1$ so that y = u^4

$\frac{du}{dx} = 2x + 1\ and\ \frac{dy}{du} = 4u^3$

But, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = 4(2x + 1)(x^2 + x - 1)^3$

$\therefore \frac{d}{dx}(x^2 + x - 1)^4 = 4(2x + 1)(x^2 + x - 1)^3$

Revision exercise 5

1. Differentiate each of the following with respect to x using chain rule

(a) $2(1 - x)^5$ $[-10(1 - x)^4]$

(b) $(x^2 + 3)^4$ $[8x(x^2 + 3)^3]$

(c) $\frac{1}{3-7x}$ $\left[\frac{7}{(3-7x)^2}\right]$

(d) $\sqrt{6x + 1}$ $\left[\frac{3}{\sqrt{6x+1}}\right]$

(e) $(6x^2 - 5)^4$ $[48x(6x^2 - 5)^3]$

(f) $(2x - 5)^{-3}$ $[-6(2x - 5)^{-4}]$

(g) $(3x + 2)^{-1}$ $[-3(3x + 2)^{-2}]$

(h) $(x^2 + 3)^{-2}$ $[-4x(x^2 + 3)^{-3}]$

(i) $(5 - 2x^3)^{-1}$ $[6x^2(5 - 2x^3)^{-2}]$

(j) $\frac{1}{3+4x}$ $\left[\frac{-4}{(3+4x)^2}\right]$

(k) $(2x - 1)^{\frac{1}{2}}$ $\left[\frac{1}{\sqrt{2x-1}}\right]$

(l) $(6 - x)^{\frac{1}{3}}$ $\left[\frac{-1}{3(6-x)^{\frac{2}{3}}}\right]$

(m) $(x^3 - 2)^{\frac{2}{3}}$ $\left[\frac{2x^2}{(6-x)^{\frac{1}{3}}}\right]$

(n) $(4 - x^5)^{-\frac{1}{5}}$ $\left[\frac{x^4}{(4-x^5)^{\frac{6}{5}}}\right]$

(o) $\sqrt{x^3 - 6x}$ $\left[\frac{3(x^2-2)}{2\sqrt{x^3-6x}}\right]$

(p) $\frac{1}{x^2-3x+5}$ $\left[\frac{3-2x^2}{(x^2-3x+5)^2}\right]$

(q) $sin\left(4x - \frac{\pi}{5}\right)$ $\left[4cos\left(4x - \frac{\pi}{5}\right)\right]$

(r) $cos^4\left(2x - \frac{\pi}{5}\right)$

$\left[-8cos^3\left(2x - \frac{\pi}{5}\right)sin\left(2x - \frac{\pi}{5}\right)\right]$

(s) $(x + 1)^{\frac{1}{2}}(x + 2)^2$ $\left[\frac{(5x+6)(x+2)}{2(x+1)^{\frac{1}{2}}}\right]$

(t) $\frac{2x^2+3x}{(x-4)^2}$ $\left[\frac{(x-4)(4x+3)-2(2x^2+3x)}{(x-4)^3}\right]$

(u) $\frac{cos2x}{1+sin2x}$ $\left[\frac{-2}{1+sin2x}\right]$

(v) $\frac{3x-1}{\sqrt{x^2+1}}$ $\left[\frac{x+3}{(x^2+1)^{\frac{3}{2}}}\right]$

2. Show that $\frac{d}{dx}\left(\frac{1+sin^2x}{cos^2x+1}\right) = \frac{3sin2x}{(cos^2x+1)^2}$

Differentiation of parametric equations

Parametric equations are expressed in terms of a third variable say t such as y = t^2 and x = 2t +1, here the parametric variable is t. Chain rule is often used to find the derivatives of these equations.

Example 11

Find the derivatives of the following in terms of parameter t.

(a) y = $3t^2$ + 2t, x= 1-2t

$\frac{dy}{dt} = 6t + 2$ and $\frac{dx}{dt} = -2$

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= (6t + 2).\frac{1}{-2}$

$= -(3t + 1)$

(b) y = $(1 + 2t)^3$, x= t^3

$\frac{dy}{dt} = 6(1 + 2t)^2$ and $\frac{dx}{dt} = 3t^2$

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$=\frac{6(1+2t)^2}{3t^2} = \frac{2(1+2t)^2}{t^2}$

(c) x= t^2, y = 4t-1

$\frac{dy}{dt} = 4$ and $\frac{dx}{dt} = 2t$

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= \frac{4}{2t} = \frac{2}{t}$

(d) x=$\frac{2}{3+\sqrt{t}}$, y = $\sqrt{t}$

$x = 2\left(3 + t^{\frac{1}{2}}\right)^{-1}$

$\frac{dx}{dt} = -2\left(3 + t^{\frac{1}{2}}\right)^{-2}.\frac{1}{2}t^{-\frac{1}{2}} = \frac{-1}{\left(3+t^{\frac{1}{2}}\right)^2 t^{\frac{1}{2}}}$

$\frac{dy}{dt} = \frac{1}{2\sqrt{t}}$

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= \frac{1}{2\sqrt{t}}.-\left(3 + t^{\frac{1}{2}}\right)^2 t^{\frac{1}{2}} =.\frac{-\left(3+t^{\frac{1}{2}}\right)^2}{2}$

(e) x = acost and y = bsint when t =$\frac{\pi}{4}$

$\frac{dx}{dt} = -asint$ and $\frac{dy}{dt} = bcost$

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$=\frac{bcost}{-asint}$

At t = $\frac{\pi}{4}$

$\frac{dy}{dx} = \frac{bcos\frac{\pi}{4}}{-asin\frac{\pi}{4}} = -\frac{b}{a}$

(f) x = asec t and y = btant when t = $\frac{\pi}{6}$

$\frac{dx}{dt} = \text{asec}\, t \tan t$ and $\frac{dy}{dt} = bsec^2t$

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= \frac{bsec^2t}{asecttant} = \frac{b}{\text{asin}\, t}$

At t = $\frac{\pi}{6}$

$\frac{dy}{dx} = \frac{b}{\text{asin}\frac{\pi}{6}} = \frac{2b}{a}$

Revision exercise 6

Find $\frac{dy}{dx}$for each of the following

(a) x=$2\sqrt{t}, y = 5t - 4$ $[5\sqrt{t}]$

(b) x=$4\sqrt{t} - t, y = t^2 - 2\sqrt{t}$ $\left[\frac{2\sqrt{t^3-1}}{2-\sqrt{t}}\right]$

(c) $x = \frac{2}{\sqrt[3]{3t-4}}, y = \sqrt[3]{6t+1}$ $\left[\sqrt[3]{\frac{(3t-4)^4}{(6t+1)^2}}\right]$

(d) $y = tan^2(3x + 1)$

$[6\tan(3x + 1)sec^2(3x + 1)]$

(e) x = t + 5, y = t^2- 2t $[2(t - 1)]$

(f) x = t^6, y = $6t^3 - 5$ $[3t^{-3}]$

(g) $x = \sqrt{t-1}, y = \frac{1}{t}$ $\left[\frac{-2\sqrt{t-1}}{t^2}\right]$

(h) $x = t^2(3t - 1), y = \sqrt{3t + 4}$

$\left[\frac{3}{2\sqrt{3t+4}(9t^2-2t)}\right]$

(i) $x = 3(2\theta - sin\theta), y = 3(1 - cos2\theta)$

$[cot\theta]$

(j) $x = cos2\theta, y = cos\theta$ $\left[\frac{1}{4}sec\theta\right]$

(k) $x = t^2 \sin 3t, y = t^2cos3t$ $\left[\frac{2-3tsin3t}{2\tan 3t+3t}\right]$

(l) $x = t + 2cost, y = t + 2cost$ $\left[\frac{1-2sint}{3+cost}\right]$

(m) $x = 1 + 2sint, y = \sin t + \cos t$

$\left[\frac{1-2\sin t}{3+\cos t}\right]$

Differentiation of implicit functions

The functions given in the form y = f(x) such as y = 2x, y = x^5 + 3x etc. are known as explicit functions whereas functions that cannot be expressed in the form y = f(x) such as y^2+2xy = 5, x^2 + 5xy + y^2 =4 etc. are known as implicit functions because y cannot be expressed easily in terms of x.

When differentiating such functions with respect to x or y, we consider each of the individual terms in the equation given

Example 12

Find $\frac{dy}{dx}$ for each of the following functions.

(a) $x^2 - 6y^3 + y = 0$

$\frac{d}{dx}(x^2) - \frac{d}{dx}(6y^2) + \frac{d}{dx}(y) = 0$

$2x - 18y^2\frac{dy}{dx} + \frac{dy}{dx} = 0$

$\frac{dy}{dx}(18y^2 - 1) = 2x$

$\frac{dy}{dx} = \frac{2x}{18y^2-1}$

(b) $x^2y = 5x + 2$

$\frac{d}{dx}(x^2y) = \frac{d}{dx}(5x) + \frac{d}{dx}(2)$

$\frac{d}{dx}(x^2y)$ is done by use of product rule

$x^2\frac{d}{dx}(y) + y\frac{d}{dx}(x^2) = \frac{d}{dx}(5x) + \frac{d}{dx}(2)$

$x^2\frac{dy}{dx} + 2xy = 5$

$\frac{dy}{dx} = \frac{5-2xy}{x^2}$

(c) $(x + y)^5 - 7x^2 = 0$

$\frac{d}{dx}(x + y)^5 - \frac{d}{dx}7x^2 = 0$

$5(x + y)^4\frac{d}{dx}(x + y) - 14x = 0$

$5(x + y)^4\left(1 + \frac{dy}{dx}\right) = 14x$

$\frac{dy}{dx} = \frac{14x}{5(x+y)^4} - 1$

$= \frac{14x-5(x+y)^4}{5(x+y)^4}$

(d) $Siny + x^2y^3 - cosx = 2y$

$\frac{d}{dx}siny + x^2\frac{d}{dx}y^3 + y^3\frac{d}{dx}x^2 - \frac{d}{dx}cox = \frac{d}{dx}2y$

$\cos y\frac{dy}{dx} + 2y^2x^2\frac{dy}{dx} + 2xy^3 + sinx = 2\frac{dy}{dx}$

$\frac{dy}{dx}(\cos y + 2y^2x^2 - 2) = -(2xy^3 + sinx)$

$$\frac{dy}{dx} = \frac{-(2xy^3 + sinx)}{(\cos y + 2y^2x^2 - 2)}$$

(e) $y^2 + x^3 - y^3 + 6 = 3y$

$\frac{d}{dx}y^2 + \frac{d}{dx}x^3 - \frac{d}{dx}y^3 + \frac{d}{dx}6 = \frac{d}{dx}3y$

$2y\frac{dy}{dx} + 3x^2 - 3y^2\frac{dy}{dx} + 0 = 3\frac{dy}{dx}$

$3x^2 = \frac{dy}{dx}(3y^2 - 2y + 3)$

$\frac{dy}{dx} = \frac{3x^2}{(3y^2-2y+3)}$

(f) $y^2 + x^3 - xy + cosy = 0$

$\frac{d}{dx}y^2 + \frac{d}{dx}x^3 - x\frac{d}{dx}y - y\frac{d}{dx}x + \frac{d}{dx}cosy = 0$

$2y\frac{dy}{dx} + 2x^2 - x\frac{dy}{dx} - y - siny\frac{dy}{dx} = 0$

$\frac{dy}{dx}(2y - x - siny) = y - 2x^2$

$\frac{dy}{dx} = \frac{y-2x^2}{(2y-x-siny)}$

Revision exercise 7

1. Find $\frac{dy}{dx}$ for each of the following functions

(a) $\frac{x^3}{x+y} = 2$ $\left[\frac{3x^2-2}{2}\right]$

(b) $2x - y^3 = 3xy$ $\left[\frac{2-3y}{3x+3y^2}\right]$

(c) $x^6 - 5xy^3 = 9xy$ $\left[\frac{6x^5-y^2-9y}{3x(3+5y^3)}\right]$

(d) $\frac{x^2}{x+y} = 2x$ $\left[\frac{x+y}{x}\right]$

(e) $\frac{y}{x^2-7y^3} = x^5$ $\left[\frac{7x^4(x^2-5y^3)}{1+21x^5y^2}\right]$

(f) $\sqrt{x} + \sqrt{y}$ $\left[\sqrt{\frac{y}{x}}\right]$

(g) $\frac{y}{x} + \frac{x}{y} = 1$ $\left[\frac{y}{x}\right]$

(h) $siny + x^2 + 4y = cosx$ $\left[\frac{-\sin x-2x}{4+cosy}\right]$

(i) $3xy^2 + \cos y^2 = 2x^3 + 5$ $\left[\frac{6x^2-3y^2}{6xy-2ysiny^2}\right]$

(j) $5x^2 - x^3siny + 5xy = 10$ $\left[\frac{10x-3x^2siny+5y}{x^3cosy-5x}\right]$

(k) $x - \cos x^2 + \frac{y^2}{x} + 3x^5 = 4x^3$ $\left[\frac{12x^4-15x^6+y^2-2x^3\sin x^2-x^2}{2xy}\right]$

(l) $\tan 5y - ysinx + 3xy^2 = 9$ $\left[\frac{ycosx-3y^2}{5sec^2 5y-sinx-6xy}\right]$

(m) $x^2 + xy + y^2 - 3x - y = 3$ $\left[\frac{3-2x-y}{x+2y-1}\right]$

(n) $y^2 - 5xy + 8x^2 = 2$ $\left[\frac{5y-16x}{2y-5x}\right]$

2. For each of the following find the gradient of the stated curve at the point specified,

(a) $xy^2 - 6y = 8$ at $(2,1)$ $\left[\frac{1}{10}\right]$

(b) $3y^4 - 7xy^2 - 12y = 5$ at $(-2,1)$ $\left[\frac{1}{4}\right]$

(c) $\frac{x^2}{x-y} = 8$ at $(4,2)$ $[0]$

(d) $\frac{2}{x} + \frac{5}{y} = 2xy$ at $\left(\frac{1}{2}, 5\right)$ $[-15]$

(e) $(x + 2y)^4 = 1$ at $(5, -2)$ $\left[-\frac{1}{2}\right]$

(f) $x^2 + 6y^2 = 10$ at $(2, -1)$ $\left[\frac{1}{3}\right]$

(g) $x^3 + 4xy = 15 + y^2$ at $(2, 1)$ $\left[-2\frac{2}{3}\right]$

Differentiation of inverse trigonometric functions

Example 13

Differentiate the following functions with respect to x

(a) $cos^{-1}x$

Let y =$cos^{-1}x$

cosy = x

$-siny\frac{dy}{dx} = 1$

$-(1 - cos^2x)^{\frac{1}{2}}\frac{dy}{dx} = 1$

$-(1 - x^2)^{\frac{1}{2}}\frac{dy}{dx} = 1$

$\frac{dy}{dx} = -\frac{1}{\sqrt{(1-x^2}}$

(b) $sin^{-1}x$

Let y = $sin^{-1}x$

siny = x

$cosy\frac{dy}{dx} = 1$

$(1 - sin^2x)^{\frac{1}{2}}\frac{dy}{dx} = 1$

$(1 - x^2)^{\frac{1}{2}}\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2}}$

(c) $tan^{-1}x$

Let y = $tan^{-1}x$

$tan\ y = x$

$sec^2y\frac{dy}{dy} = 1$

$(1 + tan^2y)\frac{dy}{dx} = 1$

$(1 + x^2)\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{1+x^2}$

(d) $cos^{-1}(-2x^2)$

Let y = $cos^{-1}(-2x^2)$

$cos\ y = (-2x^2)$

$-siny\frac{dy}{dy} = \frac{-4x}{2} = -4x$

$\sqrt{(1 - cos^2y)}\frac{dy}{dx} = 4x$

$\sqrt{(1 - (-2x^2)^2)}\frac{dy}{dx} = 4x$

$\frac{dy}{dx} = \frac{4x}{\sqrt{1-4x^4}}$

(e) $sin^{-1}\left(\frac{1-x}{1+x}\right)$

Let y = $sin^{-1}\left(\frac{1-x}{1+x}\right)$

siny =$\left(\frac{1-x}{1+x}\right)$

$cosy\frac{dy}{dx} = \frac{-(1+x)-(1-x)}{(1+x)^2}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{1-x}{1+x}\right)^2}}.\frac{-(1+x)-(1-x)}{(1+x)^2}$

$= \frac{1+x}{\sqrt{(1+x)^2-(1-x)^2}}.\frac{-(1+x)-(1-x)}{(1+x)^2}$

$= \frac{1}{\sqrt{(1+x)^2-(1-x)^2}}.\frac{-2}{(1+x)}$

$= \frac{1}{\sqrt{4x}}.\frac{-2}{(1+x)}$

$$= \frac{-1}{\sqrt{x}(1+x)}$$

Revision exercise 8

Differentiate the following with respect to x

(a) $2sec^{-1}\sqrt{x}$ $\left[\frac{1}{x\sqrt{1-x^2}}\right]$

(b) $cosec^{-1}(cotx)$ $\left[\frac{1}{cosx\sqrt{cos2x}}\right]$

(c) $cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ $\left[\frac{2}{1+x^2}\right]$

(d) $cot^{-1}x$ $\left[\frac{1}{1+x^2}\right]$

(e) $cosec^{-1}x$ $\left[\frac{-1}{\sqrt{x^2-1}}\right]$

(f) $tan^{-1}\left(\frac{2x}{1-x^2}\right)$ $\left[\frac{2}{1+x^2}\right]$

(g) $sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ $\left[\frac{-2}{1+x^2}\right]$

(h) $tan^{-1}\left(\frac{1+x}{1-x}\right)$ $\left[\frac{1}{1+x^2}\right]$

(i) $sin^{-1}(2x - 1)$ $\left[\frac{1}{\sqrt{x}(1-x)}\right]$

(j) $tan^{-1}(1 - 3x)$ $\left[\frac{-3}{2-6x+9x^2}\right]$

(k) $sin^{-1}(x^2 - 1)$ $\left[\frac{2}{\sqrt{2-x^2}}\right]$

(l) $xsin^{-1}x$ $\left[sin^{-1}x + \frac{x}{\sqrt{1-x^2}}\right]$

(m) $xtan^{-1}x$ $\left[tan^{-1}x + \frac{x}{1+x^2}\right]$

(n) $(x^2 + 1)tan^{-1}x$ $[2xtan^{-1}x + 1]$

Second derivatives

Suppose y is a function of x, the first derivative of y with respect to x is denoted as $\frac{dy}{dx}$ or f'(x)

The result of differentiating $\frac{dy}{dx}$ with respect to x is the second derivative denoted by $\frac{d^2y}{dx^2}$ or f''(x)

Note that If $\frac{d^2y}{dx^2}$ is used to determine the natures of stationary points

A stationary point on a curve occurs when $\frac{dy}{dx} = 0$ Once you have established where there is a stationary point, the type of stationary point (maximum, minimum or point of inflexion) can be determined using the second derivative.

If $\frac{d^2y}{dx^2}$ is positive, then it is a minimum point

If $\frac{d^2y}{dx^2}$ is negative, then it is a maximum point

If $\frac{d^2y}{dx^2}$ =0 then it could be maximum, maximum or point of inflection

Example 14

Determine the second derivative of each of the following

(a) x^4

$\frac{dy}{dx} = 4x^3$

$\frac{d^2y}{dx^2} = \frac{d}{dx}(4x^3) = 12x^2$

(b) cos2x

$\frac{dy}{dx} = -2sin2x$

$\frac{d^2y}{dx^2} = \frac{d}{dx}(-2sin2x) = -4cos2x$

(c) $x^2(1-x)^2$

$x^2(1-x)^2 = x^2(1-2x+x^2)$

$= x^2 - 2x^3 + x^4$

$\frac{dy}{dx} = 2x - 6x^2 + 4x^3$

$\frac{d^2y}{dx^2} = 2 - 12x + 12x^2$

(d) $xsinx$

$\frac{dy}{dx} = sinx + xcosx$

$\frac{d^2y}{dx^2} = cosx + cosx - xsinx$

$= 2xosx - xsinx$

(e) x^3sinx

$\frac{dy}{dx} = 3x^2sinx + x^3cosx$

$\frac{d^2y}{dx^2}$

$= 6xsinx + 3x^2cosx + 3x^2cosx - x^3sinx$

$= (6x - x^3)sinx + 6x^2cosx$

(f) $xtan^{-1}$x

$\frac{dy}{dx} = tan^{-1}x + \frac{x}{1+x^2}$

$\frac{d^2y}{dx^2} = \frac{x}{1+x^2} + \frac{(1+x^2)(1)-x(2x)}{(1+x^2)^2}$

$= \frac{2}{(1+x^2)^2}$

(g) If $x^2 + 3xy - y^2 = 3$, find $\frac{dy}{dx}$and $\frac{d^2y}{dx^2}$ at the point (1,1)

$2x + 3y + 3x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{2x+3y}{2y-3x}$

At (1,1)

$\frac{dy}{dx} = \frac{2(1)+3(1)}{2(1)-3(1)} = -5$

$\frac{d^2y}{dx^2} = \frac{(2y-3x)\left(2+3\frac{dy}{dx}\right)-(2x+3y)\left(2\frac{dy}{dx}-3\right)}{(2y-3x)^2}$

Substituting for x =1, y= 1and $\frac{dy}{dx} = -5$

$\frac{d^2y}{dx^2} = \frac{(2-3)(2+3(-5))-(2+3)(2(-5)-3)}{(2-3)^2}$

$= \frac{(-1)(-13)-(5)(-13)}{(-1)^2}$

$= \frac{13+65}{1} = 78$

Example 15 (parametric equation)

Find$\frac{d^2y}{dx^2}$in terms of t if

(a) $x = a(t^2-1) and\ y = 2a(t+1),$

$\frac{dx}{dt} = 2at$

$\frac{dy}{dt} = 2a$

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= \frac{2a}{2at} = \frac{1}{t}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).\frac{dt}{dx}$

$= \frac{d}{dx}(t^{-1}).\frac{dt}{dx}$

$= \frac{-1}{t^2}.\frac{1}{2at}$

$= \frac{-1}{2at^3}$

(b) $x = cost + sint\ and\ y = sint - cost$

$\frac{dx}{dt} = -sint + cost$

$\frac{dy}{dt} = cost + sint$

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= \frac{-sint+cost}{sint+cost}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{-sint+cost}{sint+cost}\right)\frac{dx}{dt}$

$= \frac{(-sint+cost)(-sint+cost)-(cost+sint)(-cost-sint}{(-sint+cost)^2(-sint+cost)}$

$= \frac{2}{(-sint+cost)^3}$

Revision exercise 9

1. Find $\frac{d^2y}{dx^2}$ of each of the following

(a) $\frac{x^2}{1+x}$ $\left[\frac{2}{(1+x)^3}\right]$

(b) $\frac{sinx}{x^2}$ $\left[\frac{(6-x^2)sinx-4cosx}{x^4}\right]$

(c) tan^2x $[4y(1+y)^2]$

(d) $tan3x$ $[18y(1+y^2)]$

(e) $xtanx$ $\left[\frac{2(x^2+y^2)(1+y)}{x^2}\right]$

(f) $sec2x$ $[4y(2y^2-1)]$

2. Find $\frac{d^2y}{dx^2}$ in terms of t or θ if

(a) x = cotθ, y = $sin^2θ$ $[2sin^{3\theta} sin3\theta]$

(b) $x = \frac{1+t^2}{1-t},\ y = \frac{2t}{1-t}$ $\left[-4\left(\frac{1-t}{1+2t-t^2}\right)^3\right]$

(c) $x = t+3, y = t^2+4$ $[2]$

(d) $x = 3-2t^2, y = \frac{1}{t}$ $\left[\frac{3}{16t^5}\right]$

(e) $x = t^2+2t, y = t^2-3t$ $\left[\frac{3}{4(t+1)}\right]$

3. Given that y = cot5x, show that

$\frac{d^2y}{dx^2} + 10y\frac{dy}{dx} = 0$

4. Given that $x = 1 - sint$ and $y = 1 - cost$ show that $y^2\frac{d^2y}{dx^2} + 1 = 0$

Differentiation of exponential functions

An exponential function is the function given in the form $y = e^x$, where y is said to be an exponential function of x.

These are differentiated using product and quotient rules.

Example 16

Differentiate each of the following with respect to x

(a) e^x

$\frac{d}{dx}(e^x) = e^x$

(b) e^{3x^2}

$Let\ u = 3x^2 and\ y = e^u$

$\frac{du}{dx} = 6x\ and\ \frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 6x.e^u$

$= 6xe^{3x^2}$

(c) e^{sinx}

Let $u = tanx$ => $y = e^u$

$\frac{du}{dx} = cosx\ and\ \frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = e^u.cosx$

$e^{tanx}cosx$

(d) e^{3x}

Let $u = 3x$ => $y = e^u$

$\frac{du}{dx} = 3\ and\ \frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = e^u.3$

$= 3e^{3x}$

(e) $y = 2e^{x^2+1}$

Let $u = x^2+1$ => $y = 2e^u$

$\frac{du}{dx} = 2x\ and\ \frac{dy}{du} = 2e^u$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 2e^u.2x$

$= 4xe^{x^2+1}$

(f) e^xcos2x

$\frac{du}{dx} = e^xcos2x + e^x(-2sin2x)$

$= e^xcos2x - 2e^xsin2x$

(g) e^xsin2x

$\frac{du}{dx} = e^xsin2x + e^x(2cos2x)$

$= e^xsn2x + 2e^xcos2x$

(h) $\frac{e^{-\frac{1}{2}\sqrt{x}}}{x^2}$

$\frac{dy}{dx} = \frac{x^2\frac{d}{dx}e^{-\frac{1}{2}\sqrt{x}} - e^{-\frac{1}{2}\sqrt{x}}\frac{d}{x}(x^2)}{x^4}$

$= \frac{e^{-\frac{1}{2}\sqrt{x}}\left(\frac{x^2}{4\sqrt{x}}+\frac{2x}{1}\right)}{x^4}$

$= \frac{e^{-\frac{1}{2}\sqrt{x}}(x+8\sqrt{x})}{4x^{\frac{7}{3}}}$

Differentiation of logarithmic functions

Logarithms of numbers to base e is called natural logarithm or napeilian logarithm.

The natural logarithm of a number say x is denoted by $\log_e x$ or Inx

Let y = $\log_e x$

$e^y = x$

$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$

$\frac{dy}{dx} = \frac{\frac{d}{dx}(x)}{e^y} = \frac{\frac{d}{dx}(e^y)}{e^y}$

Example 17

Differentiate with respect to x

(a) Inx

Let y = Inx

$\frac{dy}{dx} = \frac{\frac{d}{dx}(x)}{x} = \frac{1}{x}$

(b) In(1+2x)

Let y = In(1 + 2x)

$\frac{dy}{dx} = \frac{\frac{d}{dx}(1+2x)}{1+2x} = \frac{2}{1+2x}$

(c) In(1- x)

Let y = In(1 – x)

$\frac{dy}{dx} = \frac{\frac{d}{dx}(1-x)}{1-x} = \frac{-1}{1-x}$

(d) $In(4x^3)$

Let $y = In(4x^3)$

$\frac{dy}{dx} = \frac{\frac{d}{dx}(4x^3)}{4x^3} = \frac{12x^2}{4x^3} = \frac{3}{x}$

(e) In(tanx)

$\frac{dy}{dx} = \frac{\frac{d}{dx}(tanx)}{tanx} = \frac{sec^2x}{tanx} = secxcosecx$

$= 2cosec2x$

(f) $2y^2$

Let $q = 2y^2$

$Inq = 2y^2$=$2In(2y)$

$\frac{1}{q}\frac{dt}{dy} = 2\frac{\frac{d}{dy}(2y)}{2y} = \frac{2}{y}$

$\frac{dq}{dy} = \frac{2q}{y} = \frac{4y^2}{y} = 4y$

But $\frac{dq}{dy} = \frac{dq}{dy}.\frac{dy}{dx}$

$\frac{dq}{dx} = 4y\frac{dy}{dx}$

(g) Iny

Let q = Iny

$\frac{dq}{dy} = \frac{1}{y}$

But $\frac{dq}{dy} = \frac{dq}{dy}.\frac{dy}{dx}$

$\frac{dq}{dy} = \frac{1}{y}.\frac{dy}{dx}$

(h) 2^x

Let y = 2^x

Iny =In2^x=xIn2

$\frac{1}{y}\frac{dy}{dx} = In2$

$\frac{dy}{dx} = yIn2 = 2^xIn2$

(i) 2^{x^2}

$Iny = In2^{x^2} = x^2In2$

$\frac{1}{y}\frac{dy}{dx} = 2xInx$

$\frac{dy}{dx} = y2xInx = 2^{x^2}2xIn2$

(j) $3x^2.3^x$

Let y =$3x^2.3^x$

$Iny = In3x^2.3^x$

$= In3 + Inx^2 + In3^x$

$= In3 + 2Inx + xIn3$

$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + Inx = \frac{2+xInx}{x}$

$\frac{dy}{dx} = y\frac{2+xInx}{x} = 3x^2.3^x\left(\frac{2+xInx}{x}\right)$

$= 3x.3^x(2 + xInx)$

(k) $\sqrt[3]{\frac{x+1}{x-1}}$

$Let\ y = \sqrt[3]{\frac{x+1}{x-1}}$

$y^3 = \frac{x+1}{x-1}$

$Iny^3 = In(x + 1) - In(x - 1)$

$\frac{3y^2}{y^3}\frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{(x+1)(x-1)}$

$\frac{dy}{dx} = \frac{y}{3}.\frac{-2}{(x+1)(x-1)}$

$= \frac{(x+1)^{\frac{1}{3}}}{3(x-1)^{\frac{1}{3}}}.\frac{-2}{(x+1)(x-1)}$

$= \frac{-2}{3(x+1)^{\frac{2}{3}}(x-1)^{\frac{4}{3}}}$

Revision exercise 10

1. Differentiate with respect x

(a)	e^{2y}	$\left[2e^{2y}\frac{dy}{dx}\right]$
(b)	e^{siny}	$\left[cosye^{siny}\frac{dy}{dx}\right]$
(c)	$4x^2 + \frac{2}{e^{x^2}}$	$\left[8x - \frac{4x}{e^{x^2}}\right]$
(d)	xe^{-x}	$[e^{-x} - xe^{-x}]$
(e)	$Insinx$	$[cotx]$

(f) In(tany) $\left[secxcosec\frac{dy}{dx}\right]$

(g) $\frac{\sqrt{x^2+1}}{(2x-1)^2}$ $\left[\frac{2x^2+x+4}{(x^2+1)^{\frac{1}{2}}(2x-1)^3}\right]$

(h) $\frac{x^2e^x}{(x-1)^3}$ $\left[\frac{xe^x(x^2-2x-2)}{(x-1)^4}\right]$

(i) $\frac{sin4x}{5^{2x}}$ $\left[\frac{2sin^4x^2(4xcotx^2-In5)}{5^{2x}}\right]$

(j) $\frac{e^{x^2}\sqrt{\cos x}}{(2x+1)^3}$ $\left[\frac{e^{x^2}\sqrt{\cos x}}{(2x+1)^3}\left(2x-\frac{1}{2}tanx-\frac{6}{2x+1}\right)\right]$

(k) $\frac{2e^{-x}}{2^xcosx}$ $\left[\frac{2e^{-x}}{2^xcosx}(tanx-In2-1)\right]$

(l) $\frac{(x-1)(2-3x)}{(1+x)(x+2)}$ $\left[\frac{2(8-4x-7x^2)}{(1+x)^2(x+2)^2}\right]$

(m) $In(1+x^2)$ $\left[\frac{2x}{1+x^2}\right]$

(n) $In(x^3-2)$ $\left[\frac{3x^2}{x^3-2}\right]$

(o) $In(e^x+4)$ $\left[\frac{e^x}{e^x+4}\right]$

(p) $In(\sqrt{x})$ $\left[\frac{1}{2x}\right]$

(q) $(3-2Inx)^3$ $\left[\frac{-6(3-2Inx)^2}{x}\right]$

(r) x^2Inx $[x(1+2Inx)]$

(s) $xIn(1+x)$ $\left[\frac{x}{1+x}+In(1+x)\right]$

(t) $x^2In(3+2x)$ $\left[\frac{2x^2}{3+2x}+2xIn(3+2x)\right]$

(u) $\frac{x}{Inx}$ $\left[\frac{Inx-1}{(Inx)^2}\right]$

(v) 7^x $[7^xInx]$

(w) 2^{x^2} $[x2^{x^2}In4]$

(x) 3^{2x-1} $\left[\frac{2}{3}(3^{2x})In3\right]$

(y) e^{Inx} $[1]$

2. Given that $y=xe^{2x}$, show that $x\frac{dy}{dx}=(2x+1)y$
3. Given that $y=\frac{e^x}{e^x+1}$, show that $(1+e^x)\frac{dy}{dx}-y=0$
4. Given that $y=\frac{e^{x^2}}{x}$, show that $\frac{dy}{dx}=\frac{2e^{x^2}-y}{x}$
5. Given that e^x-e^{-x}, show that $\left(\frac{dy}{dx}\right)^2-y^2=4$
6. Given that $Ae^{4x}+Be^{-4x}$, where A and B are constants show that $\frac{d^2y}{dx^2}-16y=0$
7. Given that $y=In(Inx)$, show that $(Inx)\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{1}{x^2}=0$
8. Given that $y=In\left(\frac{1+x}{1-x}\right)$, show that $(1-x^2)\frac{dy}{dx}-2=0$
9. Given that $y=\frac{In(1+x)}{x^2}$, show that $x^2\frac{dy}{dx}+2xy=\frac{1}{1+x}$
10. Given that $y=In(I1+e^x)$, show that $\frac{d^2y}{dx^2}=e^x\left(1-\frac{dy}{dx}\right)^2$
11. Given that $y=e^{3x}sin2x$, show that $\frac{d^2y}{dx^2}+13y=6\frac{dy}{dx}$

Revision exercise 11

1. Find the derivative of $y=sin^2x$ from the first principles [2sinxcosx]
2. If δx and δy are small increment in x and y respectively and y= tan2x, write down an expression of δy in terms of x and δx. $\left[\frac{2\delta x}{cos^2x}\right]$
3. Differentiate the following with respect to x

(a) $\frac{x^3}{\sqrt{(1-2x^2)}}$ $\left[\frac{3x^2-4x^4}{(1-2x^2)^{\frac{3}{2}}}\right]$

(b) $\log_5\left(\frac{e^{tanx}}{sin^2x}\right)$ $\left[\frac{1}{In5}(sec^2x-2cotx)\right]$

(c) $(x-0.5)e^{2x}$ $[2x e^{2x}]$

(d) $(sinx)^x$ $[(sinx)^x(Insinnx+xcotx)]$

(e) $e^{\frac{-2}{x}}sin3x$ $\left[e^{\frac{-2}{x}}sin3x\left(\frac{2}{x^2}+3cot3x\right)\right]$

(f) $tan^{-1}\left(\frac{x}{1-x^2}\right)$ $\left[\frac{1+x^2}{1-x^2-x^4}\right]$

(g) $tan^{-1}\left(\frac{6x}{1-2x^2}\right)$ $\left[\frac{6+12x^2}{1-32x^2-4x^4}\right]$

(h) $(cosx)^{2x}$ $[2(cosx)^{2x}(Incosx-xtanx)]$

(i) $e^{ax}sinbx$ $[e^{ax}sinbx(a+bcosbx)]$

(j) $\frac{(x+1)^{2(x+2)}}{(x+3)^3}$ $[3(x+3)^2]$

(k) $cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ $\left[\frac{2}{1+x^2}\right]$

(l) $3xInx^2$ $[3In(x^2+2)]$

(m) $cot2x$ $[-2cosec^22x]$

(n) $(sinx)^x$ $[(sinx)^x(xcotx+Insinx)]$

(o) $\frac{(x+1)^2}{(x+4)^3}$ $\left[\frac{(5-x)(x+1)}{(x+4)^4}\right]$

(p) $\frac{3x+4}{\sqrt{2x^2+3x-2}}$ $\left[\frac{-(7x+4)}{(2x^2+3x-2)^{\frac{3}{2}}}\right]$

(q) $\log_e\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$ $\left[\frac{1}{1-x^2}\right]$

(r) $\frac{1+sin^2x}{cos^2x+1}$ $\left[\frac{3sin2x}{(cos^2x+1)^2}\right]$

(s) $tan^{-1}\left(\frac{x^2}{2}+2x^3\right)$ $\left[\frac{4x(1+6x)}{4+(x^2+4x^3)^2}\right]$

(t) e^{ax^2} $[2e^{ax^2}]$

(u) $(1-2x)^{-\frac{1}{2}}$ $\left[\frac{2x}{1-2x^2}\right]$

(v) $(x+1)^{\frac{1}{2}}(x+2)^2$ $\left[\frac{(5x+6)(x+2)}{2(x+1)^{\frac{1}{2}}}\right]$

(w) $\frac{2x^2+3x}{(x-4)^2}$ $\left[\frac{(x-4)(4x+3)-2(2x^2+3x)}{(x-4)^3}\right]$

(x) $\frac{3x-1}{\sqrt{x^2+1}}$ $\left[\frac{x+3}{(x^2+1)^{\frac{3}{2}}}\right]$

(y) $\frac{cos2x}{1+sin2x}$ $\left[\frac{-2}{1+sin2x}\right]$

(z) $In(secx+tanx)$ $[secx]$

(aa) $\left(\frac{1+2x}{1+x}\right)^2$ $\left[\frac{2(1+2x)}{(1+x)^3}\right]$

4. If $y = tan\left(\frac{x+1}{2}\right)$ show that $\frac{d^2y}{dx^2} - y\frac{dy}{dx} = 0$
5. Given that $y = e^{tanx}$, show that $\frac{d^2y}{dx^2} = 6\frac{dy}{dx}$
6. If $y = \sqrt{x}$ show that $\frac{dy}{dx} = \frac{1}{\sqrt{(x+\delta)}+\sqrt{x}}$
7. If $y = \sqrt{(5x^2+)}$, show that $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 5$
8. Given $y = In\left(1-\frac{1}{u}\right)^{\frac{1}{2}}$, $2u = \left(x-\frac{1}{x}\right)$, show that $\frac{dy}{dx} = \frac{(x+1)}{(x^2+1)(x-1)}$
9. If $y = e^{-t}\cos(t+\beta)$, show that $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$
10. Given that $y = \sqrt{\frac{1+sinx}{1-sinx}}$, show that $\frac{dy}{dx} = \frac{1}{1-sinx}$
11. Show from first principles that $\frac{d}{dx}(tanx) = sec^2x$
12. Given that x = $\frac{t^2}{1+t^3}$ and y = $\frac{t^3}{1+t^3}$, find $\frac{d^2y}{dx^2}$. $\left[\frac{6}{t}\left(\frac{1+t^3}{2-t^3}\right)^3\right]$
13. Differentiate y = $2x^2 + 3$ from first principles $[4x]$

Thank you

Dr. Bbosa Science