Integration (A-level)

It the reverse of differentiation.

During integration the following concepts should be considered.

(a) Polynomial functions;

$\int x^n dx = \frac{1}{n+1}x^{n+1} + 1 \; where \; n \neq -1$

i.e. increase the power by 1 and divide the term by the new power, e.g.

(i) $\int 1dx = \int (x^0)dx = x + c$

(ii) $\int xdx = \int x^1 dx$

$= \frac{1}{1+1}x^{1+1}$

$= \frac{1}{2}x^2$

(iii) $\int x^4 dx = \frac{1}{4+1}x^{4+1} = \frac{1}{4}x^5$

(iv) $\int 4x^3 dx = 4\int x^3 dx = \frac{4}{(3+1)}x^4 = x^4$

(v) $\int x^{-3} = \frac{1}{-3+1}x^{-3+1} = -\frac{1}{2}x^{-2} = \frac{-1}{2x^2}$

(b) Trigonometric functions, e.g.

(i) $\frac{d}{dx}(cosx) = -sinx + c$

- $\int sinx\, dx = -cosx + c$

(ii) $\frac{d}{dx}(sinx) = cosx$

- $\int cosx\, dx = sinx + c$

(iii) $\frac{d}{dx}(tanx) = sec^2 + c$

- $\int sec^2\, dx = tanx + c$

(iv) $\frac{d}{dx}(cotx) = -cosec^2 x$

- $\int cosec^2 x = -cotx + c$

Methods of integration

The choice of the method depends on judgement. Below are some of the methods:

Integration by change of variable where a derivative exist/integration by recognition or inspection

Example 1

(i) $\int x\sqrt{(x^2-2)}$dx

Solution

Let u = x^2-2

- $du = 2u \; i.e. xdx = \frac{1}{2}du$

$\int x\sqrt{(x^2-3)}\,dx = \int \sqrt{(x^2-3)}\,xdx$

$= \int \sqrt{u}.\frac{1}{2}du$

$= \frac{1}{2}\int u^{\frac{1}{2}}du$

$= \frac{1}{2}.\frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{1}{3}(x^2-3)^{\frac{3}{2}} + c$

$= \frac{1}{3}(x^2-3)\sqrt{(x^2-3)} + c$

Or let u =$\sqrt{(x^2-32)}$ => $u^2 = x^2 - 3$

2du = 2xdx i.e. xdx = udu

$\int x\sqrt{(x^2-3)}\,dx = \int \sqrt{(x^2-3)}\,xdx$

$\int u.udu = \int u^2 du$

$= \frac{1}{3}u^3 + c$

$= \frac{1}{3}(x^2-3)\sqrt{(x^2-3)} + c$

(ii) $\int xcosec^2(x^2)dx$

Solution

Let u = x^2 => du =2xdx i.e. xdx = $\frac{1}{2}du$

$\int xcosec^2(x^2)dx = \int cosec^2(x^2)xdx$

$= \frac{1}{2}\int cosec^2 udu$

$= -cotu + c$

$= -cotx^2 + c$

(iii) $\int_0^1 \frac{x^2-1}{\sqrt{(x^3-3x+5)}}dx$

Solution

Let u = $x^3 - 3x + 5$ =>$du = (3x^2 - 3)dx$

i.e. $(3x2 - 3)dx = \frac{1}{3}du$

$\therefore \int_0^1 \frac{x^2-1}{\sqrt{(x^3-3x+5)}}dx = \frac{1}{3}\int_0^1 \frac{1}{\sqrt{u}}dx$

$= \frac{1}{3}u^{\frac{1}{2}}(2) + c$

$= \frac{2}{3}\sqrt{(x^3 - 3x + 5)}$

\- $\int_0^1 \frac{x^2-1}{\sqrt{(x^3-3x+5)}}dx = \frac{2}{3}\sqrt{(x^3 - 3x + 5)}\Big|_0^1$

$= \frac{2}{3}\left(\sqrt{1-3+5} - \sqrt{5}\right)$

$= \frac{2}{3}\left(\sqrt{3} - \sqrt{5}\right) = 0.336$

(iv) $\int_0^{\frac{\pi}{4}} sec^2x\sqrt{tanx}dx$

Solution

Let u = $tanx$ => du = sec^2xdx

$\int_0^{\frac{\pi}{4}} sec^2x\sqrt{tanx}dx = \int_0^{\frac{\pi}{4}} u^{\frac{1}{2}}du$

$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^{\frac{\pi}{4}}$

$= \left[\frac{2}{3}(tanx)^{\frac{3}{2}}\right]_0^{\frac{\pi}{4}}$

$= \frac{2}{3}$

(v) $\int cosx\sqrt{1 - 2sinx}dx$

Solution

Let u = 1-2sinx => du = -2cosx

i.e. cosxdx =$-\frac{1}{2}du$

$\therefore \int cosx\sqrt{1 - 2sinx}dx = -\frac{1}{2}\int u^{\frac{1}{2}}dx$

$= \frac{1}{2}.\frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{1}{3}(1 - 2sinx)^{\frac{3}{2}} + c$

$= \frac{1}{3}(1 - 2sinx)\sqrt{(1 - 2sinx)} + c$

Integration by change of variable where a derivative is given

Example 2

(a) Using the substation x =cos2θ or otherwise, prove that

$\int_0^1 \sqrt{\left(\frac{1-x}{1+x}\right)}dx = \frac{\pi}{2} - 1$

Solution

Given x= cos2θ => dx = -2sin2θdθ

Changing limits

x	θ
0	$\frac{1}{4}\pi$
1	0

$\int_0^1 \sqrt{\left(\frac{1-x}{1+x}\right)}dx = -2\int_{\frac{\pi}{4}}^0 \sqrt{\left(\frac{1-cos2\theta}{1+cos\theta}\right)}sin2\theta\, d\theta$

$= 2\int_0^{\frac{\pi}{4}} \sqrt{\left(\frac{2sin^2\theta}{2cos^2\theta}\right)}sin2\theta\, d\theta$

$= 2\int_0^{\frac{\pi}{4}} \left(\frac{sin\theta}{cos\theta}\right)2sin\theta cos\theta\, d\theta$

$= 4\int_0^{\frac{\pi}{4}} sin^2\theta d\theta$

$= \frac{4}{2}\int_0^{\frac{\pi}{4}}(1 - cos2\theta)d\theta$

(double angle form

$= 2\left[\theta - \frac{1}{2}sin2\theta\right]_0^{\frac{\pi}{4}}$

$= 2\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{2} - 1$

(b) Use the substitution x = sinθ to evaluate

$\int_{\frac{1}{2}}^{\sqrt{\frac{1}{2}}} \frac{dx}{x^2(1-x^2)}$

Solution

Given x = sinθ => dx =cosθdθ

Changing limits

x	θ
$\frac{1}{2}$	$\frac{1}{6}\pi$
$\sqrt{\frac{1}{2}}$	$\frac{1}{4}\pi$

$\therefore \int_{\frac{1}{2}}^{\sqrt{\frac{1}{2}}} \frac{dx}{x^2(1-x^2)} = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{cos\theta d\theta}{sin^2\theta(1-sin^2\theta)}$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{d\theta}{sin^2\theta} = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} cosec^2\theta d\theta$

$= [-cot\theta]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = [cot\theta]_{\frac{\pi}{4}}^{\frac{\pi}{6}}$

$= cot\frac{\pi}{6} - cot\frac{\pi}{4} = \sqrt{3} - 1$

Revision exercise 1

Integrate the following using the suggested substitution in each case.

1. $\int x(x+4)^3 dx$, u = x + 4
$\left[\frac{1}{5}(x-1)(x+4)^4 + c\right]$
2. $\int (x-4)(x-1)^3 dx$, u = x – 1
$\left[\frac{1}{4}(4x-19)(x-1)^4 + c\right]$
3. $\int x(2x-3)^2 dx$, u = 2x – 3

$\left[\frac{1}{16}(2x+1)(2x-3)^3 + c\right]$

4. $\int (3x+1)\,(2x-5)^2 dx$, u = 2x – 5
$\left[\frac{1}{48}(18x+23)(2x-5)^3 + c\right]$

5. $\int \frac{x}{x+3} dx$, u= x+3 $[x - 3In(x+3) + c]$
6. $\int \frac{x}{(x+1)^2} dx$, u= x+1 $\left[\frac{1}{x+1} + In(x+1) + c\right]$
7. $\int \frac{x+1}{(2x-3)^3} dx$, u= 2x – 1 $\left[-\frac{4x+1}{8(2x-3)^2}\right]$
8. $\int \sqrt{(x+1)}\, dx$, u = x + 1
$\left[\frac{2}{15}(3x-2)\sqrt{(x+1)^2} + c\right]$
9. $\int x\sqrt{(x-1)}\, dx, u = \sqrt{(x-1)}$

$\left[\frac{2}{15}(3x+2)\sqrt{(x-1)^2} + c\right]$

10. $\int (x-4)\sqrt{(x+5)}$, u = x + 5
$\left[\frac{2}{5}(x-10)\sqrt{(x+1)^3} + c\right]$
11. $\int (3x-2)\sqrt{(1-2x)}dx, u = \sqrt{(1-2x)}$
$\left[\frac{1}{15}(7-9x)\sqrt{(1-2x)^2} + c\right]$
12. $\int \frac{x}{\sqrt{x+1}} dx$, u= x+1
$\left[\frac{2}{3}(x-2)\sqrt{x+1} + c\right]$
13. $\int \frac{x}{\sqrt{x-3}} dx$, u= $\sqrt{x-3}$
$\left[\frac{2}{3}(x+6)\sqrt{x-3} + c\right]$
14. $\int \frac{x-2}{\sqrt{x-4}} dx$, u= $x-4$
$\left[\frac{2}{3}(x+2)\sqrt{x-4} + c\right]$
15. $\int \frac{x+3}{\sqrt{5-x}} dx$, u= $\sqrt{5-x}$

$\left[-\frac{2}{3}(x+19)\sqrt{5-x} + c\right]$

16. Use the substitution $x = \frac{1}{u}$ to evaluate $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ $\left[\frac{\pi}{3}\right]$
17. Use the substitution u = $\sqrt{x-2}$ to evaluate $\int_3^4 \frac{3x}{\sqrt{x-2}} dx$ $[16\sqrt{2} - 4]$
18. Evaluate
(a) $\int_3^5 x(x-3)^2 dx$ [12]
(b) $\int_4^7 \frac{5-x}{\sqrt{x-3}}$dx $\left[-\frac{2}{3}\right]$
(c) $\int_1^3 (3x+1)(2-x)^4\, dx$ $\left[2\frac{4}{5}\right]$
19. By using the substitution u = $\sqrt{1+x^2}$, show that $\int_0^{\sqrt{3}} x^2\sqrt{(1+x^2)}\, dx = 3\frac{13}{15}$

Integration by change of variable where a derivative not exist

Here a term is solved by changing it to another variable

Example 3

Find

(a) $\int_5^6 x\sqrt{(x-5)}\, dx$

Solution

Let u= $\sqrt{(x-5)}$ hence $u^2 = x - 5$ => x =u^2 + 5

dx = 2udu

Changing limits

x	θ
6	1
5	0

$$\therefore \int_5^6 x\sqrt{(x-5)}\, dx = \int_0^1 (u^2+5)u.\,2udu$$
$$= 2\int_0^1 (u^4 + 5u^2)\, du$$
$$= 2\left[\frac{1}{5}u^5 + \frac{5}{3}u^3\right]_0^1$$
$$= \frac{2}{15}(3+25) = \frac{56}{15}$$

(b) $\int \frac{x-3}{\sqrt{x+1}} dx$

Solution

Let u= $\sqrt{x+1}$ i.e.$u^2 = x + 1$

- x = u^2 – 1 and dx = 2udu

$$\therefore \int \frac{x-3}{\sqrt{x+1}} dx = \int \frac{(u^2-1)}{u}.\,2udu$$
$$= 2\int (u^2 - 4)du$$
$$= 2\left(\frac{1}{3}u^3 - 4u\right) + c$$
$$= \frac{2}{3}u(u^2 - 12) + c$$
$$= \frac{2}{3}\sqrt{(x+1)}(x + 1 - 12) + c$$
$$= \frac{2}{3}(x-11)\sqrt{(x+1)} + c$$

(c) $\int (2x-1)(x+2)^3\, dx$

Solution

Let u = x+2 => x = u-2 and dx = du

$\therefore \int (2x-1)(x+2)^3\, dx$

$= \int [2(u-2) - 1u^3 du]$

$= \int (2u^4 - 5u^3)\, du$

$= \frac{2}{5}u^5 - \frac{5}{4}u^4 + c$

$= \frac{1}{20}u^4(8u-25) + c$

$= \frac{1}{20}(x+2)^4(8(x+2+-25) + c$

$= \frac{1}{20}(8x-9)(x+2)^4 + c$

(d) $\int \frac{sin\sqrt{x}}{\sqrt{x}} dx$

Solution

Let u = $\sqrt{x}$ => $x = u^2$ and dx= 2u

$\therefore \int \frac{sinu}{u}.2udx = 2\int sinudu$

$= -2$cosu + c

$= -2$cos$\sqrt{x}$ + c

Revision exercise 2

1. Integrate each of the following with respect to x using suitable substitution
   (a) x(x+3)^3
   $\left[\frac{1}{20}(4x - 3x + 3^4) + c\right]$
   (b) $x\sqrt{5-x}$
   $\left[-\frac{2}{15}\left(3x + 10\sqrt{(5-3)^3} + c\right)\right]$
   (c) $\frac{x-3}{(x+2)^2}$ $\left[\frac{5}{x+2} + In(x+2) + c\right]$
   (d) $\frac{x}{\sqrt{2x+1}}$ $\left[\frac{1}{3}(x-1)\sqrt{2x+1} + c\right]$
   (e) $(x-3)(5-2x)^4$
   $\left[\frac{1}{120}(31-10x)(5-2x)^5 + c\right]$
   (f) $\frac{x}{\sqrt{(x+1)^3}}$ $\left[\frac{2(x+2)}{\sqrt{x+1}} + c\right]$
   (g) $\frac{x+3}{(3-x)^2}$ $\left[\frac{6}{3-x} + In(3-x) + c\right]$
   (h) $x^2(x-1)^4$
   $\left[\frac{1}{105}(15x^2 + 5x + 1)(x-1)^5 + c\right]$
   (i) $x\sqrt{(1-x)^3}$
   $\left[-\frac{2}{35}(5x+2)\sqrt{(1-x)^5}\right]$

Integrations involving trigonometric functions

A. The double formulae, i.e.

- Cos2x = 2cos^{2}x – 1= 1 – 2sin^{2}x
- 1+ cos2x = 2cos^{2}x
- 1 – cos2x = 2sin^{2}x
- Sin2x =2sinxcos2

Example 4

Find the following integrals

(a) $\frac{tan\theta}{\sqrt{1+cos2\theta}} d\theta$

Solution

$\frac{tan\theta}{\sqrt{1+cos2\theta}} d\theta = \int \frac{tan\theta}{\sqrt{2cos^2\theta}} d\theta = \int \frac{sin\theta}{\sqrt{2}(cos^2\theta)} d\theta$

Let u = cosθ => du = -sinθdθ

- $d\theta = \frac{du}{-sin\theta}$

$\frac{1}{\sqrt{2}}\int \frac{sin\theta}{(cos^2\theta)} d\theta = \frac{1}{\sqrt{2}}\int \frac{sin\theta}{u^2}.\frac{du}{-sin\theta}$

$= -\frac{1}{\sqrt{2}}\int \frac{du}{u^2} = -\frac{1}{\sqrt{2}}\int u^{-2}du$

$= -\frac{1}{\sqrt{2}}(-u^{-1}) + c$

$= \frac{1}{\sqrt{2}cos\theta} + c$

(b) $\int \sqrt{1-cos4\theta}\, d\theta$

Solution

$\int \sqrt{1-cos4\theta}\, d\theta = \int \sqrt{2sin^2 2\theta} d\theta$

$= \sqrt{2}\int sin2\theta d\theta$

$= \frac{-\sqrt{2}}{2} cos2\theta + c$

(c) $\int sin3\theta cos3\theta d\theta = \frac{1}{2}\int 2sin3\theta cos3\theta d\theta$

$= \frac{1}{2}\int sin6\theta$

$= -\frac{1}{12}cos6\theta + c$

B. The factor formulae, i.e.

- $cos\alpha + cos\beta \;=\; 2cos\left(\frac{\alpha+\beta}{2}\right)cos\left(\frac{\alpha-\beta}{2}\right)$
- $cos\alpha - cos\beta \;=\; -2sin\left(\frac{\alpha+\beta}{2}\right)sin\left(\frac{\alpha-\beta}{2}\right)$
- $sin\alpha + sin\beta \;=\; 2sin\left(\frac{\alpha+\beta}{2}\right)cos\left(\frac{\alpha-\beta}{2}\right)$
- $sin\alpha - sin\beta \;=\; -2cos\left(\frac{\alpha+\beta}{2}\right)sin\left(\frac{\alpha-\beta}{2}\right)$

Example 5

Find the following integrals

(a) $\int cos2\theta cos\theta d\theta$

Solution

$$\int cos2\theta cos\theta d\theta = \frac{1}{2}\int 2cos2\theta cos\theta d\theta$$
$$= \frac{1}{2}\int (cos3\theta + cos\theta)d\theta$$
$$= \frac{1}{2}\left(\frac{1}{3}sin3\theta + sin\theta\right) + c$$
$$= \frac{1}{6}sin3\theta + \frac{1}{2}sin\theta + c$$

(b) $\int sin4\theta sin3\theta d\theta$

Solution

$$\int sin4\theta sin3\theta d\theta = \frac{1}{2}\int 2sin4\theta sin3\theta d\theta$$
$$= \frac{1}{2}\int (cos7\theta - cos\theta)d\theta$$
$$= \frac{1}{2}\left(\frac{1}{7}sin7\theta - sin\theta\right) + c$$
$$= \frac{1}{14}sin7\theta + \frac{1}{2}sin\theta + c$$

(c) $\int cos3\theta sin\theta d\theta$

Solution

$$\int cos3\theta sin\theta d\theta = \frac{1}{2}\int 2cos3\theta sin\theta d\theta$$
$$= \frac{1}{2}\int (sin4\theta - sin2\theta)d\theta$$
$$= \frac{1}{2}\left(-\frac{1}{4}cos4\theta + \frac{1}{2}cos2\theta\right) + c$$
$$= \frac{1}{4}cos2\theta - \frac{1}{8}cos4\theta + c$$

(d) $\int sin\frac{3}{2}\theta cos\frac{1}{2}\theta d\theta$

Solution

$$\int sin\frac{3}{2}\theta cos\frac{1}{2}\theta d\theta = \frac{1}{2}\int 2sin\frac{3}{2}\theta cos\frac{1}{2}\theta d\theta$$
$$= \frac{1}{2}\int (sin2\theta + sin\theta)d\theta$$
$$= \frac{1}{2}\left(-\frac{1}{2}cos2\theta - cos\theta\right) + c$$
$$= -\frac{1}{4}(cos2\theta + cos\theta) + c$$

(e) $\int sin2\theta cos\theta d\theta$

Solution

$$\int sin2\theta cos\theta d\theta = \frac{1}{2}\int 2sin2\theta cos\theta d\theta$$
$$= \frac{1}{2}\int (sin3\theta + sin\theta)d\theta$$
$$= \frac{1}{2}\left(-\frac{1}{3}cos3\theta - cos\theta\right) + c$$
$$= -\frac{1}{6}(cos2\theta + 3cos\theta) + c$$

Note

(i) The integral $\int sin2\theta cos2\theta d\theta$, where the angles are the same can be solved in two ways.

Method I: double angle formula

$$\int sin2\theta cos2\theta d\theta = \frac{1}{2}\int sin4\theta d\theta$$
$$= -\frac{1}{8}cos4\theta + c$$

Method II: the factor formula

$$\int sin2\theta cos2\theta d\theta = \frac{1}{2}\int (sin4\theta + sin0)d\theta$$
$$= -\frac{1}{8}cos4\theta + c$$

(ii) The integral of $\int sin4\theta cos2\theta$ where the angles are different, use method I because method II is inapplicable.

Revision exercise 3

1. Evaluate
   (a) $\int_0^{\frac{\pi}{2}} sin2xcosxdx$ $\left[\frac{2}{3}\right]$
   (b) $\int_0^{\frac{\pi}{6}} sin3xsinxdx$ [0.1083]
2. Integrate the following using appropriate substitution.
   (a) $\int 6xsin(x^2 - 4)\,dx$
   $[-3cos(x^2 - 4) + c]$
   (b) $\int 5xcos(5 - x^2)\,dx$
   $\left[-\frac{5}{2}sin(5 - x^2) + c\right]$
   (c) $\int 3x\sqrt{1 + x^2}\,dx$
   $\left[(1 + x^2)\sqrt{1 + x^2} + c\right]$
   (d) $\int 3x(x^2 + 6)^5 dx$ $\left[\frac{1}{4}(x^2 + 6)^6 + c\right]$
   (e) $\int \frac{x}{\sqrt{(2x^2-5)}}dx$ $\left[\frac{1}{2}\sqrt{(2x^2 - 5)}\right] + c$

Integrations of odd and even powers of trigonometric functions

(sinx, cosx, tanx, cotx, secx and cosecx)

Integration of trigonometric functions rose to odd powers

The Pythagoras theorem in trigonometry is handy namely

- $\cos^2x + \sin^2x = 1$
- $1 + \tan^2x = \sec^2x$
- $1 + \cot^2x = \mathrm{cosec}^2x$

Example 6

Integrate the following

(a) $\int cos^5xdx$

Solution

$\int cos^5xdx$=$\int cos^4xcosxdx$

$= \int (1 - sin^2x)^2 cosxdx$

$= \int (cosx - 2sin^2xcosx + sin^4xcosx)dx$

$= sinx - \frac{2}{3}sin^3x + \frac{1}{5}sin^5x + c$

(b) $\int cos^3 2xdx$

Solution

$\int cos^3 2xdx = \int sin^2 2xsin2xdx$

$= \int (1 - cos^2 2x)sin2xdx$

$= \int sin2x - cos^2 2csin2xdx$

$= -\frac{1}{2}cos2x + \frac{1}{2}\left(\frac{1}{3}\right)cos^3 2x + c$

$= \frac{1}{6}cos^3 2x - \frac{1}{2}cos2x + c$

(c) $\int_0^{\frac{\pi}{12}} cos^3 6xdx$

Solution

$\int_0^{\frac{\pi}{12}} cos^3 6xdx = \int_0^{\frac{\pi}{12}}(1 - sin^2 6x)cos6xdx$

$= \int_0^{\frac{\pi}{12}}(cos6x - sin^2 6xcos6x)\, dx$

$= \left[\frac{1}{6}sin6x - \frac{1}{6}\left(\frac{1}{3}\right)sin^3 6x\right]_0^{\frac{\pi}{12}}$

$= \frac{1}{6}sin\frac{\pi}{2} - \frac{1}{18}\left(sin\frac{\pi}{2}\right)^3 = \frac{1}{9}$

(d) $\int sin^3xdx$

Solution

$\int sin^3xdx = \int sinx(1 - cos^2x)dx$

$= \int (sinx - sixcos^2x)dx$

$= -cosx - \left(-\frac{1}{3}cos^3x\right) + c$

$= \frac{1}{3}cos^3x - cosx$

(e) $\int tan^3xdx$

Solution

$\int tan^3xdx = \int tanxtan^2x\, dx$

$= \int tanx(sec^2x - 1)dx$

$= \int tanxsec^2xdx - \int tanx\, dx$

By inspection

$\frac{d}{dx}(In(cosx) = -tanx$

$=> \int -tanxdx = In(cosx)$

Also

$\frac{d}{dx}(tan^2x) = 2tanxsec^2x$

$=> \int tanxsec^2xdx = \frac{1}{2}tan^2x$

$\therefore \int tan^3xdx = \frac{1}{2}tan^2x + In(cosx) + c$

Or

$\frac{d}{dx}In(secx) = tanx$

$=> \int tanxdx = In(secx)$

Also $\frac{d}{dx}(sec^2x) = 2tanxsec^2x$

$=> \int tanxsec^2xdx = \frac{1}{2}sec^2x$

$\therefore \int tan^3xdx = \frac{1}{2}sec^2x + In(secx) + c$

(f) $\int tan^5 2xdx$

Solution

$\int tan^5 2xdx = \int tan2xtan^4 2xdx$

$= \int tan2x(sec^2 2x - 1)^2dx$

$= \int tan2x(sec^4 2x - 2sec^2 2x) - 1)dx$

$= \int t(an2xsec^4 2x - 2tan2xsec^2 2x - tan2x)dx$

$= \frac{1}{8}sec^4 2x - \frac{1}{2}sec^2 2x + \frac{1}{2}In(sec2x) + c$

Or

$\int tan^5 2xdx = \int tan^3 2xtan^2 2xdx$

$= \int tan^3 2x(sec^2 2x - 1)dx$

$= \int tan^3 2xsec^2 2xdx - \int tan^3 2xdx$

$= \frac{1}{8}tan^4 2x - \int tan2x(sec^2 2x - 1)dx$

$= \frac{1}{8}tan^4 2x - \int tan2xsec^2 2xdx + \int tan2x\, dx$

$= \frac{1}{8}tan^4 2x - \frac{1}{4}tan^2 2x + \frac{1}{2}In(sec2x) + c$

(g) $\int cot^3xdx$

Solution

$\int cot^3xdx = \int cotxcot^2xdx$

$= \int cotx(cosec^2x - 1)dx$

$= \int cotxcosec^2xdx - \int cotxdx$

$= \frac{1}{2}cot2x - In(sinx) + c$

Note: the integration of odd powers of secx and cosec x are done using integration by parts.

Integration of trigonometric functions rose to even powers

These are worked out using double angle formulae.

Example 7

Find the integrals of the following

(a) $\int cos^2xdx$

Solution

$\int cos^2xdx = \int \frac{1}{2}(1 + cos2x)dx$

$= \int \left(\frac{1}{2} + \frac{1}{2}cos2x\right) dx$
$= \frac{x}{2} + \frac{1}{4}sin2x + c$

(b) $\int cos^4 xdx$

Solution

$\int cos^4 xdx = \int (cos^2 x)^2 dx$
$= \int \frac{1}{4}\left((1 + cos2x)\right)^2 dx$
$= \int \frac{1}{4}(1 + 2cos2x + cos^2 2x)dx$
$= \int \left(\frac{1}{4} + \frac{1}{2}cos2x\right) dx + \frac{1}{4}\int \frac{1}{2}(1 + cos4x)dx$
$= \frac{1}{4}x + \frac{1}{4}sin2x + \frac{1}{8}x + \frac{1}{32}sin4x + c$

(c) $\int sin^2 3xdx$

Solution

$\int sin^2 3xdx = \frac{1}{2}\int (1 - cos6x)dx$
$= \frac{1}{2}x - \frac{1}{12}sin6x + c$

(d) $\int tan^4 xdx$

Solution

$\int tan^4 xdx = \int tan^2 x.tan^2 xdx$
$= \int tan^2 x(sec^2 x - 1)dx$
$= \int tan^2 xsec^2 xdx - \int tan^2 x\, dx$
$= \frac{1}{3}tan^3 x - \int (sec^2 - 1)\, dx$
$= \frac{1}{3}tan^3 x - tanx + x + c$

(e) $\int sec^4 xdx$

Solution

$\int sec^4 xdx = \int sec^2 x.sec^2 xdx$
$= \int sec^2 x(tan^2 x + 1)dx$
$= \int (sec^2 tan^2 x + sec^2 x)dx$
$= \frac{1}{3}tan^3 x + tanx + c$

(f) $\int cosec^2 \left(\frac{1}{2}x\right) dx$

Solution

$\int cosec^2 \left(\frac{1}{2}x\right) dx = -2cot\frac{1}{2}x + c$

(g) $\int cot^4 xdx$

Solution

$\int cot^4 xdx = \int cot^2 x.cot^2 xdx$
$= \int cot^2 x(cosec^2 x - 1)dx$
$= \int cot^2 xcosec^2 x - \int cot^2 x\, dx$
$= \frac{1}{3}cot^3 x - \int (cosec^2 x - 1)dx$
$= \frac{1}{3}cot^3 x + cotx + x + c$

Exercise 4

1. Integrate each of the following
   (a) $sinxcos^5x$ $\left[-\frac{1}{6}cos^6 x + c\right]$
   (b) $cos^3 4x$ $\left[\frac{1}{4}sin4x - \frac{1}{2}sin^3 4x + c\right]$
   (c) $sin^3 xcos^2 x$ $\left[-\frac{1}{3}cos^3 x + \frac{1}{5}cos^5 x + c\right]$
   (d) $cos^3 xsin^4 x$ $\left[\frac{1}{5}sin^5 x - \frac{1}{7}sin^7 x + c\right]$
   (e) $cos^3 2xsin^2 2x$ $\left[\frac{1}{6}sin^3 2x - \frac{1}{10}sin^5 2x + c\right]$
   (f) $sin2xsin^2 x$ $\left[\frac{1}{2}sin^4 + c\right]$
   (g) $cos^3 xsin^3 x$ $\left[\frac{1}{4}sin^4 x - \frac{1}{6}sin^6 x + c\right]$
2. Integrate each of the following
   (a) $cot^2 2x$ $\left[\frac{1}{2}x + \frac{1}{8}sin4x + c\right]$
   (b) $cos^2 3x$ $\left[\frac{1}{2}x + \frac{1}{8}sin6x + c\right]$
   (c) $sin^3 xcos^2 x$ $\left[\frac{1}{2}x + \frac{1}{16}sin8x + c\right]$
   (d) $cos^2 6x$ $\left[\frac{1}{2}x + \frac{1}{24}sin12x + c\right]$
   (e) $sin^2\left(\frac{1}{2}x\right)$ $\left[\frac{1}{2}x - \frac{1}{2}sinx + c\right]$
   (f) $cos^4 x$ $\left[\frac{3}{8}x + \frac{1}{4}sin2x + \frac{1}{32}sin4x + c\right]$
   (g) $sin^4 2x$ $\left[\frac{3}{8}x - \frac{1}{4}sin4x + \frac{1}{64}sin8x + c\right]$
3. Integrate each of the following
   (a) $cot^2 x$ $[-cotx - x + c]$
   (b) $tan^2 2x$ $\left[\frac{1}{2}tan2x + c\right]$
   (c) $sec^2 xtan^3 x$ $\left[\frac{1}{4}tan^4 x + c\right]$
   (d) $cosec^2 2xcot^4 2x$ $\left[-\frac{1}{10}cot^5 2x\right]$
   (e) $tan^3 x$ $\left[\frac{1}{2}tan^2 x + In(cosx)\right]$
   (f) $cot^4 3x$ $\left[-\frac{1}{9}cot^3 3x + \frac{1}{3}cot3x + c\right]$
   (g) $tan^4 5x$ $\left[\frac{1}{15}tan^3 5x - \frac{1}{5}tan5x + x + c\right]$
   (h) $tan^5 2x$
   $\left[\frac{1}{8}tan^4 2x - \frac{1}{4}tan^2 2x + \frac{1}{2}In(sec2x) + c\right]$
   (i) $cosecxcot^3 x$ $\left[cosecx - \frac{1}{3}cosec^3 x + c\right]$
   (j) $tan^5 xsecx$
   $\left[secx - \frac{2}{3}sec^3 x + \frac{1}{5}sec^5 c + c\right]$
4. Find the Integral of each of the following
   (a) $cosec^2 x$ $[-cotx + c]$
   (b) $sec^2 3x$ $\left[\frac{1}{3}tan3x + c\right]$
   (c) $cosec^2\left(\frac{1}{3}x\right)$ $\left[-3cot\left(\frac{1}{3}x\right) + c\right]$
   (d) $sec^4 x$ $\left[\frac{1}{3}tan^3 x + tanx + c\right]$

(e) cosec^{4}5x $\left[-\frac{1}{15}cot^3 5x - \frac{1}{5}cot5x + c\right]$

(f) sec^{4}3x $\left[\frac{1}{9}tan^3 3x + \frac{1}{3}tan3x + c\right]$

(g) sec^{6}x $\left[\frac{1}{5}tan^5 x + \frac{2}{3}tan^3 x + tanx + c\right]$

Integration involving inverse trigonometric functions

A. From $\frac{d}{dx}\left(sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}$

- $\int \frac{1}{\sqrt{1-x^2}} = sin^{-1}x + c$

This result enables the integration of the form $\int \frac{1}{\sqrt{a^2-b^2x^2}}dx$ to be workout, i.e.

$$\int \frac{1}{\sqrt{a^2-b^2x^2}}dx = \int \frac{1}{a\sqrt{1-\frac{b^2x^2}{a^2}}}dx = \int \frac{1}{a\sqrt{1-\left(\frac{bx}{a}\right)^2}}dx$$

Let $\frac{bx}{a} = sinu; dx = \frac{a}{b}cosudu$

- $\int \frac{1}{\sqrt{a^2-b^2x^2}}dx = \int \frac{1}{a\sqrt{1-sin^2u}}.\frac{a}{b}cosudu$
$= \int \frac{1}{acosu}.\frac{a}{b}cosudu$
$= \frac{1}{b}\int du = \frac{1}{b}u + c$
$= \frac{1}{b}sin^{-1}\left(\frac{bx}{a}\right) + c$

Example 8

Integrate the following

(a) $\int \frac{1}{\sqrt{4-9x^2}}dx$

Solution

$\int \frac{1}{\sqrt{4-9x^2}}dx = \int \frac{1}{2\sqrt{\left(1-\left(\frac{3x}{2}\right)^2\right)}}dx$

Let sinu =$\frac{3x}{2}$, dx =$\frac{2}{3}cosudu$

- $\int \frac{1}{\sqrt{4-9x^2}}dx = \int \frac{1}{2\sqrt{(1-sin^2u)}}.\frac{2}{3}cosudu$
$= \int \frac{1}{2cosu}.\frac{2}{3}cosudu$
$= \frac{1}{3}\int du = \frac{1}{3}u + c$
$= \frac{1}{3}sin^{-1}\left(\frac{3x}{2}\right) + c$

(b) $\int \frac{1}{\sqrt{1-16x^2)}}dx$

Solution

$\int \frac{1}{\sqrt{1-16x^2}}dx = \int \frac{1}{\sqrt{(1-(4x)^2)}}dx$

Let sinu =$4x$, dx =$\frac{1}{4}cosudu$

- $\int \frac{1}{\sqrt{(1-(4x)^2)}}dx = \int \frac{1}{\sqrt{(1-sin^2u)}}.\frac{1}{4}cosudu$
$= \int \frac{1}{cosu}.\frac{1}{4}cosudu$
$= \frac{1}{4}\int du = \frac{1}{4}u + c$
$= \frac{1}{4}sin^{-1}(4x) + c$

(c) $\int \frac{4x^2}{\sqrt{1-x^6)}}dx$

Solution

$\int \frac{4x^2}{\sqrt{1-x^6)}}dx = \int \frac{4x^2}{\sqrt{(1-(x^3)^2)}}dx$

Let sinu =x^3, dx =$\frac{1}{3x^2}cosudu$

- $\int \frac{4x^2}{\sqrt{(1-(x^3)^2)}}dx = \int \frac{4x^3}{\sqrt{(1-sin^2u)}}.\frac{1}{3x^2}cosudu$
$= \int \frac{1}{cosu}.\frac{4}{3}cosudu$
$= \frac{4}{3}\int du = \frac{4}{3}u + c$
$= \frac{4}{3}sin^{-1}(x^3) + c$

(d) $\int \frac{1}{\sqrt{36-4x^2}}dx$

Solution

$\int \frac{1}{\sqrt{36-4x^2}}dx = \int \frac{1}{6\sqrt{\left(1-\left(\frac{2x}{6}\right)^2\right)}}dx$

Let sinu =$\frac{2x}{6} = \frac{1}{3}x$, dx =$3cosudu$

- $\int \frac{1}{\sqrt{36-4x^2}}dx = \int \frac{1}{6\sqrt{(1-sin^2u)}}.3cosudu$
$= \int \frac{1}{6cosu}.3cosudu$
$= \frac{1}{2}\int du = \frac{1}{2}u + c$
$= \frac{1}{3}sin^{-1}\left(\frac{x}{3}\right) + c$

(e) $\int \frac{1}{\sqrt{25-9x^2}}dx$

Solution

$\int \frac{1}{\sqrt{25-9x^2}}dx = \int \frac{1}{5\sqrt{\left(1-\left(\frac{3x}{5}\right)^2\right)}}dx$

Let sinu =$\frac{3x}{5}$, dx =$\frac{5}{3}cosudu$

- $\int \frac{1}{\sqrt{25-9x^2}}dx = \int \frac{1}{5\sqrt{(1-sin^2u)}}.\frac{5}{3}cosudu$
$= \int \frac{1}{5cosu}.\frac{5}{3}cosudu$
$= \frac{1}{3}\int du = \frac{1}{3}u + c$
$= \frac{1}{3}sin^{-1}\left(\frac{3x}{5}\right) + c$

(f) $\int \frac{1}{\sqrt{3-2x-x^2}}dx$

Solution

$3 - 2x - x^2 = -(x^2 + 2x - 3)$

By completing squares

$-(x^2+2x-3) = -[(x+1)^2-3-1]$

$= 4-(x+1)^2$

$\int \frac{1}{\sqrt{3-2x-x^2}}dx = \int \frac{1}{\sqrt{4-(x+1)^2}}dx$

$\int \frac{1}{\sqrt{4-(x+1)^2}}dx = \int \frac{1}{2\sqrt{1-\left(\frac{x+1}{2}\right)^2}}$

Let sinu $=\frac{x+1}{2}$, dx $=2cosudu$

- $\int \frac{1}{\sqrt{4-(x+1)^2}}dx = \int \frac{1}{2\sqrt{(1-sin^2u)}}.2cosudu$

$= \int \frac{1}{2cosu}.2cosudu$

$= \int du = u + c$

$= sin^{-1}\left(\frac{x+1}{2}\right) + c$

B. From $\frac{d}{dx}\left(tan^{-1}x\right) = \frac{1}{1+x^2}$

- $\int \frac{1}{1+x^2}dx = tan^{-1}x + c$

This result enables the integration of the form $\int \frac{1}{\sqrt{a^2+b^2x^2}}dx$ to be workout, i.e

$\int \frac{1}{a^2+b^2x^2}dx = \int \frac{1}{a^2\left(1+\frac{b^2x^2}{a^2}\right)}dx$

$= \int \frac{1}{a^2\left(1+\left(\frac{bx}{a}\right)^2\right)}dx$

Let $\frac{bx}{a}$= tanu, dx=$\frac{a}{b}sec^2udu$

$=> \int \frac{1}{a^2\left(1+\left(\frac{bx}{a}\right)^2\right)}dx = \int \frac{1}{a^2(1+tan^2u)}.\frac{a}{b}sec^2udu$

$= \frac{1}{ab}\int \frac{1}{sec^2u}.sec^2udu$

$= \frac{1}{ab}\int du = \frac{1}{ab}u + c$

$= \frac{1}{ab}tan^{-1}\left(\frac{bx}{a}\right) + c$

Example 9

Find

(a) $\int \frac{1}{9+25x^2}dx$

Solution

Comparing $\int \frac{1}{9+25x^2}$ with$\int \frac{1}{a^2+b^2x^2}dx$

a=3 and b =5

$\int \frac{1}{9+25x^2} = \frac{1}{3x5}\left[tan^{-1}\left(\frac{5}{3}x\right)\right] + c$

$= \frac{1}{15}\left[tan^{-1}\left(\frac{5}{3}x\right)\right] + c$

(b) $\int \frac{1}{5+9x^2}dx$

Solution

Comparing $\int \frac{1}{5+9x^2}$ with$\int \frac{1}{a^2+b^2x^2}dx$

a=$\sqrt{5}$ and b =3

$\int \frac{1}{9+25x^2} = \frac{1}{\sqrt{5}x3}\left[tan^{-1}\left(\frac{3}{\sqrt{5}}x\right)\right] + c$

$= \frac{1}{3\sqrt{5}}\left[tan^{-1}\left(\frac{3\sqrt{5}}{5}x\right)\right] + c$

(c) $\int \frac{1}{1+2x+4x^2}dx$

Solution

$1+2x+4x^2 = 4x^2+2x+1$

$= 4\left(x^2+\frac{1}{2}x+\frac{1}{4}\right)$

$= 4\left[\left(x+\frac{1}{4}\right)^2+\frac{1}{4}-\frac{1}{16}\right]$

$= 4\left(x+\frac{1}{4}\right)^2+\frac{3}{4}$

$\int \frac{1}{1+2x+4x^2dx} = \int \frac{1}{4\left(x+\frac{1}{4}\right)^2+\frac{3}{4}}dx = \int \frac{1}{\frac{3}{4}+4\left(x+\frac{1}{4}\right)^2}$dx

Comparing $\int \frac{1}{\frac{3}{4}+4\left(x+\frac{1}{4}\right)^2}$ with$\int \frac{1}{a^2+b^2x^2}dx$

a=$\frac{\sqrt{3}}{2}$ and b =2

$\int \frac{1}{\frac{3}{4}+4\left(x+\frac{1}{4}\right)^2} = \frac{1}{\frac{\sqrt{3}}{2}x2}\left[tan^{-1}\left(\frac{2\left(x+\frac{1}{4}\right)}{\frac{\sqrt{3}}{2}}x\right)\right] + c$

$= \frac{\sqrt{3}}{3}\left[tan^{-1}\left(\frac{4x+1}{\sqrt{3}}x\right)\right] + c$

(d) $\int_0^1 \frac{1}{3x^2+6x+4}\ dx$

Solution

$3x^2+6x+4 = 3\left(x^2+2x+\frac{4}{3}\right)$

$= 3\left[(x+1)^2+\frac{4}{3}-1\right]$

$= 3\left[(x+1)^2+\frac{1}{3}\right]$

$= 3(x+1)^2+1$

$\int \frac{1}{3x^2+6x+4}dx = \int \frac{1}{1+3(x+1)^2}$

Comparing $\int \frac{1}{1+3(x+1)^2}$ with$\int \frac{1}{a^2+b^2x^2}dx$

a=1 and b =$\sqrt{3}$

$\int \frac{1}{1+3(x+1)^2} = \frac{1}{\sqrt{3}}\left[tan^{-1}\left(\frac{\sqrt{3}(x+1)}{1}x\right)\right] + c$

$= \frac{1}{\sqrt{3}}\left[tan^{-1}\left(\sqrt{3}(x+1)\right)\right] + c$

$\therefore \int_0^1 \frac{1}{3x^2+6x+4}\ dx = \left[\frac{1}{\sqrt{3}}\left[tan^{-1}\left(\sqrt{3}(x+1)\right)\right]\right]_0^1$

$= \frac{1}{\sqrt{3}}\left[tan^{-1}\left(2\sqrt{3}\right) - tan^{-1}\left(\sqrt{3}\right)\right]$

$= 0.44$

Revision exercise 5

Find

(a) $\int \frac{1}{9+x^2}dx$ $\left[\frac{1}{3}tan^{-1}\left(\frac{1}{3}x\right)+c\right]$

(b) $\int_{-2}^{2}\frac{1}{4+x^2}dx$ [0.7854]

(c) $\int_{-\sqrt{3}}^{3}\frac{1}{x^2+3}dx$ [1.833]

(d) $\int \frac{1}{4-x^2}dx$ $[sin^{-1}x+c]$

(e) $\int \frac{1}{\sqrt{16-x^2}}dx$ $\left[sin^{-1}\left(\frac{x}{4}\right)+c\right]$

(f) $\int \frac{1}{\sqrt{49-x^2}}dx$ $\left[sin^{-1}\left(\frac{x}{7}\right)+c\right]$

(g) $\int \frac{1}{\sqrt{25-4x^2}}dx$ $\left[\frac{1}{2}sin^{-1}\left(\frac{2}{4}x\right)+c\right]$

(h) $\int \frac{3}{9+x^2}dx$ $\left[tan^{-1}\left(\frac{x}{3}\right)+c\right]$

(i) $\int \frac{1}{25+x^2}dx$ $\left[\frac{1}{5}tan^{-1}\left(\frac{x}{5}\right)+c\right]$

(j) $\int \frac{2}{100+9x^2}dx$ $\left[\frac{1}{15}tan^{-1}\left(\frac{3}{10}x\right)+c\right]$

(k) $\int \frac{1}{3-2x+x^2}dx$ $\left[\frac{1}{\sqrt{2}}tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+c\right]$

(l) $\int \frac{1}{x^2\sqrt{4-x^2}}dx$ $\left[-\frac{1}{4}cotsin^{-1}\left(\frac{x}{2}\right)+c\right]$

Integration of exponential and logarithmic functions.

A. From $\frac{d}{dx}e^x = e^x$

- $\int e^x dx = e^x + c$

Example 10

Find

(a) $\int xe^{x^2}dx$

Solution

Let u = $3x^2$ => du = 6xdx i.e. xdx =$\frac{1}{6}du$

$\int xe^{x^2}dx = \frac{1}{6}\int e^u du = \frac{1}{6}e^u + c$

$\therefore \int xe^{x^2}dx = \frac{1}{6}e^{x^2} + c$

(b) $\int secxtanxe^{secx}dx$

Solution

Let u = secx => du=secxtanxdx

$\int secxtanxe^{secx}dx = \int e^u du = e^u + c$

$\therefore \int secxtanxe^{secx}dx = e^{secx} + c$

(c) $\int \frac{e^{cotx}}{sin^2x}dx$

Solution

Let u = cotx

- Du = -cosec^{2}x = $-\frac{1}{sin^2x}$

$\int \frac{e^{cotx}}{sin^2x}dx = -\int e^u du = -e^u + c$

$\therefore \int \frac{e^{cotx}}{sin^2x}dx = -e^{cotx} + c$

(d) $\int \frac{e^{-\frac{1}{x}}}{x^2}dx$

Solution

Let u = $-\frac{1}{x}$ => du = $\frac{1}{x^2}dx$

$\int \frac{e^{-\frac{1}{x}}}{x^2}dx = \int e^u du = -e^u + c$

$\therefore \int \frac{e^{-\frac{1}{x}}}{x^2}dx = e^{-\frac{1}{x}} + c$

B. From $\frac{d}{dx}(\boldsymbol{Inx}) = \frac{1}{x}$

- $\int \frac{1}{x}\boldsymbol{dx} = \boldsymbol{Inx + c \equiv InAx}$

This result shows that

$\int \frac{f'(x)}{f(x)}\boldsymbol{dx} = \boldsymbol{In[f(x)] + c}$ **i.e.**

- $\int \boldsymbol{cot2xdx} = \int \frac{cos2x}{sin2x}\boldsymbol{dx}$

$= \frac{1}{2}\boldsymbol{In(sin2x) + c}$

- $\int \frac{a}{b+cx}\boldsymbol{dx} = \frac{a}{c}\boldsymbol{In(b+cx) + k}$

Example 11

Find

(a) $\int \frac{1}{3x+4}dx$

Solution

Let u = 3x+4 => du 3dx i.e. dx=$\frac{1}{3}du$

$\therefore \int \frac{1}{3x+4}dx = \frac{1}{3}\int du = \frac{1}{3}Inu + c$

$= \frac{1}{3}In(3x+4) + c$

(b) $\int \frac{x}{1-5x^2}dx$

Solution

Let u = $1-5x^2$

=> du -10xdx i.e. dx=$-\frac{1}{10x}du$

$\therefore \int \frac{x}{1-5x^2}dx = \frac{1}{10}\int \frac{1}{u}du = \frac{1}{10}Inu + c$

$= \frac{1}{3}In(1-5x^2) + c$

(c) $\int tan^3xdx$

Solution

$\int tan^3xdx = \int tan^2xtanxdx$ (an odd power)

$= \int (sec^2x - 1)tanxdx$

$= \int sec^2xtanxdx - \int tanxdx$

For $\int sec^2xtanxdx$

Let u = tanx, => du = sec^{2}xdx

$\therefore \int sec^2xtanxdx = \int udu = \frac{1}{2}tan^2x + c$

For$\int tanxdx = \int \frac{sinx}{cosx}dx$

Let u =cosx, => du = -sinxdx

$\therefore \int tanxdx = -\int \frac{1}{u}du = -Inu + c$

= -In(cosx)+c =In(secx)+c

$\therefore \int tan^3xdx = \frac{1}{2}tan^2x + In(secx) + c$

(d) $\int_0^1 \frac{x+1}{3+4x^2}dx$

Solution

$\int_0^1 \frac{x+1}{3+4x^2}dx = \int_0^1 \frac{1}{3+4x^2}dx + \int_0^1 \frac{1}{3+4x^2}dx$

$= \left[\frac{1}{8}In(3 + 4x^2 + \frac{1}{2\sqrt{3}}tan^{-1}\left(\frac{2}{\sqrt{3}}\right)\right]_0^1$

$= \frac{1}{8}In\left(\frac{7}{3}\right) + \frac{1}{2\sqrt{3}}tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$

(e) $\int_a^{2a} \frac{x^3}{x^4+a^4}dx$

Solution

$\int_a^{2a} \frac{x^3}{x^4+a^4}dx = \frac{1}{4}[In(x^4 + a^4]_a^{2a}$

$= \frac{1}{4}(In17a^4 - In2a^4)$

$= \frac{1}{4}In\left(\frac{17}{2}\right)$ =0.535

C. From $\frac{d}{dx}a^x = a^xIna$

⇨ $\int a^xdx = \frac{1}{Ina}a^x + c$

It follows that $\int 2^xdx = \frac{2^x}{In2} + c$

Example 12

Integrate

(a) $\int x^2 2^{3x^2}dx$

Solution

Let u = 3x^3, => du = $9x^2$ i.e. $x^2dx = \frac{1}{9}du$

$\int x^2 2^{3x^2}dx = \frac{1}{9}\int 2^udu = \frac{1}{9}\frac{2^u}{In2} + c$

$= \frac{1}{9}\frac{2^{3x^2}}{In2} + c$

(b) $\int cosx.5^{sinx}dx$

Solution

Let u = sinx, => du = cosxdx

$\int cosx.5^{sinx}dx = \int 5^udu = \frac{5^u}{In5} + c$

$= \frac{5^{sinx}}{In5} + c$

(c) $\int \frac{3^{cotx}}{sin^2x}dx$

Solution

Let u = cotx, => du=-cosec^{2}x

$\int \frac{3^{cotx}}{sin^2x}dx = -\int 3^{cotx}cosec^2xdx$

$= \int 3^udu = \frac{3^u}{In3} + c$

$= \frac{3^{cotx}}{In3} + c$

Revision exercise 6

1. Find the following integrals

(a) $\int e^x(3 + e^x)^2dx$ $\left[\frac{1}{3}(3 + e^x)^3 + c\right]$

(b) $\int 2e^x(e^x - 4)^3dx$ $\left[\frac{1}{2}(e^x - 4)^4 + c\right]$

(c) $\int \frac{4e^{-2x}}{(1+e^{-2x})^2}dx$ $\left[\frac{2}{1+e^{-2x}} + c\right]$

(d) $\int \frac{(e^{-x}+7)^2}{e^x}dx$ $\left[-\frac{1}{3}(e^{-x} + 7)^3 + c\right]$

(e) $\int e^x\sqrt{4 + e^x}\,dx$ $\left[\frac{2}{3}\sqrt{(4 + e^x)^3 + c}\right]$

(f) $\int e^{5x}\sqrt{e^{5x} + 2}\,dx$ $\left[\frac{2}{15}\sqrt{(e^{5x} + 2)^3} + c\right]$

(g) $\int \frac{e^{3x}}{\sqrt{e^{3x}-1}}$ $\left[\frac{2}{3}\sqrt{e^{3x} - 1} + c\right]$

(h) $\int \frac{1}{2e^x\sqrt{1-e^{-x}}}dx$ $\left[\sqrt{1 - e^{-x}} + c\right]$

(i) $\int 5^xdx$ $\left[\frac{5^x}{In5} + c\right]$

(j) $\int 3^{2x}$ $\left[\frac{3^{2x}}{In9} + c\right]$

(k) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}$ $\left[2e^{\sqrt{x}} + c\right]$

(l) $\int x^2e^{x^3}$ $\left[\frac{1}{3}e^{x^3}\right]$

(m) $\int 4^xdx$ $\left[\frac{4x}{In4}\right]$

(n) $\int x10^xdx$ $\left[\frac{x10^x}{In10} - \frac{10^x}{(In10)^2} + c\right]$

2. Evaluate

(a) $\int_1^3 e^xdx$ $[e(e^2 - 1)]$

(b) $\int_0^3 e^{-x}dx$ $\left[1 - \frac{1}{e^3}\right]$

(c) $\int_1^2 2e^{(2x+1)}$ $[e^3(e^2 - 1)]$

(d) $\int_{-1}^1 2e^{(1-2x)}$ $\left[e^3 - \frac{1}{e}\right]$

(e) $\int_0^1\left(4xe^{x^2} + 1\right)dx$ $[2e - 1]$

Integration involving partial fractions

There are three established types of partial fractions depending on the nature of the denominator.

A. Denominators with linear factors e.g. 3x – 1, x + 2 and 3x – 4.

Each linear factor (ax + b) in the denominator has a corresponding partial fraction of the form $\frac{c}{(ax+b)}$, where a, b and c are constants.

Example 13

(a) Express each of the following in partial fraction. Hence find the integral of each with respect to x.

(i) $\frac{x-1}{(x+1)(x-2)}$

Solution

Let $\frac{x-1}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$

Multiplying by (x+1)(x-2)

⇨ x – 1 = A(x – 2) + B(x+1)

then we find the values of A and B

Putting x = 2: 1 = 3B, => B = $\frac{1}{3}$

Putting x = -1: -2 = -3A, => A = $\frac{2}{3}$

$$\therefore \frac{x-1}{(x+1)(x-2)} = \frac{\frac{2}{3}}{(x+1)} + \frac{\frac{2}{3}}{(x-2)}$$

$$= \frac{2}{3(x+1)} + \frac{1}{3(x-2)}$$

Hence,

$$\int \frac{x-1}{(x+1)(x-2)}dx = \frac{2}{3}\int \frac{1}{(x+1)}dx + \frac{1}{3}\int \frac{1}{(x-2)}dx$$

$$= \frac{2}{3}In(x+1) + \frac{1}{3}In(x-2) + c$$

$$= \frac{2}{3}In(x+1)^2(x-2) + c$$

(ii) $\frac{1}{x^3-9x}$

Solution

$$\frac{1}{x^3-9x} = \frac{1}{x(x^2-9)} = \frac{1}{x(x-3)(x+3)}$$

⇨ $\frac{1}{x^3-9x} = \frac{A}{x} + \frac{B}{(x-3)} + \frac{C}{(x+3)}$

Multiplying through with $x(x-3)(x+3)$

1= $A(x^2-9) + B(x^2+3x) + C(x^2-3x)$

Putting x = 0; 1 = -9A => $A = -\frac{1}{9}$

Putting x = 3; 1 = 18B => $B = \frac{1}{18}$

Putting x = -3; 1 = 18C => $C = \frac{1}{18}$

⇨ $\frac{1}{x^3-9x} = -\frac{1}{9x} + \frac{1}{18(x-3)} + \frac{1}{18(x+3)}$

Hence,

$$\int \frac{1}{x^3-9x}dx$$

$$= -\frac{1}{9}\int \frac{1}{x}dx + \frac{1}{18}\int \frac{1}{(x-3)}dx + \frac{1}{18}\int \frac{1}{(x+3)}dx$$

$$=-\frac{1}{9}Inx + \frac{1}{18}In(x+3) + \frac{1}{18}In(x-3)+c$$

$$=\frac{1}{18}(In(x+3) + In(x-3) - 2Inx) + c$$

$$=\frac{1}{18}\left[In\frac{(x+3)(x-3)}{x^2}\right] + c$$

(iii) $\frac{2x+1}{(x-1)(3x^2+7x+2)}$

Solution

$$\frac{2x+1}{(x-1)(3x^2+7x+2)} = \frac{2x+1}{(x-1)(x+2)(3x+1)}$$

$$\frac{2x+1}{(x-1)(3x^2+7x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{1}{(3x+1)}$$

Multiplying by $(x-1)(x+2)(3x+1)$

2x + 1= A(x+2)(3x+1)+B(x-1)(3x+1)+C(x-1)(x+2)

Putting x = 1; 3 = 12A => $A = \frac{1}{4}$

Putting x = -2; -3 = 15B => $B = -\frac{1}{5}$

Putting x = $-\frac{1}{3}$; $\frac{1}{3} = -\frac{20}{9}$C => $C = -\frac{3}{20}$

$$\therefore \frac{2x+1}{(x-1)(3x^2+7x+2)} = \frac{1}{4(x-1)} - \frac{1}{5(x+2)} - \frac{3}{20(3x+1)}$$

Hence,

$$\int \frac{2x+1}{(x-1)(3x^2+7x+2)}dx$$

$$= \frac{1}{4}\int \frac{1}{(x-1)}dx - \frac{1}{5}\int \frac{1}{(x+2)}dx - \frac{3}{20}\int \frac{1}{(3x+1)}dx$$

$$= \frac{1}{4}In(x-1) - \frac{1}{5}In(x+2) - \frac{3}{20}In(3x+1)$$

$$= \frac{1}{20}In\frac{(x-1)^5}{(x+2)^4(3x+1)^3}$$

(iv) $\frac{2x^2-x+1}{(x^2-1)(x+2)}$

Solution

$\frac{2x^2-x+1}{(x^2-1)(x+2)} = \frac{2x^2-x+1}{(x+1)(x-1)(x+2)}$

⇨ $\frac{2x^2-x+1}{(x^2-1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x+2)}$

Multiplying through by (x+1)(x-1)(x+2)

$2x^2 - x+1= A(x-1)(x+2)+B(x+1)(x+2)+C(x+1)(x-1)$

Putting x = -1; 4 = -2A => $A = -2$

Putting x = 1; 2 = 6B => $B = \frac{1}{3}$

Putting x = -2; 11 = 3C => $C = \frac{11}{3}$

$\therefore \frac{2x^2-x+1}{(x^2-1)(x+2)} = \frac{1}{3(x-1)} - \frac{2}{(x+1)} + \frac{11}{3(x+2)}$

Hence,

$\int \frac{2x^2-x+1}{(x^2-1)(x+2)} dx$

$= \frac{1}{3}\int \frac{1}{(x-1)} dx - 2\int \frac{1}{(x+1)} dx + \frac{11}{3}\int \frac{1}{(x+2)} dx$

$= \frac{1}{3}In(x-1) - 2In(x+1) + \frac{11}{3}In(x+2) + c$

(b) Evaluate $\int_1^3 \frac{x^2+1}{x^3+4x^2+3x} dx$

Solution

$\frac{x^2+1}{x^3+4x^2+3x} = \frac{x^2+1}{x(x+1)(x+3)}$

Let $\frac{x^2+1}{x^3+4x^2+3x} = \frac{A}{x} + \frac{B}{(x-3)} + \frac{C}{(x+3)}$

Multiplying with x(x – 3)(x + 3)

$x^2+ 1 = A(x – 3)(x+3)+ B(x)(x+3)+C(x)(x-3)$

Putting x = 0; 1 = 3A => $A = \frac{1}{3}$

Putting x = -1; 2 = -2B => $B = -1$

Putting x = -3; 10 = 6C => $C = \frac{5}{3}$

$\therefore \frac{x^2+1}{x^3+4x^2+3x} = \frac{1}{3x} - \frac{1}{(x+1)} + \frac{5}{3(x+3)}$

Hence

$\int_1^3 \frac{x^2+1}{x^3+4x^2+3x} dx$

$= \frac{1}{3}\int_1^3 \frac{1}{x} dx - \int_1^3 \frac{1}{(x+1)} dx + \frac{5}{3}\int_1^3 \frac{1}{(x+3)} dx$

$= \left[\frac{1}{3}Inx - In(x+1) + \frac{5}{3}In(x+3)\right]_1^3$

$= \left\{\frac{1}{3}In3 - In4 + \frac{5}{3}In6\right\} - \left\{\frac{1}{3}In1 - In2 + \frac{5}{3}In4\right\}$

$= 0.3488$

B. Denominators with linear factors
Quadratic factors

Each quadratic factors ($ax^2+bx +c$) has a corresponding partial fraction of the form $\frac{Ax+B}{(ax^2+bx+c)}$ where a, b, c and A and B are constants.

Example 14

(a) Express $\frac{7x^2+2x-28}{(x-6)(x^2+3x+5)}$ in partial fraction.

Solution

Let $\frac{7x^2+2x-28}{(x-6)(x^2+3x+5)} = \frac{A}{x-6} + \frac{Bx+c}{x^2+3x+5}$

Multiplying through by $(x – 6)(x^2+3x+5)$

$7x^2+2x-28= A(x^2+3x+5)+ (Bx +C)(x – 6)$

Putting x = 6; 236= 59A, => A = 4

Equating coefficients of x^2

7 = A + B

7 = 4 + B; => B = 3

Equating constants

-28 = 5A – 6C

-28 = 20 – 6C

C = 8

$\therefore \frac{7x^2+2x-28}{(x-6)(x^2+3x+5)} = \frac{4}{x-6} + \frac{3x+8}{x^2+3x+5}$

(b) Find the integral of $f(x) = \frac{2x-1}{(x-1)(x^2+1)}$

Solution

Let $\frac{2x-1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)}$

Multiplying through by $(x-1)(x^2+1)$

$2x – 1 = A(x^2+1)+ (Bx + C)(x-1)$

Putting x = 1; 1 = 2A => A =$\frac{1}{2}$

Putting x= 0; -1 = A –C => C = $\frac{3}{2}$

Putting x= -1; 2A + 2B – 2C => B = $-\frac{1}{2}$

$\therefore \frac{2x-1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} + \frac{-\frac{1}{2}x+\frac{3}{2}}{(x^2+1)}$

$\frac{2x-1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} + \frac{3-x}{2(x^2+1)}$

Note the values of x = 0 and x = -1 are conveniently chosen, but the constants B and C by expansion of the expression and equating constants, i.e.

-1 = A –C => C = $\frac{3}{2}$

2 = C – B

B = $\frac{3}{2} - 2 = -\frac{1}{2}$

Thus,

$\int \frac{2x-1}{(x-1)(x^2+1)}dx$

$= \frac{1}{2}\int \frac{1}{(x-1)}dx + \frac{3}{2}\int \frac{1}{(x^2+1)}dx - \frac{1}{2}\int \frac{x}{(x^2+1)}dx$

$= \frac{1}{2}In(x-1) + \frac{3}{2}tan^{-1}x - \frac{1}{4}In(x^2+1) + c$

(c) Evaluate

(i) $\int_2^3 \frac{3+3x}{x^3-1}$dx

Solution

Note memorize the identities

$x^3 - 1 = (x-1)(x^2+x+1)$

$x^3 + 1 = (x-1)(x^2-x+1)$

Then

$\frac{3+3x}{x^3-1} = \frac{3+3x}{(x-1)(x^2+x+1)}$

$Let\,\frac{3+3x}{x^3-1} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)}$

Multiplying through by (x – 1)(x^2+x+1)

3+3x = A(x^2+x+1) + (Bx+C)(x – 1)

Putting x = 1, 6 = 3A, => A = 2

By expanding and equating coefficients

x^2: A + B = 0, => B = 0 – 2 = -2

x^0: A – C = 3, => C = 2 – 3 = -1

$\therefore \frac{3+3x}{x^3-1} = \frac{2}{(x-1)} - \frac{2x+1}{(x^2+x+1)}$

$\int_2^3 \frac{3+3x}{x^3-1}dx = 2\int_2^3 \frac{1}{(x-1)}dx - \int_2^3 \frac{2x+1}{(x^2+x+1)}$

$= [2In(x-1) - In(x^2+x+1)]_2^3$

$= 2In(2) + In\left(\frac{7}{13}\right)$

$= 0.7673$

(ii) $\int_2^3 \frac{x^2}{x^4-1}dx$

Solution

$\frac{x^2}{x^4-1} = \frac{x^2}{(x-1)(x+1)(x^2+1)}$

Let$\frac{x^2}{(x-1)(x+1)(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{Cx+D}{(x^2+1)}$

By multiplying through by (x – 1)(x+1)(x^2+1)

x^2= A(x+1)(x^2+1)+ B(x -1)(x^2+1) + (Cx+D)(x^2-1)

By equating coefficients

x^3: A + B + C = 0 …………………… (i)

x^2: A – B + D= 1 ……………………..(ii)

x^1: A + B – C = 0 ……………………..(iii)

x^0: A – B – D = 0 ……………………..(iv)

Eqn. (ii) – Eqn. (iv)

2D = 2 => D = $\frac{1}{2}$

Eqn.(i)+ (iii)

2A+ 2B = 0 ………………………… (v)

Eqn. (ii) + Eqn. (iv)

2A – 2B = 1…………………………(vi)

Eqn. (v)+ Eqn. (vi)

4A = 1 => A = $\frac{1}{4}$

Eqn. (v)

B = $-\frac{1}{4}$

Eqn. (i)

C = 0

$\therefore \frac{x^2}{x^4-1} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} + \frac{1}{2(x^2+1)}$

$\int \frac{x^2}{x^4-1}dx$

$= \frac{1}{4}\int \frac{1}{(x-1)}dx - \frac{1}{4}\int \frac{1}{(x+1)}dx + \frac{1}{2}\int \frac{1}{(x^2+1)}dx$

$= \frac{1}{4}In(x-1) - \frac{1}{4}In(x+1) + \frac{1}{2}tan^{-1}x + c$

$\int_2^3 \frac{x^2}{x^4-1}dx$

$= \left[\frac{1}{4}In(x-1) - \frac{1}{4}In(x+1) + \frac{1}{2}tan^{-1}x\right]_2^3$

$= \frac{1}{4}\left[In\frac{x-1}{x+1} + 2tan^{-1}x\right]_2^3$

$= \frac{1}{4}\left\{\left[In\frac{2}{4} + 2tan^{-1}3\right] - \left[In\frac{1}{3} + 2tan^{-1}2\right]\right\}$

$= \frac{1}{4}\left[In\frac{1}{2} - In\frac{1}{3} + 2(tan^{-1}3 - tan^{-1}2)\right]$

$= \frac{1}{4}(0.405 + 0.1\pi)$

$= 0.18$

(iii) $\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)}dx$

Solution

Let $\frac{x^2+6}{(x^2+4)(x^2+9)} = \frac{Ax+B}{(x^2+4)} + \frac{Cx+D}{(x^2+9)}$

Multiplying by $(x^2+4)(x^2+9)$

$x^2+6 = (Ax+B)(x^2+9) + (Cx+D)(x^2+4)$

$x^2+6 = (A+C)x^3+(B+D)x^2+(9A+4C)x+9B+4D$

Equating coefficients

x^3: A + C = 0 ……………….(i)

x^2: B+D = 1………………….(ii)

x^1: 9A + 4C = 0 …………..(iii)

x^0: 9B + 4D = 6 ………….(iv)

Solving simultaneously

A= C= 0; B=$\frac{2}{5}$ and D =$\frac{3}{5}$

$\therefore \frac{x^2+6}{(x^2+4)(x^2+9)} = \frac{2}{5(x^2+4)} + \frac{3}{5(x^2+9)}$

$\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)}dx$

$= \frac{2}{5}\int_0^1 \frac{1}{(x^2+4)}dx + \frac{3}{5}\int_0^1 \frac{1}{(x^2+9)}dx$

$= \frac{1}{5}\left[tan^{-1}\frac{1}{2}x + tan^{-1}\frac{1}{3}x\right]_0^1$

$= \frac{1}{5}tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right) = 0.1571$

C. Repeated factors

Each repeated factor $(ax^2+ b)^n$ in the denominator has corresponding partial fraction of the form: $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$, where a, b, A_i are constants (i = 1, 2, ….n)

Example 15

Express each of the follow in partial fraction and hence find their integrals.

(a) $\frac{4x-9}{(x-3)^2}$

Solution

Let $\frac{4x-9}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$

Multiplying through by $(x-3)^2$

4x – 9 = A(x-3) + B = Ax – 3A + B

Equating coefficients

x^1: x = 4

x^0: -3A + B = 4; B = 3

$\therefore \frac{4x-9}{(x-3)^2} = \frac{4}{x-3} + \frac{3}{(x-3)^2}$

Hence

$\int \frac{4x-9}{(x-3)^2}dx = 4\int \frac{1}{x-3}dx + 3\int (x-3)^{-2}dx$

$= 4In(x-3) - \frac{3}{x-3} + c$

(b) $\frac{3x-14}{x^2-8x+16}$

Solution

$\frac{3x-14}{x^2-8x+16} = \frac{3x-14}{(x-4)^2}$

Let $\frac{3x-14}{(x-4)^2} = \frac{A}{x-4} + \frac{B}{(x-4)^2}$

Multiplying through by $(x-4)^2$

3x – 14 = A(x-4) + B = Ax – 4A + B

Equating coefficients

x^1: x = 3

x^0: -4A + B = -14; B = -2

$\therefore \frac{3x-14}{(x-3)^2} = \frac{3}{x-3} - \frac{2}{(x-4)^2}$

Hence

$\int \frac{3x-14}{(x-4)^2}dx = 3\int \frac{1}{x-4}dx - 2\int (x-4)^{-2}dx$

$= 3In(x-4) + \frac{2}{x-4} + c$

(c) $\frac{2x^2-5x+7}{(x-2)(x-1)^2}$

Solution

Let $\frac{2x^2-5x+7}{(x-2)(x-1)^2} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

Multiplying through by $(x-2)(x-1)^2$

$2x^2-5x+7 = A(x-1)^2+B(x-2)(x-1)+C(x-2)$

Putting x = 1: 4 = -C, => C = -4

Putting x = 2: A = 5

Putting x = 0, 7 =A – 2B – 2C; B = -2

$\therefore \frac{2x^2-5x+7}{(x-2)(x-1)^2} = \frac{5}{x-2} - \frac{2}{x-1} - \frac{4}{(x-1)^2}$

Hence

$\int \frac{2x^2-5x+7}{(x-2)(x-1)^2} dx$

$= 5\int \frac{1}{x-2} dx - 2\int \frac{1}{x-1} dx - 4\int (x-1)^{-2}\, dx$

$= 5In(x-2) - 2In(x-1) - \frac{4}{x-1} + c$

(d) $\frac{7x+2}{3x^3+x^2}$

Solution

$\frac{7x+2}{3x^3+x^2} = \frac{7x+2}{x^2(3x+1)}$

Let$\frac{7x+2}{x^2(3x+1)} = \frac{A}{(3x+1)} + \frac{B}{x} + \frac{C}{x^2}$

Multiplying through by $x^2(3x+1)$

7x+ 2 = Ax^2 +Bx(3x+1)+C(3x+1)

Putting x = 0; c = 2

Putting x = -$\frac{1}{3}$; $\frac{A}{9} = 2 - \frac{7}{3}$ => A = -3

Putting x= -1; -5 = A + 2B- 2C, => B = 1

$\therefore \frac{7x+2}{x^2(3x+1)} = \frac{-3}{(3x+1)} + \frac{1}{x} + \frac{2}{x^2}$

Hence

$\int \frac{7x+2}{x^2(3x+1)} dx$

$= -\int \frac{3}{(3x+1)} dx + \int \frac{1}{x} dx + 2\int x^{-2} dx$

$= -In(3x+1) + Inx - \frac{2}{x} + c$

$= In\frac{x}{3x+3} - \frac{2}{x} + c$ln

Integration of improper fractions

Improper fractions are those whose index of the numerator is equal to or greater than that of the denominators.

They are first changed to proper fraction by long division or otherwise, before being integrated.

Example 16

(a) Express $\frac{5x^2-71}{(x+5)(x-4)}$ in partial fractions.

Hence find $\int \frac{5x^2-71}{(x+5)(x-5)} dx$

Solution

$\frac{5x^2-71}{(x+5)(x-4)} = \frac{5x^2-71}{x^2+x-20}$

Using long division

$$\begin{array}{r|l} & 5 \\ \hline x^2+x-20 & 5x^2+0x-71 \\ - & 5x^2+5x-100 \\ \hline & -5x+29 \end{array}$$

=> $\frac{5x^2-71}{(x+5)(x-4)} = 5 + \frac{-5x+29}{x^2+x-20}$

Let$\frac{-5x+29}{(x+5)(x-4)} = \frac{A}{x+5} + \frac{B}{x-4}$

Multiplying through by (x+5)(x-4)

-5x + 29 = A(x – 4)+ B(x + 5)

Putting x = 4, B = 1

Putting x = -5; A = -6

$\therefore \frac{-5x+29}{(x+5)(x-4)} = \frac{-6}{x+5} + \frac{1}{x-4}$

Hence

$\int \frac{5x^2-71}{(x+5)(x-4)} dx$

$= 5\int dx - 6\int \frac{1}{x+5} dx + \int \frac{1}{x-4} dx$

$= 5x - 6In(x+5) + In(x-4) + c$

(b) Evaluate $\int_0^1 \frac{3-2x}{1+x} dx$

Solution

$\frac{3-2x}{1+x} = \frac{-2x+3}{x+1}$

Using long division

$$\begin{array}{r|l} & -2 \\ \hline x+1 & -2x+3 \\ & -2x-2 \\ \hline & \quad 5 \end{array}$$

$\therefore \frac{3-2x}{1+x} = -2 + \frac{5}{x+1}$

Hence

$\int_0^1 \frac{3-2x}{1+x} dx = -2\int_0^1 dx + 5\int_0^1 \frac{1}{x+1} dx$

$= [-2x + 5In(x+1)]_0^1$

= -2+ 5ln2

= 1.4657

Revision exercise 7

1. Express the following into partial fraction
   (a) $\frac{8x}{x^2-4x-12}$ $\left[\frac{6}{x-6}+\frac{2}{x+2}\right]$
   (b) $\frac{x^4-x^3+x^2+1}{x^3+x}$ $\left[x-1+\frac{1}{x}+\frac{x-1}{x^2+1}\right]$
   (c) $\frac{5x-1}{2x^2+x}-10$ $\left[\frac{3}{2x+5}+\frac{1}{x-2}\right]$
   (d) $\frac{2x^2-7x+1}{(2x+1)(2x-1)(x-2)}$
   $\left[\frac{1}{2x+1}+\frac{2}{3(2x-1)}-\frac{1}{3(x-2)}\right]$
   (e) $\frac{6x+7}{(x^2+2)(x+3)}$ $\left[\frac{x+3}{x^2+2}-\frac{1}{x+3}\right]$
   (f) $\frac{5x+7}{(x+1)^2(x+2)}$ $\left[\frac{3}{x+1}+\frac{2}{(x+1)^2}-\frac{3}{x+2}\right]$
   (g) $\frac{2x^3+3x^2-x-4}{x^2(x+1)}$ $\left[2+\frac{3}{x}+\frac{4}{x^2}-\frac{2}{x+1}\right]$
2. Find
   (a) $\int\frac{x^2}{x^4-1}dx$ $\left[\frac{1}{4}In\left(\frac{x-1}{x+1}\right)+tan^{-1}x+c\right]$
   (b) $\int\frac{x^2-4}{(x+1)^2(x-5)}dx$
   $\left[\frac{5}{12}In(x+1)-\frac{1}{2(x+1)}+\frac{7}{12}In(x-5)\right]$

(c) $\int\frac{3x^2+x+1}{(x-2)(x+1)^3}dx$

$\left[\frac{5}{9}In(x-2)-\frac{5}{9}In(x+1)-\frac{4}{3(x+1)}+\frac{1}{2(x+1)^2}\right]$

(d) $\int\frac{x^4-x^3+x^2+1}{x^3+x}dx$

$\left[\frac{x^2}{2}-x+Inx+tan^{-1}x-\frac{1}{2}In(1+x^2)+c\right]$

(e) $\int\frac{5x-1}{2x^2+x-10}dx$

$\left[\frac{3}{2}In(2x+5)+In(x-2)+c\right]$

(f) $\int\frac{x^2-9x+2}{(x11)(x-1)(x-2)}dx$

$[2In(x+1)+3In(x-1)-4In(x-2)]+c$

(g) $\int\frac{9x+7}{(2x^2+3)(x+2)}dx$

$\left[\frac{1}{2}In(2x^2+3)+\frac{5}{\sqrt{10}}tan^{-1}\left(\sqrt{\frac{2}{3}}x\right)-In(x+2)+c\right]$

(h) $\int\frac{7+5x-6x^2}{(2x+1)^2(x+2)}dx$

$\left[\frac{3}{2}In(2x+1)-\frac{1}{2x+1}-3In(x+2)+c\right]$

(i) $\int\frac{x^2+7x-14}{(x+5)(x-3)}dx$

$[x+3In(x+5)+2In(x-3)+c]$

3. Evaluate

(a) $\int_0^2\frac{3x^4+7x^3+8x^2+53-186}{(x+4)(x^2+9)}dx$ [-4.5489]

(b) $\int_2^3\frac{x^2}{x^4-1}dx$ [0.18]

(c) $\int_1^3\frac{x^2+1}{x^3+4x^2+3x}$dx [0.3489]

(d) $\int_0^1\frac{x^3}{x^2+1}dx$ [0.1535]

(e) $\int_6^7\frac{x^2-4}{(x+1)^2(x-5)}dx$ [0.4689]

(f) $\int_3^4\frac{3x^2+x+1}{(x-2)(x+1)^3}dx$ [0.3165]

(g) $\int_0^2\frac{8x}{x^2-4x-12}dx$ [1.05]

Integration by parts

This stems from differentiating the product of a function, y = uv,

$\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}$

$u\frac{dv}{dx}=\frac{d}{dx}(uv)-v\frac{du}{dx}$

$\int u\frac{dv}{dx}dx=\int\frac{d}{dx}(uv)dx-\int v\frac{du}{dx}dx$

$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$

Or simply $\int udv=uv-\int vdu$

The function chosen as u should be easily differentiated whereas the other function chosen as v should be easily integrated.

The above expression of the integration by parts can be summarized by using a technique of integration by parts

This is summarized in the table below

Sign	Differentiate	Integrates
+	u_1	$\frac{dv}{dx}$
-	u_2	v_1
+	u_3	v_2
-	u_4	v_3

NB: the signs change as +, -, + etc.

The u function is differentiated until a zero value is obtained otherwise we continue with differentiation.

The integral of the function is equal to the sum of result shown in the table above.

Integration by parts is applied in the following areas:

A. Integration products of polynomials by parts

Example 17

(a) Find

(i) $\int x(x+2)^3dx$

Solution

Let u = x and $\frac{dv}{dx} = (x+2)^3$

$\frac{du}{dx} = 1$; v = $\frac{1}{4}(x+2)^4$

From $\int udv = uv - \int vdu$

$\int x(x+2)^3dx$
=$\frac{1}{4}x(x+2)^4 - \int 1.\frac{1}{4}(x+2)^4dx$
=$\frac{1}{4}x(x+2)^4 - \frac{1}{4}\int(x+2)^4dx$
=$\frac{1}{4}x(x+2)^4 - \frac{1}{20}(x+2)^5 + c$
=$\frac{1}{2o}(x+2)^4(5x - x - 2) + c$
=$\frac{1}{2o}(x+2)^4(4x-2) + c$
=$\frac{1}{1o}(x+2)^4(x-1) + c$
Or by using basic techniques

Sign	Differentiate	Integrates
+	x	$(x+2)^3$
-	1	$\frac{1}{4}(x+2)^4$
+	0	$\frac{1}{20}(x+2)^5$

$\int x(x+2)^3dx$ =$\frac{1}{4}x(x+2)^4 - \frac{1}{20}(x+2)^5 + c$
=$\frac{1}{1o}(x+2)^4(x-1) + c$
$\therefore \int x(x+2)^3dx$=$\frac{1}{1o}(x+2)^4(x-1) + c$

(ii) $\int(x+3)(x-4)^5dx$

Solution

Let u = (x+3) and $\frac{dv}{dx} = (x-4)^5$

$\frac{du}{dx} = 1$; v = $\frac{1}{6}(x-4)^6$

$\int(x+3)(x-4)^5dx$
=$\frac{1}{6}(x+3)(x-4)^6 - \frac{1}{6}\int 1.(x-4)^6$dx
=$\frac{1}{6}(x+3)(x-4)^6 - \frac{1}{6}\int(x-4)^6dx$
=$\frac{1}{6}(x+3)(x-4)^6 - \frac{1}{42}(x-4)^7 + c$
=$\frac{1}{42}(x-4)^6((7(x+3) - x + 4) + c$
=$\frac{1}{42}(x-4)^6(6x+25) + c$

Sign	Differentiate	Integrates
+	x+3	$(x-4)^5$
-	1	$\frac{1}{6}(x+2)^6$
+	0	$\frac{1}{42}(x+2)^7$

$\int(x+3)(x-4)^5dx$
=(x + 3)$(x-4)^5$- $\frac{1}{42}x(x+2)^7 + c$
$\therefore \int(x+3)(x-4)^5dx$
=$\frac{1}{42}(x-4)^6(6x+25) + c$

(iii) $\int \frac{3x-4}{(x+2)^4}dx$

Solution

$\int \frac{3x-4}{(x+2)^4}dx = \int(3x-4)(x+2)^{-4}$dx

Let u =(3x – 4) and $\frac{dv}{dx} = (x+2)^{-4}$

$\frac{du}{dx} = 3$; v = $-\frac{1}{3}(x+2)^{-3}$

$\int \frac{3x-4}{(x+2)^4}dx$

$= -\frac{1}{3}(3\text{x} - 4)(x+2)^{-3} - \int 3.-\frac{1}{3}(x+2)^{-3}dx$

$= -\frac{1}{3}(3\text{x} - 4)(x+2)^{-3} + \int(x+2)^{-3}dx$

$= -\frac{1}{3}(3\text{x} - 4)(x+2)^{-3} - \frac{1}{2}(x+2)^{-2} + c$

$= \frac{4-3x}{3(x+2)^3} - \frac{1}{2(x+2)^2} + c$

$= \frac{2(4-3x)-3(x+2)}{6(x+2)^3} + c = \frac{2-9x}{6(x+2)^3} + c$

$\therefore \int \frac{3x-4}{(x+2)^4}dx = \frac{2(4-3x)-3(x+2)}{6(x+2)^3} + c = \frac{2-9x}{6(x+2)^3} + c$

Sign	Differentiate	Integrates
+	3x-4	$(x-4)^{-4}$
-	3	$\frac{1}{3}(x-4)^{-3}$
+	0	$\frac{1}{6}(x-4)^{-2}$

$\int \frac{3x-4}{(x+2)^4}dx$

$= -\frac{1}{3}(3x-4)(x-4)^{-3} - \frac{1}{2}(x-4)^{-2} + c$

$= \frac{2-9x}{6(x+2)^3} + c$

(b) Evaluate

(i) $\int_0^2 x(x-3)^2dx$

Solution

Sign	Differentiate	Integrates
+	x	$(x-3)^2$
-	1	$\frac{1}{3}(x-3)^3$
+	0	$\frac{1}{12}(x-3)^4$

$\int x(x-3)^2dx = \frac{1}{3}x(x-3)^3 - \frac{1}{12}(x-3)^4$+c

$= \frac{1}{12}(x-3)^3(4x-x+3)+c$

$= \frac{1}{12}(x-3)^3(3x+3)+c$

$= \frac{1}{4}(x-3)^3(x+1)+c$

⇨ $\int_0^2 x(x-3)^2dx = \left[\frac{1}{4}(x-3)^3(x+1)\right]_0^2$

$= \frac{-3}{4} - \frac{-27}{4} = \frac{24}{4} = 6$

(i) $\int_3^6 \frac{x}{\sqrt{x-2}}dx$

Solution

Sign	Differentiate	Integrates
+	x	$(x-2)^{-\frac{1}{2}}$
-	1	$2(x-2)^{\frac{1}{2}}$
+	0	$\frac{4}{3}(x-2)^{\frac{3}{2}}$

$\int \frac{x}{\sqrt{x-2}}dx = 2x\,(x-2)^{\frac{1}{2}} - \frac{4}{3}(x-2)^{\frac{3}{2}}$+c

$= \frac{2}{3}(x-2)^{\frac{1}{2}}[3x-2(x-2)]+c$

$= \frac{2}{3}(x-2)^{\frac{1}{2}}(x+4)+c$

⇨ $\int_3^6 \frac{x}{\sqrt{x-2}}dx = \left[\frac{2}{3}(x-2)^{\frac{1}{2}}(x+4)\right]_3^6$

$= \left[\frac{2}{3}(6-2)^{\frac{1}{2}}(6+4)\right] - \left[\frac{2}{3}(3-2)^{\frac{1}{2}}(3+4)\right]$

$= \frac{2}{3}(20-7) = \frac{26}{3} = 8\frac{2}{3}$

Revision exercise 8

1. Integrate

(a) $\int(x-1)(x+2)^2$dx

$\left[\frac{1}{4}(x-2)(x+2)^3+c\right]$

(b) $\int(3x-1)(2x+3)^2$dx

$\left[\frac{1}{48}(18x-17)(2x+3)^3+c\right]$

(c) $\int(2-5x)(4-x)^4$dx

$\left[\frac{1}{30}(25x+8)(4-x)^5\right]$

(d) $\int \frac{x-2}{(2x-3)^2}$dx

$\left[\frac{1}{4}In(2x-3)+\frac{1}{4(2x-3)}+c\right]$

(e) $\int \frac{x+4}{\sqrt{3x-2}}$dx

$\left[\frac{2}{27}(3x+40)\sqrt{3x-2}+c\right]$

(f) $\int \frac{3x+1}{\sqrt{1-2x}}$dx

$In\left(\frac{2-x}{5-x}\right)+\frac{1}{5-x}+c$

2. Evaluate

(a) $\int_{-1}^1 x^2(x+3)^3dx$ $\left[\frac{108}{5}\right]$

(b) $\int_3^6 \frac{x^2}{\sqrt{1-2x}}dx$ $\left[\frac{586}{15}\right]$

B. Integration products of polynomials and circular/trigonometric functions by parts

Example 18

(a) Find

(i) $\int xsin\,xdx$

Solution

Let u = x and $\frac{dv}{dx} = sinx$

$\frac{du}{dx} = 1$, v = -cosx

$\int xcos\,xdx = -xcos\,x - \int 1.-cosx$

$= -xcos\,x + sinx + c$

Or: by using basic technique

Sign	Differentiate	Integrates
+	x	$sinx$
-	1	$-cosx$
+	0	$-sinx$

$\int xcos\,xdx = -xcos\,x + sinx + c$

(ii) $\int x^2cosxdx$

Solution

Let u = x^2 and $\frac{dv}{dx} = cosx$

$\frac{du}{dx} = 2x$, v = sinx

$\int x^2cosxdx = x^2sinx - 2\int xsinxdx$+c

Let u = x and $\frac{dv}{dx} = sinx$

$\frac{du}{dx} = 1$, v = -cosx

$\int x^2 cosxdx$
$= x^2 sinx - 2[-xcosx - \int -cosxdx] + c$
$= x^2 sinx - 2[-xcosx + \int cosxdx] + c$
$= x^2 sinx - 2[-xcosx + sinx] + c$
$= x^2 sinx + 2xcosx - 2sinx + c$
Or using basic technique

Sign	Differentiate	Integrates
+	x^2	$cosx$
-	2x	$sinx$
+	2	$-cosx$
-	0	$-sinx$

$\int x^2 cosxdx$ =x^2sins+ 2xcosx – 2sinx + c

(iii) $\int x^2 sin^2 xdx$

Solution

Let u = x^2 and $\frac{dv}{dx} = sin^2 x = \frac{1}{2}(1 - cos2x)$

$\frac{du}{dx} = 2x$, v = $\frac{1}{2}\left(x - \frac{1}{2}sin2x\right)$

$\int x^2 sin^2 xdx$
=$\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \int 2x.\frac{1}{2}\left(x - \frac{1}{2}sin2x\right)dx$
=$\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \int x^2 dx + \frac{1}{2}\int xsin2xdx$
=$\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \frac{1}{3}x^3 + \frac{1}{2}\int xsin2xdx$
Let u = x and $\frac{dv}{dx} = sin2x$
$\frac{du}{dx} = 1$and v = -$\frac{1}{2}$cos2x
$\int xsin2xdx$ =-$\frac{1}{2}$xcos2x+$\frac{1}{4}$sin2x+c
Substituting for$\int xsin2xdx$
$\int x^2 sin^2 xdx$
=$\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \frac{1}{3}x^3 + \frac{1}{2}\left[-\frac{1}{2}\text{xcos2x} + \frac{1}{4}\text{sin2x}\right]$
=$\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \frac{1}{3}x^3 - \frac{1}{4}\text{xcos2x} + \frac{1}{8}\text{sin2x} + c$
$= \frac{1}{6}x^3 - \frac{1}{4}x^2 sin2x - \frac{1}{4}xcos2x + \frac{1}{8}sin2x + c$

Or by using basic technique

Sign	Differentiate	Integrates
+	x^2	$sin^2 x = \frac{1}{2}(1 - cos2x)$
-	2x	$\frac{1}{2}x - \frac{1}{4}sins2x$
+	2	$\frac{1}{4}x^2 + \frac{1}{8}cos2x$
-	0	$\frac{1}{12}x^3 + \frac{1}{16}sin2x$

$\int x^2 sin^2 xdx$
$= \frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - 2x\left(\frac{1}{4}x^2 + \frac{1}{8}cos2x\right) +$
$2\left(\frac{1}{12}x^3 + \frac{1}{16}sin2x\right)$
$= \frac{1}{6}x^3 - \frac{1}{4}x^2 sin2x - \frac{1}{4}xcos2x + \frac{1}{8}sin2x + c$

(iv) $\int xcos^2 xdx$

Solution

$\int xcos^2 xdx$
Let u = x and $\frac{dv}{dx} = cos^2 x = \frac{1}{2} + \frac{1}{2}cos2x$
$\frac{du}{dx} = 1$, v =$\frac{1}{2}\left(x + \frac{1}{2}sin2x\right)$
$\int xcos^2 xdx$
$= \frac{1}{2}x\left(x + \frac{1}{2}sin2x\right) - \int 1.\frac{1}{2}\left(x + \frac{1}{2}sin2x\right)$ dx
$= \frac{1}{2}x\left(x + \frac{1}{2}sin2x\right) - \frac{1}{2}\int xdx - \frac{1}{4}\int sin2x\text{dx}$
$= \frac{1}{2}x^2 + \frac{1}{4}sin2x - \frac{1}{4}x^2 + \frac{1}{8}cos2x + c$
$= \frac{1}{4}x^2 + \frac{1}{4}sin2x + \frac{1}{8}cos2x + c$
(b) Evaluate
(i) $\int_0^{\frac{\pi}{2}} x^2 cosxdx$

Solution

$\int_0^{\frac{\pi}{2}} x^2 cosxdx$

=$[x^2 sinx + 2xcosx - 2sinx]_0^{\frac{\pi}{2}}$

=$\left[\frac{\pi^2}{4}sin\frac{\pi}{2} + 2.\frac{\pi}{2}cos\frac{\pi}{2} - 2sin\frac{\pi}{2}\right] - 0$

=$\left(\frac{\pi^2}{4} - 2\right) = 0.4674$

(ii) $\int_0^{\frac{\pi}{4}} xtan^2 xdx$

Solution

Let u = x and $\frac{dv}{dx} = tan^2 x = sec^2 x - 1$

$\frac{du}{dx} = 1$; v = tanx - x

$\int xtan^2 xdx$=xtanx - x^2-$\int(tanx - x)dx$
=xtanx - x^2+Incosx+$\frac{1}{2}x^2 + c$

=xtanx +Incosx$-\frac{1}{2}x^2 + c$

Or by using basic technique

Sign	Differentiate	Integrates
+	x	$tan^2x = sec^2x - 1$
-	1	tanx – x
+	0	-Incosx$-\frac{1}{2}x^2 +$

$\int xtan^2xdx$=xtanx - x^2+Incosx+$\frac{1}{2}x^2 + c$

=xtanx +Incosx$-\frac{1}{2}x^2 + c$

Hence;

$$\int_0^{\frac{\pi}{4}} xtan^2xdx = \left[\text{xtanx } + \text{Incosx} - \frac{1}{2}x^2\right]_0^{\frac{\pi}{4}}$$
$$= \left[\frac{\pi}{4}\tan\frac{\pi}{4} + \text{Incos}\frac{\pi}{4} - \frac{1}{2}\left(\frac{\pi}{4}\right)^2\right] - 0$$
$$= 0.1304$$

Revision Exercise 9

1. Integrate each of the following
   (a) $\int xsin2xdx$
   $\left[-\frac{x}{2}cos2x + \frac{1}{4}sin2x + c\right]$
   (b) $\int x^2sinxdx$
   $[-x^2cosx + 2xsinx + 2cosx + c]$
   (c) $\int (x+1)^2sinxdx$
   $[(1 - 2x - x^2)cosx = 2(x+1)sinx + c]$
   (d) $\int x^2sinxcosxdx$
   $\left[\frac{1}{8}cos2x(1 - 2x^2) + \frac{1}{4}xsin2x + c\right]$
   (e) $\int x^3 \cos x^2\, dx$
   $\left[\frac{1}{2}x^2 \sin x^2 + \frac{1}{2}\cos x^2 + c\right]$
   (f) $\int (xcosx)^2dx$
   $\left[\frac{1}{6}x^3 + \frac{1}{8}(2x^2 - 1) + \frac{1}{4}xsin2x + c\right]$
2. Evaluate
   (a) $\int_0^{\pi} x^2sinxdx$ $[5.8696]$
   (b) $\int_0^{\pi} x^2cos2xdx$ $[0.0584]$
   (c) $\int_0^{\frac{\pi}{4}} xtan^2xdx$ $[0.1304]$

C. Integration products of polynomials and exponential functions by parts

Examples 19

(a) Find

(i) $\int xe^xdx$

Solution

Let u = x and $\frac{dv}{dx} = e^x$

$\frac{du}{dx} = 1$; v =e^x

$\int xe^xdx = xe^x - \int 1.e^xdx$

$= xe^x - e^x + c$

Or by using basic technique

Sign	Differentiate	Integrates
+	x	e^x
-	1	e^x
+	0	e^x

$\int xe^xdx = xe^x - e^x + c$

(ii) $\int xe^{-x}dx$

Solution

Let u = x and $\frac{dv}{dx} = e^{-x}$

$\frac{du}{dx} = 1$; v =$-e^{-x}$

$\int xe^xdx = -xe^{-x} - \int 1.-e^{-x}dx$
$= -xe^{-x} + \int e^{-x}dx$

$= -xe^x - e^x + c$

Or by using basic technique

Sign	Differentiate	Integrates
+	x	e^{-x}
-	1	$-e^{-x}$
+	0	e^x

$\int xe^xdx = -xe^x - e^x + c$

(iii) $\int xe^{3x}dx$

Solution

Let u = x and $\frac{dv}{dx} = e^{3x}$

$\frac{du}{dx} = 1$; v =$\frac{1}{3}e^{3x}$

$\int xe^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$

Or by using basic technique

Sign	Differentiate	Integrates
+	x	e^{3x}
-	1	$\frac{1}{3}e^{3x}$
+	0	$\frac{1}{9}e^{3x}$

$\int xe^xdx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$

(b) Find

(i) $\int x.2^x$dx

Solution

Let u = x and $\frac{dv}{dx} = 2^x$

$\frac{du}{dx} = 1$; v = $\frac{2^x}{In2}$

$\int x.2^x$dx=$\frac{x.2^x}{In2} - \frac{1}{In2}\int 2^x dx$

$=\frac{x.2^x}{In2} - \frac{1}{In2}\left(\frac{2^x}{In2}\right) + c$

$=\frac{2x}{In2}(x-1) + c$

(ii) $\int 3^{\sqrt{(2x-1)}}$dx

Solution

Let p =$\sqrt{(2x-1)}$, p^2 = 2x – 1

2pdp = 2dx

pdp = dx

⇨ $\int 3^{\sqrt{(2x-1)}}$dx=$\int 3^p\,.pdp$

Let u =p and $\frac{dv}{dp} = 3^p$

$\frac{du}{dp} = 1$, v = $\frac{3^p}{In3}$

$\int 3^p\,.pdp = \frac{3^p.p}{In3} - \frac{1}{In3}\int 3^p dp$

$= \frac{3^p.p}{In3} - \frac{1}{In3}\left(\frac{3^p}{In3}\right) + c$

∴ $\int 3^{\sqrt{(2x-1)}}$dx

$= \frac{\sqrt{(2x-1)}3^{\sqrt{(2x-1)}}}{In3} - \frac{1}{In3}\left(\frac{3^{\sqrt{(2x-1)}}}{In3}\right) + c$

$= \frac{3^{\sqrt{(2x-1)}}}{In3}\left(\sqrt{(2x-1)} - \frac{1}{In3}\right) + c$

(c) Evaluate

(i) $\int_0^1 xe^{-x}dx$

Solution

$\int_0^1 xe^{-x}dx = [-xe^{-x} - e^{-x}]_0^1$

$= (-e^{-1} - e^{-1}) - (0 - e^0)$

$= -2e^{-1} + 1$

$= 1 - \frac{2}{e}$

$= 0.2642$

(ii) $\int_0^1 xe^{3x}dx$

Solution

$\int_0^1 xe^{3x}dx = \left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right]_0^1$

$= \left[\frac{1}{3}e^3 - \frac{1}{9}e^3\right] - \left[0 - \frac{1}{9}e^0\right]$

$= \frac{2}{9}e^3 + \frac{1}{9} = 4.5746$

Revision exercise 10

1. Integrate each of the following with respect to x

   (a) xe^{3x} $\left[\frac{e^{3x}}{9}(3x-1) + c\right]$

   (b) x^2e^x $[e^x(x^2 - 2x + 2) + c]$

   (c) $x^3e^{x^2}$ $\left[\frac{e^{x^2}}{2}(x^2 - 1) + c\right]$

   (d) x^2e^{-2x} $\left[-\frac{e^{-2}}{4}(2x^2 + 2x + 1) + c\right]$

   (e) $\frac{x^2}{e^{-x^3}}$ $\left[-\frac{1}{3}e^{-x^3} + c\right]$

   (f) $e^x(3 + e^x)^2$ $\left[\frac{1}{3}(3 + e^x)^3 + c\right]$

2. Evaluate each of the following

   (a) $\int_0^1 x^2\,e^{2x}dx$ [1.5973]

   (b) $\int_0^1 (x-1)e^x dx$ [2]

D. Integration products of polynomials and inverse trigonometric functions by parts

Example 20

(a) Find

(i) $\int sin^{-1}xdx$

Solution

$\int sin^{-1}xdx = \int 1.sin^{-1}xdx$

Let u=$sin^{-1}x$ and $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$; v = x

$\int sin^{-1}xdx = xsin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}dx$

For $\int \frac{x}{\sqrt{1-x^2}}dx$

Let u = 1 – x^2

Du= -2x

$-\frac{1}{2x}du = dx$

$\int \frac{x}{\sqrt{1-x^2}}dx = \int \frac{x}{u^{\frac{1}{2}}}.-\frac{1}{2x}du$

$= -\frac{1}{2}\int u^{-\frac{1}{2}}du$

$= -\frac{1}{2}\left[2u^{\frac{1}{2}} + c\right] = -u^{\frac{1}{2}} + c$

By substitution

$\int sin^{-1}xdx = xsin^{-1}x + u^{\frac{1}{2}} + c$

$\therefore \int sin^{-1}xdx = xsin^{-1}x + \sqrt{1-x^2} + c$

(ii) $\int cos^{-1}\left(\frac{x}{a}\right)dx$

Solution

$\int cos^{-1}\left(\frac{x}{a}\right)dx = \int 1.cos^{-1}\left(\frac{x}{a}\right)dx$

Let u = $cos^{-1}\left(\frac{x}{a}\right)$ and $\frac{dv}{dx} = 1$

$\frac{du}{dx} = -\frac{1}{\sqrt{a^2-x^2}}$

$\int cos^{-1}\left(\frac{x}{a}\right)dx = xcos^{-1}\left(\frac{x}{a}\right) + \int \frac{1}{\sqrt{a^2-x^2}}$ dx

For $\int \frac{x}{\sqrt{a^2-x^2}}dx$

Let u = $a^2 - x^2$

Du= -2x

$-\frac{1}{2x}du = dx$

$\int \frac{x}{\sqrt{a^2-x^2}}dx = \int \frac{x}{u^{\frac{1}{2}}}.-\frac{1}{2x}du$

$= -\frac{1}{2}\int u^{-\frac{1}{2}}du$

$= -\frac{1}{2}\left[2u^{\frac{1}{2}} + c\right] = -u^{\frac{1}{2}} + c$

By substitution

$\int cos^{-1}\left(\frac{x}{a}\right)dx = xcos^{-1}x + u^{\frac{1}{2}} + c$

$\therefore \int cos^{-1}\left(\frac{x}{a}\right)dx = xcos^{-1}x + \sqrt{a^2-x^2} + c$

(iii) $\int xtan^{-1}x\,dx$

Solution

Let u = $tan^{-1}x$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{1+x^2}$; v = $\frac{1}{2}x^2$

$\int xtan^{-1}x\,dx = \frac{1}{2}x^2tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}$dx

For$\int \frac{x^2}{1+x^2}$dx $= \int \left(1 - \frac{1}{1+x^2}\right)dx$

$= \int dx - \int \frac{1}{1+x^2}dx$

$= x - tan^{-1}x + c$

By substitution

$\int xtan^{-1}x\,dx$

$= \frac{1}{2}x^2tan^{-1}x - \frac{1}{2}[x - tan^{-1}x] + c$

$= \frac{1}{2}x^2tan^{-1}x - \frac{1}{2}x + \frac{1}{2}tan^{-1}x + c$

$= \frac{1}{2}[(x^2+1)tan^{-1}x - x] + c$

(b) Evaluate $\int_0^1 xsin^{-1}xdx$

Solution

Let u = $sin^{-1}x$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$; v= $\frac{1}{2}x^2$

$\int xsin^{-1}xdx = \frac{1}{2}x^2sin^{-1}x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx$

For $\int \frac{x^2}{\sqrt{1-x^2}}dx$

Let x = sinθ => dx = cosθdθ

$\int \frac{x^2}{\sqrt{1-x^2}}dx = \int \frac{sin^2\theta}{\sqrt{1-sin^2\theta}}.cos\theta d\theta$

$= \int \frac{sin^2\theta}{cos\theta}.cos\theta d\theta$

$= \int sin^2\theta\,d\theta$

$= \frac{1}{2}\int(1 - cos2\theta)\,d\theta$

$= \frac{1}{2}\theta - \frac{1}{4}sin2\theta + c$

$= \frac{1}{2}\theta - \frac{1}{4}(2sin\theta cos\theta) + c$

$= \frac{1}{2}\theta - \frac{1}{2}(sin\theta cos\theta) + c$

$= \frac{1}{2}sin^{-1}x - \frac{1}{2}x\sqrt{1-x^2} + c$

$\therefore \int xsin^{-1}xdx$

$= \frac{1}{2}x^2 sin^{-1}x - \frac{1}{4}sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2}$

$\int_0^1 xsin^{-1}xdx$

$= \left[\frac{1}{2}x^2 sin^{-1}x - \frac{1}{4}sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2}\right]_0^1$

$= \left[\frac{1}{2}.1.sin^{-1}1 - \frac{1}{4}sin^{-1}(1)\right] - (0)$

$= \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8}$

Revision exercise 11

1. Find the following integrals

(a) $\int tan^{-1}3xdx$

$\left[xtan^{-1}x - \frac{1}{6}In(1+9x^2) + c\right]$

(b) $\int \frac{xsin^{-1}x}{\sqrt{1-x^2}}dx$

$\left[xsin^{-1}x + \sqrt{1-x^{2+c}}\right]$

(c) $\int sec^{-1}x\,dx$

$\left[xsec^{-1}x - In\left(x+\sqrt{x^2-1}\right) + c\right]$

(d) $\int cot^{-1}xdx$

$\left[xcot^{-1}x + \frac{1}{2}In(1+x^2) + c\right]$

2. Evaluate

(a) $\int_0^1 sin^{-1}xdx$ $\left[\frac{\pi}{2} - 1\right]$

(b) $\int_0^1 cos^{-1}xdx$ $[1]$

E. Integration products of polynomials and logarithmic functions by parts

Example 21

(a) Integrate

(i) $\int Inx^2dx$

Solution

$\int Inx^2dx = \int 1.Inx^2dx$

Let u = $\text{In}x^2$, $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{2x}{x^2} = \frac{2}{x}$; v = x

$\int Inx^2dx = xInx^2 - 2\int x.\frac{1}{x}dx$

$= xInx^2 - 2x + c$

$= 2xInx - 2x + c$

$\therefore \int Inx^2dx = 2xInx - 2x + c$

(ii) $\int xIn(x^2-1)dx$

Solution

Let u = $In(x^2-1)$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{2x}{x^2-1}$; $v = \frac{1}{2}x^2$

$\int xIn(x^2-1)dx$

$= \frac{1}{2}x^2In(x^2-1) - \int \frac{1}{2}x^2.\frac{2x}{x^2-1}dx$

$= \frac{1}{2}x^2In(x^2-1) - \int \frac{x^3}{x^2-1}dx$

For $\int \frac{x^3}{x^2-1}dx$

By using long division

$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$

$\Rightarrow \int \frac{x^3}{x^2-1}dx = \int xdx + \int \frac{x}{x^2-1}dx$

$= \frac{1}{2}x^2 + \frac{1}{2}In(x^2-1) + c$

$\therefore \int xIn(x^2-1)dx$

$= \frac{1}{2}x^2In(x^2-1) - \frac{1}{2}x^2 - \frac{1}{2}In(x^2-1) + c$

(iii) $\int x^{-3}Inxdx$

Solution

Let u = Inx and $\frac{dv}{dx} = x^{-3}$

$\frac{du}{dx} = \frac{1}{x}$ and v = $-\frac{1}{2}x^{-2}$

$\int x^{-3}Inxdx = -\frac{1}{2}x^{-2}Inx + \frac{1}{2}\int \frac{1}{x}.x^{-2}$ dx

$= -\frac{1}{2}x^{-2}Inx + \frac{1}{2}\int x^{-3}dx$

$= -\frac{1}{2}x^{-2}Inx - \frac{1}{4}x^{-2} + c$

$= -\frac{1}{4}x^{-2}(Inx + 1) + c$

(b) Evaluate $\int_1^{10} x\log_{10}xdx$

Solution

Changing from base 10 to base e

$\log_{10}x = \frac{Ine}{In10}$

$\int_1^{10} x\log_{10}xdx = \frac{1}{In10}\int_1^{10} xInxdx$

Let u= Inx; $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x}$; $v = \frac{1}{2}x^2$

$$\frac{1}{In10}\int_1^{10} xInxdx = \frac{1}{In10}\left[\frac{1}{2}x^2Inx - \frac{1}{4}x^2\right]_1^{10}$$

$= \frac{1}{In10}\left[(50In10 - 25) - \frac{1}{4}\right]$

$= \frac{1}{In10}\left[50In10 - \frac{99}{4}\right]$ $= 50 - \frac{99}{4In10}$

Revision exercise 12

1. Integrate each of the following
   (a) $xInx$ $\left[\frac{x^2}{4}(2Inx-1)+c\right]$
   (b) x^2Inx $\left[\frac{x^2}{9}(3Inx-1)+c\right]$
   (c) $\sqrt{x}Inx$ $\left[\frac{2}{9}\sqrt{x^3}(3Inx-2)+c\right]$
   (d) $(Inx)^2$ $[x(2-2Inx+(Inx)^2)+c]$
   (e) $\frac{Inx}{x^2}$ $\left[-\frac{1}{x}(Inx+1)+c\right]$
   (f) $3^x x$ $\left[\frac{3^x}{(In3)^2}(xIn3-1)+c\right]$
   (g) $x(Inx)^2$
   $\left[\frac{1}{4}x^2(1-2Inx+2(Inx)^2)+c\right]$
2. Evaluate the following
   (a) $\int_2^4 x^3Inxdx$ [70.9503]
   (b) $\int_2^4(x-1)In(2x)dx$ [1.0794]
   (c) $\int_1^4\frac{Inx}{x^2}dx$ [0.4034]

F. Integration of products of exponential and trigonometric functions by parts

Example 22

(a) Find
(i) $\int e^{-x}sinxdx$

Solution

Taking I = $\int e^{-x}sinxdx$

Let u = e^{-x}, $\frac{dv}{dx}=sinx$

$\frac{du}{dx}=-e^{-x}$; v = -cosx

$=> I=-e^{-x}cosx-\int -e^{-x}.-cosxdx$

$I=-e^{-x}cosx-\int e^{-x}.cosxdx$ …(*)

For $\int e^{-x}.cosxdx$

Let u = e^{-x}, $\frac{dv}{dx}=cosx$

$\frac{du}{dx}=-e^{-x}$; v = sinx

$\int e^{-x}.cosxdx=e^{-x}sinx-\int -e^{-x}sinx$

$=e^{-x}sinx+I$ ………(**)

Substituting for (**) in equation (*)

$I=-e^{-x}cosx-e^{-x}sinx-I$

$2I=-e^{-x}cosx-e^{-x}sinx-I+A$

$I=-\frac{1}{2}e^{-x}(cosx+sinx)+c$

$\therefore \int e^{-x}sinxdx=-\frac{1}{2}e^{-x}(cosx+sinx)+c$

Or by using basic technique

sign	Differentiate	integrate
+	e^{-x}	sinx
-	$-e^{-x}$	-cosx
+	e^{-x}	-sinx

$I=-e^{-x}cosx-e^{-x}sinx-\int e^{-x}sinx$

$I=-e^{-x}cosx-e^{-x}sinx-I$

$2I=-e^{-x}cosx-e^{-x}sinx+A$

$I=-\frac{1}{2}e^{-x}(cosx+sinx)+c$

$\therefore \int e^{-x}sinxdx=-\frac{1}{2}e^{-x}(cosx+sinx)+c$

(ii) $\int e^{2x}cos3xdx$

Solution

Taking I = $\int e^{2x}cos3xdx$

Let u = e^{2x}, $\frac{dv}{dx}=cos3x$

$\frac{du}{dx}=2e^{2x}$; v = $\frac{1}{3}$sin3x

$I=\frac{1}{3}e^{2x}sin3x-\int 2e^{2x}.\frac{1}{3}sin3xdx$

$I=\frac{1}{3}e^{2x}\text{sin3x}-\frac{2}{3}\int e^{2x}\text{sin3xdx}$ … (*)

For $\int e^{2x}$sin3xdx

Let u = e^{2x}, $\frac{dv}{dx}=sin3x$

$\frac{du}{dx}=2e^{2x}$; v =- $\frac{1}{3}$cos3x

$\int e^{2x}\,\text{sin3xdx}$

$=-\frac{1}{3}e^{2x}\text{cos3x}-\int 2e^{2x}.-\frac{1}{3}\text{cos3xdx}$

$=-\frac{1}{3}e^{2x}\text{cos3x}+\frac{2}{3}\int e^{2x}\text{cos3xdx}$

$=-\frac{1}{3}e^{2x}\text{cos3x}+\frac{2}{3}I$ …………………(**)

Substituting (**) into (*)

$I=\frac{1}{3}e^{2x}\text{sin3x}-\frac{2}{3}\left(-\frac{1}{3}e^{2x}\text{cos3x}+\frac{2}{3}I\right)$

$= \frac{1}{3}e^{2x}$sin3x+$\frac{2}{9}e^{2x}$cos3x + $\frac{4}{9}I$ + c

$\frac{13}{9}I = \frac{1}{3}e^{2x}$sin3x + $\frac{2}{9}e^{2x}$cos3x + A

$I = \frac{3}{13}e^{2x}$sin3x+$\frac{2}{13}e^{2x}$cos3x + c

$I = \frac{1}{13}e^{2x}(3sin3x + 2cos3x) + c$

$\therefore \int e^{2x}cos3xdx = \frac{1}{13}e^{2x}(3sin3x + 2cos3x) + c$

(iii) $\int e^{3x}sin2xdx$

Solution

Taking I =$\int e^{3x}sin2xdx$

Let u = e^{3x}, $\frac{dv}{dx} = sin2x$

$\frac{du}{dx} = 3e^{3x}$; v = $-\frac{1}{2}$cos2x

$I = -\frac{1}{2}e^{3x}cos2x - \int 3e^{3x}.-\frac{1}{2}cos2xdx$

$I = -\frac{1}{2}e^{3x}$cos2x+$\frac{3}{2}\int e^{3x}$cos2xdx ... (*)

For $\int e^{3x}$cos2xdx

Let u = e^{3x}, $\frac{dv}{dx} = cos2x$

$\frac{du}{dx} = 3e^{3x}$; v = $\frac{1}{2}$sin2x

$\int e^{3x}$ cos2xdx

$= \frac{1}{2}e^{3x}$sin2x$- \int 3e^{3x}.\frac{1}{2}$sin2xdx

$= \frac{1}{2}e^{3x}$sin2x$-\frac{3}{2}\int e^{3x}$sin2xxdx

$= \frac{1}{2}e^{3x}$sin2x$-\frac{3}{2}I$ ....(**)

Substituting (**) into (*)

$I = -\frac{1}{2}e^{3x}cos2x + \frac{3}{2}\left(\frac{1}{2}e^{3x}\text{sin2x} - \frac{3}{2}I\right)$

$= -\frac{1}{2}e^{3x}cos2x + \frac{3}{4}e^{3x}\text{sin2x} - \frac{9}{4}I$ + c

$\frac{13}{4}I = -\frac{1}{2}e^{3x}cos2x + \frac{3}{4}e^{3x}$sin2x + A

$I = -\frac{2}{13}e^{3x}$cos2x+$\frac{3}{13}e^{3x}$sin2x + c

$I = \frac{1}{13}e^{3x}(3sin2x - 2cos2x) + c$

$\therefore \int e^{3x}sin2xdx = \frac{1}{13}e^{2x}(3sin2x - 2cos3x) + c$

Or using basic technique

sign	Differentiate	integrate
+	e^{3x}	sin2x
-	$3e^{3x}$	$-\frac{1}{2}$cos2x
+	$9e^{3x}$	$-\frac{1}{4}$sinx

$I = -\frac{1}{2}e^{3x}cos2x + \frac{3}{4}e^{3x}\text{sin2x} - \frac{9}{4}I$ + c

$\frac{13}{4}I = -\frac{1}{2}e^{3x}cos2x + \frac{3}{4}e^{3x}$sin2x + A

$I = -\frac{2}{13}e^{3x}$cos2x+$\frac{3}{13}e^{3x}$sin2x + c

$I = \frac{1}{13}e^{3x}(3sin2x - 2cos2x) + c$

$\therefore \int e^{3x}sin2xdx = \frac{1}{13}e^{2x}(3sin2x - 2cos3x) + c$

(b) Evaluate $\int_0^{\infty} e^{-2x}sin3xdx$

Solution

Taking I =$e^{-2x}sin3xdx$

Let u = e^{-2x}, $\frac{dv}{dx} = sin3x$

$\frac{du}{dx} = -2e^{-2x}$; v = $-\frac{1}{3}$cos3x

$I = -\frac{1}{3}e^{-2x}$cos3x-$\int 2e^{2x}.\frac{1}{3}$sin3xdx

$I = \frac{1}{3}e^{3x}$sin3x$-\frac{2}{3}\int e^{-2x}$cos3xdx ...... (*)

For $\int e^{-2x}$cos3xdx

Let u = e^{-2x}, $\frac{dv}{dx} = cosx$

$\frac{du}{dx} = -2e^{2x}$; v = $\frac{1}{3}$sin3x

$\int e^{-2x}$cos3xdx

$= -\frac{1}{3}e^{-2x}$sin3x+$\frac{2}{3}\int e^{2x}$sins3xdx

$= -\frac{1}{3}e^{-2x}$sin3x+$\frac{2}{3}I$ ....(**)

Substituting (**) into (*)

$I = -\frac{1}{3}e^{-2x}$cos3x$-\frac{2}{3}\left(\frac{1}{3}e^{-2x}\text{cos3x} + \frac{2}{3}I\right)$

$= -\frac{1}{3}e^{-2x}$cos3x$-\frac{2}{9}e^{-2x}\text{sin3x} - \frac{4}{9}I$ + c

$\frac{13}{9}I = -\frac{1}{3}e^{-2x}$scos3x - $\frac{2}{9}e^{-2x}$sin3x + c

$I = -\frac{1}{13}e^{-2x}(3cos3x + 2sin3x) + c$

$\therefore \int e^{-2x}sin3xdx$

$= -\frac{1}{13}e^{-2x}(3cos3x + 2sin3x) + c$

⇨ $\int_0^{\infty} e^{-2x}sin3xdx$

$=\left[-\frac{1}{13}e^{-2x}(3cos3x + 2sin3x)\right]_0^{\infty}$

$=\frac{3}{13}$ since $e^{\infty} = 0$

Revision exercise 13

Integrate each of the following with respect to x

(a) $e^x cosx$ $\left[\frac{1}{2}e^x(sinx + cosx) + c\right]$

(b) $e^x sinxx$ $\left[\frac{1}{2}e^x(sinx - cosx) + c\right]$

(c) $e^{ax} cosbx$ $\left[\frac{e^{ax}}{b^2+a^2}(acosbx + bsinbx) + c\right]$

(d) $e^{3x} sin2x$ $\left[\frac{1}{13}e^{3x}(3sin2x - 2cos2x) + c\right]$

G. Integration of products of trigonometric functions by parts

A student should take note of the following

(i) $\int tanxdx = In(\sec x)$+ c

Proof

$\frac{d}{dx}In(sex)dx = \frac{secxtanx}{secx} = tanx$

Hence$\int tanxdx = In(\sec x)$+ c

(ii) $\int cosecxdx = -In(cosecx + cotx) + c$

Proof

$\frac{d}{dx}In(cosecx + cotx)dx = \frac{-cosecxcotx-cosec^2x}{cosecx+cotx}$

$= \frac{-cosecx(cotx-cosecx)}{cosecx+cotx}$

$= -$cosecx

$\therefore \int cosecxdx = -In(cosecx + cotx) + c$

(iii) $\int secxdx = In(secx + tanx) + c$

Proof

$\frac{d}{dx}In(secx + tanx)dx = \frac{secxtanx+sec^2x}{secx+tanx}$

$= \frac{secx(In(secx+tanx))}{secx+tanx}$

=secx

$\therefore \int secxdx = In(secx + tanx) + c$

(iv) $\int cotxdx = In(sinx) + c$

Proof

$\frac{d}{dx}In(sinx) = \frac{cosx}{sinx} = cotx$

$\therefore \int cotxdx = In(sinx) + c$

Example 22

(a) Find

(i) $\int sec^3xdx$

Solution

Taking $I = \int sec^3xdx = \int secxsec^2xdx$

Let u = secx and $\frac{dv}{dx} = sec^2x$

$\frac{du}{dx} = secxtanx$; v = tanx

$I = secxtanx - \int(secxtanx)tanxdx$

$= secxtanx - \int secxtan^2xdx$

$= secxtanx - \int secx(sec^2x - 1)dx$

$= secxtanx - \int sec^3xdx + \int secx\, dx$

$= secxtanx -$ I + In(secx+tanx)+c

2I = $secxtanx$ + In(secx+tanx)+c

$I = \frac{1}{2}[secxtanx + \text{In}(\text{secx} + \text{tanx})] + c$

$\therefore \int sec^3xdx$

$= \frac{1}{2}[secxtanx + \text{In}(\text{secx} + \text{tanx})] + c$

(ii) $\int cosec^3xdx$

Taking $I = \int cosec^3xdx = \int cosecxcosec^2xdx$

Let u = cosecx and $\frac{dv}{dx} = cosec^2x$

$\frac{du}{dx} = cosecxcotx$; v = -cotx

$I = -cotxcosecx - \int(secxcotx)cotxdx$

$= -cosecxcotx - \int secxcot^2xdx$

$= -cosecxcotx - \int cosecx(cosec^2x - 1)dx$

$= -cosecxtanx - \int cosec^3xdx + \int cosecx\, dx$

$= -cotxcosecx - I + \int cosecx\, dx$+A

$2I = -cotxcosex) - In(cosecx + cotx)$+c

$I = -\frac{1}{2}[cotxcosecx + In(cosecx + cotx)] + c$

$\therefore \int cosec^3xdx$

$= -\frac{1}{2}[cotxcosecx + In(cosecx + cotx)] + c$

(b) Show that $\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} sec^3xtanx = \frac{8}{27}(9 - \sqrt{3})$

Solution

$\int sec^3xtanxdx = \int sec^2xsecxtanx$

Let u = sec^2x and $\frac{dv}{dx} = secxtanx$

$\frac{du}{dx} = 2sec^2x; v = secx$

$\int sec^3xtanxdx = sec^3x - 2\int sec^3xtanxdx$

$I - sec^3x - 2I + c$

$3I = sec^3x$

$I = \frac{1}{3}sec^3x + c$

$\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} sec^3xtanx = \left[\frac{1}{3}sec^3x\right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \frac{1}{3}\left[sec^3\left(\frac{\pi}{3}\right) - sec^3\left(-\frac{\pi}{6}\right)\right]$

$= \frac{1}{3}\left[8 - \frac{8}{3\sqrt{3}}\right] = \frac{1}{3}\left[8 - \frac{8\sqrt{3}}{9}\right]$

$= \frac{8}{3}\left[1 - \frac{\sqrt{3}}{9}\right]$

$= \frac{8}{27}\left[9 - \sqrt{3}\right]$

Revision exercise 14

Integrate each of the following with respect to x

1. sec^3x

   $\left[\frac{1}{2}[secxtanx + \text{In}(\text{secx} + \text{tanx})] + c\right]$
2. $cosec^3x$

   $\left[-\frac{1}{2}[cotxcosecx + In(cosecx + cotx)] + c\right]$
3. sec^3xtanx

   $\left[\frac{1}{3}sec^3x\right]$

Integration using t- substitution

Case 1

We know that if t = $tan\frac{1}{2}\theta$, then

$\sin\theta = \frac{2t}{1+t^2}$ and

$cos\theta = \frac{1-t^2}{1+t^2}$

Generally

If t = $tan\frac{1}{2}k\theta$, then

$\sin k\theta = \frac{2t}{1+t^2}$ and

$cosk\theta = \frac{1-t^2}{1+t^2}$

Example 23

Find

(a) $\int cosecxdx$

Solution

Let t = tan$\frac{1}{2}x$

$dt = sec^2\frac{1}{2}xdx$

$2dt = (1 + t^2)dx$

$dx = \frac{2}{1+t^2}dt$

$\int cosecxdx = \int \frac{1}{sinx}dx = \int \frac{1+t^2}{2t}.\frac{2}{1+t^2}dt$
$= \int \frac{1}{t}dt = Int + c$

$\therefore \int cosecxdx = In\left(tan\frac{1}{2}x\right) + c$

(b) $\int secxdx$

Solution

Let t = tan$\frac{1}{2}x$

$dt = sec^2\frac{1}{2}xdx$

$2dt = (1 + t^2)dx$

$dx = \frac{2}{1+t^2}dt$

$\int secxdx = \int \frac{1}{cosx}dx = \int \frac{1+t^2}{1-t^2}.\frac{2}{1+t^2}dt$
$= \int \frac{2}{1-t^2}dt = \int \frac{2}{(1+t)(1-t)}dt$

Let $\frac{2}{(1+t)(1-t)} = \frac{A}{1+t} + \frac{B}{1-t}$

2= A(1 – t) + B(1 + t)

Putting t = 1, B = 1

Putting t = -1; A= 1

$=>\frac{2}{(1+t)(1-t)} = \frac{1}{1+t} + \frac{1}{1-t}$

$\int \frac{2}{1-t^2}dt = \int \frac{1}{1+t}dt + \int \frac{1}{1-t}dt$

$= In(1+t) - In(1-t) + c = In\left(\frac{1+t}{1-t}\right) + c$

$\therefore \int secxdx = In\left(\frac{1+tan\frac{1}{2}x}{1-tan\frac{1}{2}x}\right) + c$

(c) $\int sec3xdx$

Solution

Let t = $tan\frac{1}{2}(3x) = \tan\frac{3}{2}x$

$2dt = 3(1+t^2)dx$

$dx = \frac{2}{3(1+)}dt$

$\int sec3xdx = \int \frac{1}{cos3x}dx = \int \frac{1+t^2}{1-t^2}.\frac{2}{3(1+t^2)}dt$
$= \frac{2}{3}\int \frac{1}{1-t^2}dt = \frac{1}{3}\int \frac{2}{1-t^2}dt$

Let$\frac{2}{(1+t)(1-t)} = \frac{A}{1+t} + \frac{B}{1-t}$

2= A(1 – t) + B(1 + t)

Putting t = 1, B = 1

Putting t = -1; A= 1

$=>\frac{2}{(1+t)(1-t)} = \frac{1}{1+t} + \frac{1}{1-t}$

$\int \frac{2}{1-t^2}dt = \int \frac{1}{1+t}dt + \int \frac{1}{1-t}dt$

$= In(1+t) - In(1-t) + c = In\left(\frac{1+t}{1-t}\right) + c$

$\therefore \int sec3xdx = \frac{1}{3}In\left(\frac{1+tan\frac{1}{2}x}{1-tan\frac{1}{2}x}\right) + c$

(d) $\int \frac{1}{3-2cosx}dx$

Solution

Let t = $tan\frac{1}{2}x$

$dt = sec^2\frac{1}{2}xdx$

$2dt = (1+t^2)dx$

$dx = \frac{2}{1+t^2}dt$

$\int \frac{1}{3-2cosx}dx = \int \frac{1}{3-2\left(\frac{1-t^2}{1+t^2}\right)}.\frac{2}{1+t^2}dt = 2\int \frac{1}{1+5t^2}dt$

$= \frac{2}{\sqrt{5}}tan^{-1}(\sqrt{5}t) + c = \frac{2\sqrt{5}}{5}tan^{-1}(\sqrt{5}t) + c$

$\therefore \int \frac{1}{3-2cosx}dx = \frac{2\sqrt{5}}{5}tan^{-1}\left(\sqrt{5}tan\frac{1}{2}x\right) + c$

(e) $\int \frac{2}{3sin2x+4}dx$

Solution

Let t = $tanx$

$dt = sec^2xdx$

$dt = (1+t^2)dx$

$dx = \frac{1}{1+t^2}dt$

$\int \frac{2}{3sin2x+4}dx = \int \frac{2}{3\left(\frac{2t}{1+t^2}\right)+4}.\frac{1}{1+t^2}dt$

$= \int \frac{1}{2t^2+3t+2}dt = \int \frac{1}{\frac{7}{8}+2\left(t+\frac{3}{4}\right)^2}dt$

$= \frac{1}{\sqrt{2}}.\frac{\sqrt{8}}{\sqrt{7}}tan^{-1}\frac{\sqrt{2}\left(t+\frac{3}{4}\right)}{\sqrt{\left(\frac{7}{8}\right)}} + c$

$= \frac{2}{\sqrt{7}}tan^{-1}\frac{(4t+3)}{\sqrt{7}} + c$

$\therefore \int \frac{2}{3sin2x+4}dx = \frac{2\sqrt{7}}{7}tan^{-1}\left(\frac{4tanx+3}{\sqrt{7}}\right) + c$

(f) $\int \frac{2}{3+5cos\frac{1}{2}x}$

Solution

Let t = $tan\frac{1}{4}x$

$dt = \frac{1}{4}sec^2\frac{1}{4}xdx$

$dt = \frac{1}{4}(1+t^2)dx$

$dx = \frac{4}{1+t^2}dt$

(g) $\int \frac{2}{3+5cos\frac{1}{2}x} = \int \frac{2}{3+5\left(\frac{1-t^2}{1+t^2}\right)}.\frac{4}{1+t^2}dt$
$= \int \frac{2}{4-t^2}dt = \int \frac{2}{(2+t)(2-t)}dt$

Let $\frac{2}{(2+t)(2-t)} = \frac{A}{(2+t)} + \frac{B}{2-t}$

2= A(2 – t)+ B(2 + t)

Putting t = 2; B= $\frac{1}{2}$

Putting t = -2; A= $\frac{1}{2}$

$$\int \frac{2}{3+5cos\frac{1}{2}x}dx = \frac{1}{2}\int \frac{1}{2+t}dt + \frac{1}{2}\int \frac{1}{2-t}dt$$

$$= \frac{1}{2}In(2+t) - \frac{1}{2}In(2-t) + c$$

$$= \frac{1}{2}In\left(\frac{2+t}{2-t}\right) + c$$

$$\therefore \int \frac{2}{3+5cos\frac{1}{2}x}dx = \frac{1}{2}In\left(\frac{2+tan\frac{1}{4}x}{2-tan\frac{1}{4}x}\right) + c$$

Case II

When integrating fractional trigonometric functions containing the square of sinx, cosx, etc.

We use the

t-substitution, t = tanx

For sin^2kx or cos^2kx, we use t = tanx

Example 24

Find the integrals of the following

(a) $\int \frac{1}{4sin^2x-9cos^2x}dx$

Solution

Dividing numerator and denominator by cos^2x

$$\int \frac{1}{4sin^2x-9cos^2x}dx = \int \frac{sec^2x}{4tan^2x-9}dx$$

$$= \int \frac{1+tan^2x}{4tan^2x-9}dx$$

Let t = tanx

$$dt = sec^2xdx = (1+t^2)dx$$

$$dx = \frac{dt}{(1+t^2)}$$

$$\int \frac{1+tan^2x}{4tan^2x-9}dx = \int \frac{1+t^2}{4t^2-9}.\frac{dt}{(1+t^2)}$$

$$= \int \frac{1}{(2t+3)(2t-3)}dt$$

Let $\frac{1}{(2t+3)(2t-3)} = \frac{A}{(2t+3)} + \frac{B}{(2t-3)}$

1 = A(2t – 3) + B(2t + 3)

Putting t = $\frac{3}{2}$; B = $\frac{1}{6}$

Putting t = $-\frac{3}{2}$; A = $-\frac{1}{6}$

$$\Rightarrow \int \frac{1}{(2t+3)(2t-3)}dt$$
$$= \frac{1}{6}\int \frac{B}{(2t-3)}dt + \frac{1}{6}\int \frac{1}{(2t+3)}dt$$
$$= \frac{1}{6}.\frac{1}{2}In(2t-3) - \frac{1}{6}.\frac{1}{2}In(2t+3) + c$$
$$= \frac{1}{12}In\left(\frac{2t-3}{2t+3}\right) + c$$
$$\int \frac{1}{4sin^2x-9cos^2x}dx = \frac{1}{12}In\left(\frac{2tanx-3}{2tanx+3}\right) + c$$

(b) $\int \frac{1}{3+4sin^2 5x}dx$

Solution

Dividing by the numerator and denominator $cos^2 5x$

$$\int \frac{1}{3+4sin^2 5x}dx = \int \frac{sec^2 5x}{3sec^2 5x-4}dx$$
$$= \int \frac{1+tan^2 5x}{3+7tan^2 5x}dx$$

Let t = tan5x

$$dt = sec^2 5xdx = 5(1+t^2)dx$$

$$dx = \frac{dt}{5(1+t^2)}$$

$$\int \frac{1}{3+4sin^2 5x}dx = \int \left(\frac{1+t^2}{3+7t^2}\right).\frac{dt}{5(1+t^2)}$$
$$= \frac{1}{5}\int \frac{1}{3+7t^2}dt$$
$$= \frac{1}{5}.\frac{1}{\sqrt{7}}.\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{\sqrt{7}}{\sqrt{3}}t\right) + c$$

$$= \frac{1}{5\sqrt{21}}tan^{-1}\left(\frac{\sqrt{7}}{\sqrt{3}}tan5x\right) + c$$

$$\therefore \int \frac{1}{3+4sin^2 5x}dx = \frac{\sqrt{21}}{105}tan^{-1}\left(\frac{\sqrt{21}}{3}tan5x\right) + c$$

(c) $\int \frac{sin^2 3x}{1+cos^2 3x}dx$

Solution

Dividing numerator and denominator by $cos^2 3x$

$$\int \frac{sin^2 3x}{1+cos^2 3x}dx = \int \frac{tan^2 3x}{sec^2 3x+1}dx = \int \frac{tan^2 3x}{2+tan^2 3x}dx$$

Let t = tan3x

$$dt = 3sec^2 3xdx = 3(1+t^2)dx$$

$dx = \frac{dt}{3(1+t^2)}$

$\int \frac{tan^2 3x}{2+tan^2 3x} dx = \int \frac{t^2}{2+t^2} . \frac{dt}{3(1+t^2)}$

$= \frac{1}{3} \int \frac{t^2}{(2+t^2)(1+t^2)} dt$

Let $\frac{t^2}{(2+t^2)(1+t^2)} = \frac{Ax+B}{(2+t^2)} + \frac{Cx+D}{(1+t^2)}$

By equating coefficients and solving simultaneously

A = 2, C = -1, B = D = 0

$\int \frac{sin^2 3x}{1+cos^2 3x} dx = \int \frac{2}{(2+t^2)} dt - \int \frac{1}{(1+t^2)} dt$

$= \frac{1}{3}\left[\frac{2}{\sqrt{2}} tan^{-1}\left(\frac{t}{\sqrt{2}}\right) - tan^{-1}t\right] + c$

$= \frac{1}{3}\left[\frac{2}{\sqrt{2}} tan^{-1}\left(\frac{tan3x}{\sqrt{2}}\right) - tan^{-1}(tan3x)\right] + c$

$= \frac{\sqrt{2}}{3} tan^{-1}\left(\frac{tan3x}{\sqrt{2}}\right) - \frac{1}{3} tan^{-1}(tan3x) + c$

(d) $\int \frac{1}{cos2x - 3sin^2 x} dx$

Solution

$\int \frac{1}{cos2x-3sin^2 x} dx = \int \frac{1}{1-5sin^2 x} dx$

Dividing the numerator and denominator by $cos_2 x$

$\int \frac{1}{sec^2 - 5tan^2 x} dx$

Let t = tanx

$dt = sec^2 x dx = (1+t^2)dx$

$dx = \frac{dt}{(1+t^2)}$

$\int \frac{1}{1-5sin^2 x} dx = \int \frac{1}{1-4t^2} . \frac{dt}{(1+t^2)}$

$= \int \frac{1}{1-4t^2} dt = \int \frac{1}{(1+2t)(1-2t)} dt$

Let $\frac{1}{(1+2t)(1-2t} = \frac{A}{1+2t} + \frac{B}{1-2t}$

1 = A(1 – 2t) + B (1 + 2t)

Putting t = $\frac{1}{2}$; B = $\frac{1}{2}$

Putting t = $-\frac{1}{2}$; A = $\frac{1}{2}$

$\int \frac{1}{(1+2t)(1-2t)} dt = \frac{1}{2} \int \frac{dt}{1+2t} + \frac{1}{2} \int \frac{dt}{1-2t}$

$= \frac{1}{2}\left[\frac{1}{2} In(1+2t) - \frac{1}{2} In(1-2t)\right] + c$

$= \frac{1}{4} In\left(\frac{1+2t}{1-2t}\right) + c$

$\therefore \int \frac{1}{cos2x-3sin^2 x} dx = \frac{1}{4} In\left(\frac{1+2tanx}{1-2tanx}\right) + c$

Revision exercise 14

1. Integrate the following

(a) $\int \frac{4}{3+5sinx} dx$ $\left[\frac{3tan\frac{1}{2}x+1}{tan\frac{1}{2}x+3} + c\right]$

(b) $\int \frac{1}{4+5cosx} dx$ $\left[\frac{1}{3} In\left(\frac{3+tan\frac{1}{2}x}{3-tan\frac{1}{2}x}\right) + c\right]$

(c) $\int \frac{1}{1+5sin2x} dx$ $\left[-\frac{1}{1+tanx} + c\right]$

(d) $\int \frac{4}{5+3cos\frac{1}{2}x} dx$ $\left[tan^{-1}\left(\frac{1}{2} tan\frac{1}{4}x\right) + c\right]$

(e) $\int \frac{4}{2+sin\frac{1}{2}x} dx$

$\left[\frac{8\sqrt{3}}{9} tan^{-1}\left(\frac{2tan\frac{1}{4}x+1}{\sqrt{3}}\right) + c\right]$

2. Integrate each of the following

(a) $\int \frac{1}{1+2sin^2 x} dx$

$\left[\frac{\sqrt{3}}{3} tan^{-1}(\sqrt{3}tanx) + c\right]$

(b) $\int \frac{1}{1-10sin^2 x} dx$

$\left[\frac{1}{6} In(1+3tanx) - \frac{1}{6} In(1-3tanx)\right] + c$

(c) $\int \frac{sin^2 x}{1+cos^2 x} dx$

$\left[\sqrt{2} tan^{-1}\left(\frac{\sqrt{2}}{2} tanx\right) - x + c\right]$

(d) $\int \frac{4}{cos^2 x + 9sin^2 x} dx$

$\left[\frac{4}{3} tan^{-1}(3tanx) + c\right]$

(e) $\int \frac{1+sinx}{cos^2 x} dx$

$[tanx + secx + c]$

(f) $\int \frac{1}{1+tanx} dx$

$\left[\frac{1}{2}x + \frac{1}{2} In(cosx + sinx) + c\right]$

3. Evaluate

(a) $\int_0^{\frac{\pi}{2}} \frac{3}{1+sinx} dx$ $[3]$

(b) $\int_0^{\frac{2\pi}{3}} \frac{3}{5+4cosx} dx$ $\left[\frac{1}{3}\pi\right]$

(c) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3}{4+5cosx} dx$ $[2In2]$

(d) $\int_0^{\frac{\pi}{2}} \frac{5}{3sinx+4cosx} dx$ $[In6]$

Integration of special cases involving splitting the numerator

Case 1

When a fractional integrand with quadratic denominator expressed in the form of $\frac{f(x)}{g(x)}$ is such that g(x) cannot be factorized or written in simple partial fractions, it is normally very useful to express it as a fraction by splitting the numerator.

i.e. Numerator = A(derivative of denominator + B

Example 25

Find the integral of each of the following

(a) $\int \frac{2x-1}{4x^2+3}dx$

Solution

Numerator=$A\left[\frac{d}{dx}(4x^2+3)\right]+B$

2x – 1 =A(8x)+B

Putting x = 0, B= -1

Putting x = 1, A = $\frac{1}{4}$

$\int \frac{2x-1}{4x^2+3}dx = \frac{1}{4}\int \frac{8x}{4x^2+3}dx - \int \frac{1}{4x^2+3}dx$

$= \frac{1}{4}In(4x^2+3) - \frac{\sqrt{3}}{6}tan^{-1}\left(\frac{2\sqrt{3}}{3}x\right) + c$

(b) $\int \frac{2x+3}{x^2+2x+10}dx$

Solution

Numerator=$A\left[\frac{d}{dx}(x^2+2x+10)\right]+B$

2x +3 =A(2x+2)+B

Putting x = -1, B= 1

Putting x = 0, A = 1

$\int \frac{2x+3}{x^2+2x+10}dx$

$= \frac{1}{4}\int \frac{2x+2}{x^2+2x+10}dx + \int \frac{1}{x^2+2x+10}dx$

$= In(x^2+2x+10) + \int \frac{1}{9+(x+1)^2}$dx

$= In(x^2+2x+10) + \frac{1}{3}tan^{-1}\left(\frac{x+1}{3}\right) + c$

(c) $\int \frac{x}{x^2+3x+5}dx$

Solution

Numerator=$A\left[\frac{d}{dx}(x^2+3x+5)\right]+B$

x =A(2x+3)+B

Putting x = $-\frac{3}{2}$, B= $-\frac{3}{2}$

Putting x = 0, A = $-\frac{1}{2}$

$\int \frac{x}{x^2+3x+5}dx$

$= \frac{1}{2}\int \frac{2x+3}{x^2+3x+5}dx - \frac{3}{2}\int \frac{1}{x^2+3x+5}dx$

$= \frac{1}{2}In(x^2+3x+5) - \frac{3}{2}\int \frac{1}{\frac{11}{4}+\left(x+\frac{3}{2}\right)^2}$dx

$= \frac{1}{2}In(x^2+3x+5) - \frac{3}{\sqrt{11}}tan^{-1}\left(\frac{2x+3}{\sqrt{11}}\right) + c$

(d) $\int \frac{1-2x}{9-(x+2)^2}dx$

Solution

$\int \frac{1-2x}{\sqrt{9-(x+2)^2}}dx$

$= \int \frac{1}{\sqrt{9-(x+2)^2}}dx - \int \frac{2x}{\sqrt{9-(x+2)^2}}dx$

$= sin^{-1}\left(\frac{x+2}{3}\right) - \int \frac{2x}{\sqrt{9-(x+2)^2}}dx$

For $\int \frac{2x}{\sqrt{9-(x+2)^2}}dx$

Let sinu = $\frac{x+2}{3}$

3sinu= x +2

3cosudu=dx

$\int \frac{2x}{\sqrt{9-(x+2)^2}}dx = \int \frac{2(3sinu-2)}{\sqrt{9-9sin^2u}}.3cosudu$

$= \int \frac{6sinu-4}{3\sqrt{(1-sin^2u)}}.3cosudu$

$= \int(6sinu-4)du$

$= -6cosu - 4u + c$

$= -6\sqrt{1-\left(\frac{x+2}{3}\right)^2} - 2sin^{-1}\left(\frac{x+2}{3}\right) + c$

Substituting for $\int \frac{2x}{\sqrt{9-(x+2)^2}}dx$

$\int \frac{1-2x}{\sqrt{9-(x+2)^2}}dx$

$= sin^{-1}\left(\frac{x+2}{3}\right) + -6\sqrt{1-\left(\frac{x+2}{3}\right)^2} - 4sin^{-1}\left(\frac{x+2}{3}\right) + c$

$= 5sin^{-1}\left(\frac{x+2}{3}\right) + 6\sqrt{1-\left(\frac{x+2}{3}\right)^2} + c$

Case II

When finding the integral of fractional trigonometric function expressed in the form $\int \frac{acosx+bsinx}{c\ cosx+dsinx}$, a, b, c and d are constants, we split the numerator as:

Numerator = A(derivative of denominator)+ (denominator)

Example 26

1. Find

(a) $\int \frac{2cosx+9sinx}{3cosx+sinx}dx$

Solution

Let 2cosx+ 9sinx =A$\frac{d}{dx}$(3cosx+sinx)+B(3cosx+sinx)

2cosx+ 9sinx =A(-3sinx+cosx)+B(3cosx+sinx)

2cosx+ 9sinx =(A+3B)cosx +(-3A+B)sinx

Equating coefficients:

For cosx: A+3B = 2 ………………. (i)

For sinx: -3A+B = 9…………………(ii)

Solving Eqn. (i) and Eqn. (ii) simultaneously

A= $-\frac{5}{2}$ and B = $\frac{3}{2}$

⇨ $\int \frac{2cosx+9sinx}{3cosx+sinx}dx$

$=-\frac{5}{2}\int \frac{-3sinx+cosx}{3cosx+sinx}dx + \frac{3}{2}\int \frac{3cosx+sinx}{3cosx+sinx}$

$=-\frac{5}{2}$In(3cosx+sinx)+ $\frac{3}{2}x + c$

(b) $\int \frac{3sinx}{4cosx-sinx}dx$

Solution

Let 3sinx = A$\frac{d}{dx}$(4cosx–sinx) + B(4cosx–sinx)

3sinx = A(-4sinx–cosx) + B (4cosx – sinx)

3sinx = (-A+B)cosx +(-4A-B)sinx

Equating coefficients

For cosx: -A + 4B = 0 …………………. (i)

For sin x: -4A- B = 3 …………………. (ii)

Solving Eqn. (i) and Eqn. (ii) simultaneously

A= $-\frac{12}{17}$ and B= $-\frac{3}{17}$

$\int \frac{3sinx}{4cosx-sinx}dx$

$= -\frac{12}{17}\int \frac{-4sinx-cosx}{4cosx-sinx}dx - \frac{3}{17}\int \frac{4cosx-sinx}{4cosx-sinx}dx$

$= -\frac{12}{17}In(4cosx - sinx) - \frac{3}{17}x$+c

2. Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cosx-sinx}{3cosx+2sinx}dx$

Solution

Let 3sinx = A$\frac{d}{dx}$(3cosx+2sinx) + B(3cosx+2sinx)

cosx-sinx = A(-3sinx+2cosx) + B (3cosx +2sinx)

cosx- sinx = (2A+3B)cosx +(-3A+2B)sinx

Equating coefficients

For cosx: 2A + 3B = 1 …………………. (i)

For sin x: -3A+2B = -1 …………………. (ii)

Solving Eqn. (i) and Eqn. (ii) simultaneously

A= $\frac{5}{13}$ and B= $-\frac{1}{13}$

$\int \frac{cosx-sinx}{3cosx+2sinx}dx$

$=\frac{5}{13}\int \frac{2cosx-3sinx}{3cosx+2sinx}dx + \frac{1}{13}\int \frac{3cosx+2sinx}{3cosx+2sinx}dx$

$=\frac{5}{13}In(3cosx + 2sinx) + \frac{1}{13}x + c$

$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cosx-sinx}{3cosx+2sinx}dx$

$=\left[\frac{5}{13}In(3cosx + 2sinx) + \frac{1}{13}x + c\right]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}$

$=\left[\frac{5}{13}In\left(3cos\frac{\pi}{2} + 2sin\frac{\pi}{2}\right) + \frac{1}{13}.\frac{\pi}{2}\right] - \left[\frac{5}{13}In\left(3cos\frac{\pi}{6} + 2sin\frac{\pi}{6}\right) + \frac{1}{13}.\frac{\pi}{6}\right]$

$=\left[\frac{5}{13}In2 + \frac{\pi}{26}\right] - \left[\frac{5}{13}In\frac{2+\sqrt{3}}{2} + \frac{\pi}{78}\right]$

$=\frac{5}{13}In\left(\frac{4}{2+3\sqrt{3}}\right) + \frac{\pi}{39}$

Revision exercise 15

1. Integrate each of the following

(a) $\int \frac{x+2}{x^2+2x+4}dx$

$\left[\frac{1}{2}In(x^2 + 2x + 4) + \frac{\sqrt{3}}{3}tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + c\right]$

(b) $\int \frac{x}{x^2-x+3}dx$

$\left[\frac{1}{2}In(x^2 - x + 3) + \frac{2}{\sqrt{11}}tan^{-1}\left(\frac{2x-1}{\sqrt{11}}\right) + c\right]$

(c) $\int \frac{2(x+1)}{x^2+4x+8}dx$

$\left[In(x^2+4x+8) - tan^{-1}\left(\frac{x+2}{2}\right) + c\right]$

(d) $\int \frac{5x+7}{x^2+4x+8}dx$

$\left[\frac{5}{2}In(x^2+4x+8) - \frac{3}{2}tan^{-1}\left(\frac{x+2}{2}\right) + c\right]$

2. Integrate the following

(a) $\int \frac{cosx-2sinx}{3cosx+4sinx}dx$

$\left[\frac{2}{5}In(4sinx+3cox) - \frac{1}{5}x + c\right]$

(b) $\int \frac{cosx}{2cosx-sinx}dx$

$\left[-\frac{1}{5}In(2cosx-sinx) + \frac{2}{5}x + c\right]$

(c) $\int \frac{cosx}{cosx-2sinx}dx$

$\left[-\frac{2}{5}In(cosx-2sinx) - \frac{1}{5}x + c\right]$

(d) $\int \frac{2cosx+sinx}{4cosx+3sinx}dx$

$\left[-\frac{14}{15}In(4cosx+3sinx) - \frac{11}{5}x + c\right]$

Revision exercise 16: general topical revision questions

1. Find

(a) $\int sinxdx$ $\left[xsin^{-1}x + \sqrt{1-x^2} + c\right]$

(b) $\int xsec^2xdx$ $[xtanx + Incosx + c]$

(c) $\int \frac{x^2}{\sqrt{1-x^2}}dx$ $\left[\sqrt{1-x^2}\left(\frac{-2-x^2}{3}\right) + c\right]$

(d) $\int In(x^2-4)dx$

$\left[xIn(x^2-4) - 2x + 2\left(In\frac{x+2}{x-2}\right) + c\right]$

(e) $\int \frac{dx}{3-2cosx}dx$ $\left[\frac{2}{\sqrt{5}}tan^{-1}\left(\sqrt{5}tan\frac{x}{2}\right) + c\right]$

(f) $\int 3^{\sqrt{2x-1}}dx$

$\left[\frac{3^{\sqrt{2x-1}}}{In3}\left(\sqrt{2x-1} - \frac{1}{In3}\right) + c\right]$

(g) $\int sin^2xdx$ $\left[\frac{1}{2}\left(x - \frac{1}{2}sin2x\right) + c\right]$

(h) $\int tan^3xdx$ $\left[\frac{1}{2}tan^2x - Incosx + c\right]$

(i) $\int \frac{4x^2}{\sqrt{1-x^6}}dx$ $\left[\frac{4}{3}sin^{-1}(x^3) + c\right]$

(j) $\int \frac{x^2}{x^4-1}dx$ $\left[\frac{1}{4}In\left(\frac{x-1}{x+1}\right) + \frac{1}{2}tan^{-1}x + c\right]$

(k) $\int \frac{2x}{\sqrt{x^2+4}}dx$ $\left[2\sqrt{x^2+4} + c\right]$

(l) $\int xInxdx$ $\left[\frac{x^2}{2}Inx - \frac{x^2}{4} + c\right]$

(m) $\int x^3e^{x^4}dx$ $\left[\frac{1}{4}e^{x^4} + c\right]$

(n) $\int \frac{1}{1+sin^2x}$ $\left[\frac{\sqrt{2}}{2}tan^{-1}(\sqrt{2}tanx) + c\right]$

(o) $\int Inxdx$ $[x(Inx-1) + c]$

(p) $\int x^2sin2xdx$

$\left[-\frac{1}{2}x^2cos2x + \frac{1}{2}xsin2x + \frac{1}{4}cos2x + c\right]$

(q) $\int Inx^2dx$ $[2x(Inx-1) + c]$

(r) $\int \frac{dx}{e^x-1}$ $[In(1-e^{-x}) + c]$

(s) $\int \frac{x^2}{(1+x^2)^{\frac{1}{2}}}dx$ $\left[\frac{1}{3}(1+x^2)^{\frac{1}{2}}(x^2-2) + c\right]$

(t) $\int \frac{dx}{1-cosx}$ $\left[-cot\left(\frac{x}{2}\right) + c\right]$

(u) $\int \frac{x^4-x^3+x^2+1}{x^3+x}dx$

$\left[\frac{x^2}{2} - x + Inx + tan^{-1}x - \frac{1}{2}In(1+x^2) + c\right]$

(v) $\int \frac{sin^{-1}2x}{\sqrt{1-4x^2}}dx$ $\left[\left(\frac{sin^{-1}2x}{2}\right)^2 + c\right]$

(w) $\int x(1-x^2)^{\frac{1}{2}}dx$ $\left[\frac{1}{3}(1-x^2)^{\frac{3}{2}} + c\right]$

(x) $\int \frac{1+\sqrt{x}}{2\sqrt{x}}$dx $\left[\sqrt{x} + \frac{x}{2} + c\right]$

(y) $\int x^2e^xdx$ $[x^2e^x - 2xe^x + 2e^x + c]$

(z) $\int \frac{dx}{x^2\sqrt{(25-x^2)}}dx$ $\left[-\frac{1}{25}\left(\frac{5\sqrt{25-x^2}}{x^2}\right) + C\right]$

2. Evaluate

(a) $\int_0^{\frac{\pi}{2}} xcos^2xdx$ [0.3669]

(b) $\int_1^{\sqrt{3}}(x+tanx)dx$ [1.0003]

(c) $\int_0^{\frac{\pi}{2}} sin2xcosxdx$ $\left[\frac{2}{3}\right]$

(d) $\int_1^3 \frac{x^2+1}{x^3+4x^2+3x}dx$ [0.3489]

(e) $\int_0^1 \frac{x}{\sqrt{1+x}}dx$ [0.3905]

(f) $\int_0^{\frac{\pi}{2}} \frac{cosx}{1+sin^2x}dx$ [0.7854]

(g) $\int_0^{\frac{\pi}{6}} sinxsin3xdx$ [0.1083]

(h) $\int_0^1 \frac{x^3}{x^2+1}dx$ [0.15345]

(i) $\int_0^{\sqrt{\frac{\pi}{2}}} 2xcosx^2dx$ [1]

(j) $\int_0^2 \frac{8x}{x^2-4x-12}dx$ [1.05]

(k) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+sinx+cosx}$ $[In2]$

(l) $\int_0^{\frac{\pi}{2}} sin2xcosxdx$ $\left[\frac{2}{3}\right]$

(m) $\int_4^6 \frac{dx}{x^2-2x-3}$ [0.1905]

(n) $\int_0^{\frac{\pi}{2}} xsin^22xdx$ $\left[\frac{\pi^2}{16}\right]$

(o) $\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{x^4-x^2}} dx$ [2]

(p) $\int_1^3 \frac{3x^2+4x+1}{x^3+2x^2+x} dx$ $[In12]$

(q) $\int_0^{\frac{\pi}{2}} x^2 sinxdx$ $[\pi - 2]$

(r) $\int_0^1 xe^{2x}dx$ $[2.0973]$

(s) $\int_{\frac{\pi}{3}}^{\pi} xsinxdx$ $[2.7992]$

(t) $\int_{\frac{1}{2}}^{1} 10x\sqrt{(1- x^2)}\, dx$ $[2.165]$

(u) $\int_0^{\frac{\pi}{2}} sin5xcos3xdx$ $\left[\frac{1}{2}\right]$

(v) $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{9+4x^2} dx$ $\left[\frac{\pi}{36}\right]$

3. Show that

(a) $\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)} = \frac{\pi}{20}$

(b) $\int_0^{\frac{\pi}{2}} xtan^2xdx = \frac{1}{32}(8\pi - \pi^2 - 16\log_e 2)$

(c) $\int_2^4 xInxdx = 14In2 - 3$

(d)

4. Given that

$\frac{3x^3+2x^2-6x-2}{(x^2+x-2)(x^2-2)} = \frac{1}{x+2} + \frac{B}{x-1} + \frac{Cx+D}{x^2-2}$

Determine the values of A, B, C, D

Hence evaluate $\int_3^4 \frac{3x^3+2x^2-6x-2}{(x^2+x-2)(x^2-2)}$dx

$[A = B = C = 1, D = 0; 2.4770]$

5. Use the substitution of x = $\frac{1}{u}$ to evaluate

$\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ $\left[\frac{\pi}{3}\right]$

6. Express $\frac{x^3-3}{(x-2)(x^2+1)}$ as partial fractions

$\left[\frac{x^3-3}{(x-2)(x^2+1)} = 1 + \frac{1}{x-2} + \frac{x+1}{x^2+1}\right]$

Hence find $\int \frac{x^3-3}{(x-2)(x^2+1)} dx$

$\left[x + In(x-2) + \frac{1}{2}In(x^2+1) + tan^{-1}x + c\right]$

7. Express f(x) = $\frac{2x^2-x+14}{(4x^2-1)(x+3)}$ in partial fraction

$\left[\frac{2x^2-x+14}{(4x^2-1)(x+3)} = \frac{-3}{2x+1} + \frac{2}{2x-1} + \frac{1}{x+3}\right]$

Hence evaluate$\int_1^3 f(x)dx$ $[0.7440]$

8. Using the substitution 2x+1 = u, find

$\int_0^1 \frac{xdx}{(2x+1)^2}$ $\left[\frac{1}{18}\right]$

9. Express

(a) f(x) = $\frac{6x}{(x-2)(x+4)^2}$ in partial fraction

$\left[\frac{6x}{(x-2)(x+4)^2} = \frac{1}{3(x-2)} - \frac{1}{3(x+4)} + \frac{4}{(x+4)^2}\right]$

Hence evaluate$\int f(x)dx$

$$\left[\frac{1}{3}In\left(\frac{x-2}{x+4}\right) - \frac{4}{(x+4)} + c\right]$$

(b) f(x) = $\frac{3x^2+x+1}{(x-2)(x+1)^2}$ in partial fraction

$\frac{3x^2+x+1}{(x-2)(x+1)^2}$

$= \frac{5}{9(x-2)} - \frac{5}{9(x+2)} + \frac{4}{3(x+1)^2} - \frac{1}{(x+1)^3}$

Hence evaluate

$\int_3^4 \frac{3x^2+x+1}{(x-2)(x+1)^2} dx$ $[0.317]$

(c)

10. Using the substitution x = 3sinθ, evaluate

(a) $\int_0^3 \sqrt{\left(\frac{3+x}{3-x}\right)}\, dx$ $[7.7125]$

(b) $\int_0^{\pi} \frac{dx}{3+5cosx}$ $[0.2747]$

(e)

11. Use t= tan$\frac{1}{2}x$ to evaluate

(a) $\int_0^{\frac{\pi}{2}} \frac{dx}{3-cosx}$ $[0.6755]$

(b)

12. Given that $\int_0^a (x^2 + 2x - 6)\, dx = 0$, find the value of a [a=-6]

13. Use the substitution $x^2 = \theta$ to find

$\int \frac{x}{1+\cos x^2} dx$ $\left[\frac{1-3x}{3(x+1)^{\frac{5}{3}}(x-1)^{\frac{4}{5}}}\right]$

14. Resolve $y = \frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)}$ into partial fraction

$\left[\frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)} \equiv \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{4}{(x^2+2)}\right]$

Hence find $\int ydx$

$\left[In(x-1) + \frac{-2}{(x-1)} + \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c\right]$

15. Express f(x) = $\frac{1}{x^2(x-1)}$ in partial fraction

$\left[\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}\right]$

Hence evaluate$\int_2^3 f(x)dx$ [0.12102]

Application of integration

Like differentiation, integration has a wide spectrum of application, some of which are discussed below

Acceleration, velocity, displacement

Given the acceleration, a, of a particle, its velocity, v and displacement, s can be computed as long as the initial values are known.

Acceleration, a = $\frac{dv}{dt}$ => v = $\int adt$

Also, velocity v = $\frac{ds}{dt}$ => s = $\int vdt$

Example 27

The acceleration of a particle after t seconds is given by a = 5 + t.

If initially, the particle is moving at $1ms^{-1}$, find the velocity after 2s and the distance it would have covered by then

Given $\frac{dv}{dt} = 5 + t$

⇨ $dv = (5 + t)dt$
$v = 5t + \frac{1}{2}t^2 + c$

Whet t = 0, v = 1, =>c = 1

$\therefore v = 5t + \frac{1}{2}t^2 + 1$

When t = 2s

$v = 5(2) + \frac{1}{2}(2)^2 + 1$= $13ms^{-1}$.

And $\frac{ds}{dt} = 5t + \frac{1}{2}t^2 + 1$

$ds = \left(5t + \frac{1}{2}t^2 + 1\right)dt$

s = $\frac{5}{2}t^2 + \frac{1}{6}t^3 + t + c$

when t = 0, s= 0 => c = 0

$\therefore$s = $\frac{5}{2}t^2 + \frac{1}{6}t^3 + t$

At t =2s

s = $\frac{5}{2}(2)^2 + \frac{1}{6}(2)^3 + 2$=$13\frac{1}{3}$m

Example 28

A particle with a velocity $(2i+3j)ms^{-1}$ is accelerated uniformly at the rate of $(3ti – 2j)ms^{-1}$ from the origin. Find

(i) The speed reached by the particle at t = 4s.
Solution
Given a= 3ti – 2j
v = $\int adt$ =$\int(3ti - 2j)dt$
=$\frac{3}{2}t^2i - 2tj + c$
At t= 0, 2i+3j
c =2i+3j
By substitution
$v = \left(\frac{3}{2}t^2 + 2\right)i + (-2t + 3)j$
At t = 4s
$v = \left(\frac{3}{2}(4)^2 + 2\right)i + (-2(4) + 3)j$
= $(26i – 5j)ms^{-1}$
Speed = $|v| = \sqrt{26^2 + (-5)^2}$ =$26.5ms^{-1}$

(ii) The distance travelled by the particle after 2s.

Solution

r = $\int vdt$

r= $\int \left(\left(\frac{3}{2}(4)^2 + 2\right)i + (-2(4) + 3)j\right)dt$

$= \left(\frac{3}{6}t^3 + 2t\right)i + \left(\frac{-2}{2}t^2 + 3t\right)j + \text{c}$

At t = 0, r = 0; => c = 0

$\therefore r = \left(\frac{3}{6}t^3 + 2t\right)i + \left(\frac{-2}{2}t^2 + 3t\right)j$

At t = 2

$\therefore r = \left(\frac{8}{2} + 4\right)i + (-4 + 6)j$

$|r| = \sqrt{8^2 + 2^2}$ =8.25m

Hence the distance = 8.25m

Example 29

A particle has initial position of (7i+5j)m. the particle moves with constant velocity of (ai+bi)ms-1 and 3s later its position is (10i – j)m. fins the values of a and b.

Solution

Given v = ai +bj

r =$\int vdt = \int(ai + bi)dt + c$

$= ati + btj + c$

at t = 0; r = c =(7i+5j)m

$\therefore r = (at + 7)i + (bt + 5)j$

After 3s

$10i - j = (3a + 7)i + (3b + 5)j$

Equating corresponding vectors

For i: 10 = 3a+7 => a = 1

For j: -1 = 3b +5 => b = -2

$\therefore a = 1\ and\ b = -2$

Example 30

A particle of mass 2kg, initially at rest at (0, 0, 0) is acted on by a force $\begin{pmatrix} 2t \\ 2t \\ 4t \end{pmatrix} N$. Find

(i) its acceleration at time t
from F = Ma
$\begin{pmatrix} 2t \\ 2t \\ 4t \end{pmatrix} = 2a => a = \begin{pmatrix} t \\ t \\ 2t \end{pmatrix}$

(ii) its velocity after 3s
velocity v =$\int adt = \int \begin{pmatrix} t \\ t \\ 2t \end{pmatrix} dt$

v=$\begin{pmatrix} \frac{t^2}{2} \\ \frac{t^2}{2} \\ t^2 \end{pmatrix} + c$

at t = 0, v = 0 => c= 0

$\therefore$v=$\begin{pmatrix} \frac{t^2}{2} \\ \frac{t^2}{2} \\ t^2 \end{pmatrix}$

At t = 3s

$v = \frac{9}{2}i + \frac{9}{2}j + 9k$

(iii) the distance of the particle travelled after 3s.
r= $\int vdt$ =$\int \left(\frac{t^2}{2}i + \frac{t^2}{2}j + t^2k\right) dt$
= $\left(\frac{t^3}{6}i + \frac{t^3}{6}j + \frac{1}{3}t^3k\right) + c$

At t = 0, r = 0 => c = 0

$\therefore r = \left(\frac{t^3}{6}i + \frac{t^3}{6}j + \frac{1}{3}t^3k\right)$

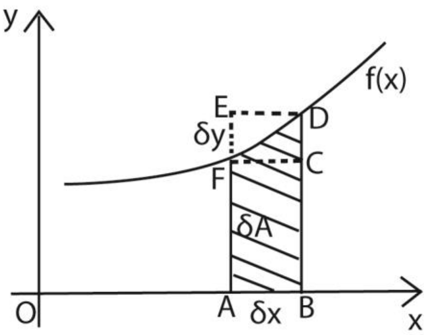
At t = 3

$r = \left(\frac{3^3}{6}i + \frac{3^3}{6}j + \frac{1}{3}.3^3k\right) = \left(\frac{9}{2}i + \frac{9}{2}j + 9k\right)$

$|r| = \sqrt{\left(\frac{9}{2}\right)^2 + \left(\frac{9}{2}\right)^2 + 9^2}$ =11.02m

Area under a curve

If the area under the curve y = f(x) for α≤ x ≤β is required, a small strip can be used for analysis



Suppose the shaded region is δA, the area of the shaded strip lies between areas of the rectangles ABCF and AVDE.

i.e. Area of ABCF ≤ δA ≤ ABDE.

$y\delta x \le \delta A \le (y + \delta y)dx$

Dividing by δx

$y \le \frac{\delta A}{\delta x} \le (y + \delta y)$

$\lim_{\delta x \to 0} \frac{\delta A}{\delta x} \to \frac{dA}{dx}$ and $\delta y \to 0$

Hence $\frac{dA}{dx} = y$

Integrating both sides with respect to x

$\int \frac{dA}{dx} dx = \int ydx$

Now for the interval $\alpha \le x \le \beta$

$A = \int_{\alpha}^{\beta} ydx\ Or\ A = \int_{\alpha}^{\beta} f(x)dx$

Note: when finding the area under the curve, it is advisable that you sketch the curve first in order to establish the required region.

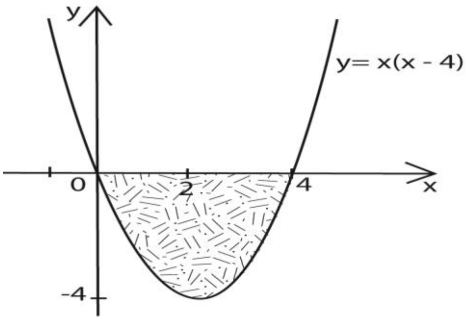
Area between the curve and the x-axis

Example 31

(i) Find the area enclosed by y = x(x – 4) and x-axis

Solution

By sketching the graph y = x(x – 4) with the x-axis we have



Area required = $\int_0^4 x(x-4)dx$

$=\int_0^4 x^2 - 4xdx$

$=\left[\frac{x^3}{3} - 2x^2\right]_0^4$

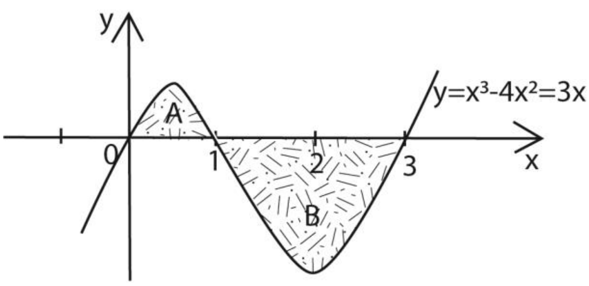
$=\frac{64}{3} - 32 = \frac{-32}{3}$

Hence the area under the curve is $\frac{32}{3}$ $sq.units$ (- sign indicates that the area is below the x-axis).

(ii) Find the area enclosed by the curve y = x^3 – 4x^2 + 3x and the x-axis from x = 0 and x = 3

Solution

By sketching the graph y = x^3 – 4x^2 + 3x with the x-axis we have



Required area = A+ B

Area A $=\int_0^1 (x^3 - 4x^2 + 3x)\,dx$

$=\left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right]_0^1$

$=\left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right) - (0) = \frac{5}{12}$

Area B $=\int_1^3 (x^3 - 4x^2 + 3x)\,dx$

$=\left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right]_1^3$

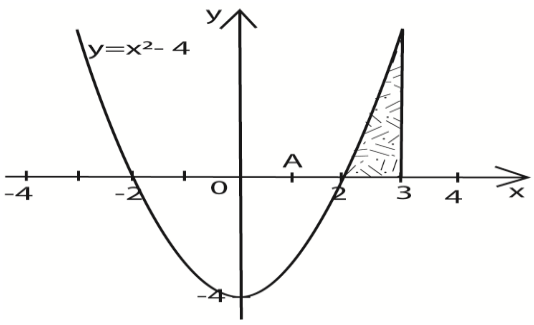
$=\left(\frac{81}{4} - 36 + \frac{27}{2}\right) - \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right) = -\frac{8}{3}$

Area$=\frac{5}{12} + \frac{8}{3} = \frac{37}{12} sq.units$

(iii) Find the area between y=x^2 – 4, the x-axis and line x = 3.

Solution

By sketching the graph of y=x^2 – 4 with the x-axis, we have



Required $= \int_2^3 (x^2 - 4)\,dx$

$= \frac{1}{3}\left[x^3 - 4x\right]_2^3$

$= \frac{7}{3}$ sq. units

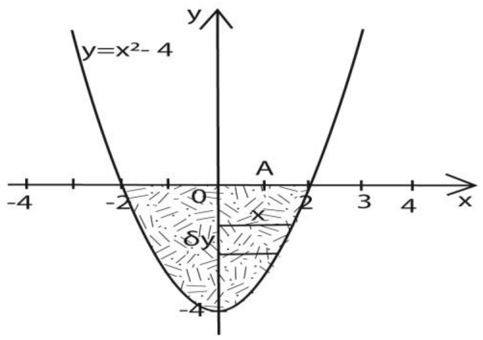
Area between the curve and the y-axis

This involves finding the area under the curve with respect to y or by subtracting the area under the curve with the x-axis from the rectangle (s) formed.

Example 32

Find the area enclosed by the curve y = x^2- 4 and the y = x2 -4 and y-axis between

(i) y = -4 and y = 0
Solution



1st Approach

Integrating with respect to x

Required area $= \int_{-2}^{2}(x^2-4)dx$

$= \frac{1}{3}[x^3-4x]_{-2}^{2}$

$= \left(\frac{8}{3}-8\right)-\left(\frac{-8}{3}+8\right)$

$= \left(\frac{16}{3}-16\right)$sq. units

$= \frac{-32}{3}$

Hence the required area is $\frac{32}{3}\,sq.units$

2nd approach

$y = x^2 - 4$

x = $(y+4)^{\frac{1}{2}}$

Required area = $2\int_{-1}^{0}xdy$

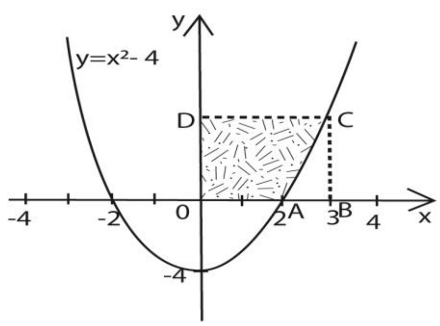
= $2\int_{-1}^{0}(y+4)^{\frac{1}{2}}dy$

= $2\left[\frac{2}{3}(y+4)^{\frac{3}{2}}\right]_{-4}^{0}$

= $2\frac{2}{3}[(8)-(0)]$

= $\frac{32}{3}$ sq. units

(ii) y = 0 and y = 5



1st approach

Required area= 2 x shaded region

Required area = $2\int_{0}^{5}xdy$

= $2\int_{0}^{5}(y+4)^{\frac{1}{2}}dy$

= $2\left[\frac{2}{3}(y+4)^{\frac{3}{2}}\right]_{0}^{5}$

= $2\frac{2}{3}[(2)-(8)]$

= $\frac{76}{3}$ sq. units

2nd approach

Required area = 2 x shaded area

=2[Area of OBCD – area of ABC]

=2[(3 x 5) – $\int_{2}^{3}(x^2-4)dx$]

=2[15– $\frac{7}{3}$]=$\frac{76}{3}$sq.units

Area between two curves

Suppose we want to find the area between two intersecting functions f(x) and g(x), required it to

(i) find the point of intersection of the functions
(ii) sketch the functions f(x) and g(x)

Note if f(x) is above g(x), then the required area

$= \int f(x)dx - \int g(x)dx$

Example 33

Find the area enclosed between the curves

(a) $y = x^2 - 4$ and $y = 4 - x^2$

Solution

Finding the points of intersection

$x^2 - 4 = 4 - x^2$

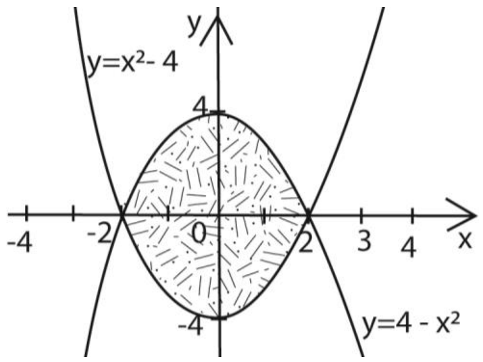
$2x^2 = 8$

x = 2 or x = 2

when x = 2, y = 0

when x = -2, y = 0

The sketch of the functions:



Required area

$= \int_{-2}^{2}[(4-x^2)-(x^2-4)]dx$

$= \int_{-2}^{2}(8-2x^2)dx$

$= \left[8x - \frac{2x^3}{3}\right]_{-2}^{2}$

$= \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right)$

$= \frac{32}{3} + \frac{32}{3} = \frac{64}{3}$ sq. units

(b) y =2x^2+7x+3 and y=9+4x-x^2

Solution

Finding the points of intersection

$2x^2 + 7x + 3 = 9 + 4x - x^2$

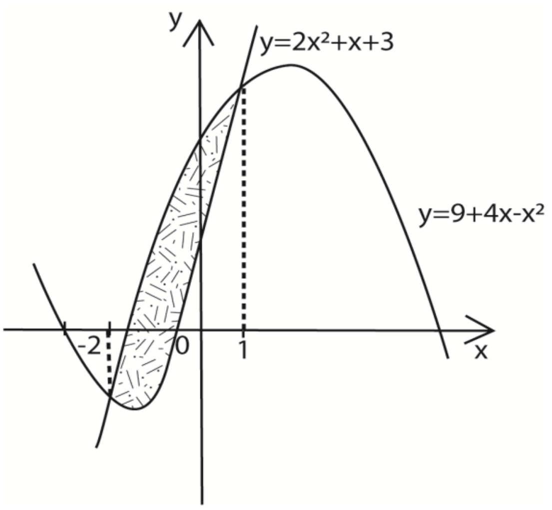
$3x^2 + 3x - 6 = 0$

$(x+2)(x-1) = 0$

x = -2 of x = 1

When x = -2, y = -3

When x = 1, y = 12



Required area

$= \int_{-2}^{1}[(9+4x-x^2)-(2x^2+7x+3)]\,dx$

$= \int_{-2}^{1}(6-3x-3x^2)\,dx$

$= \left[6x - \frac{3x^2}{2} - x^3\right]_{-2}^{1}$

$= \left(6 - \frac{3}{2} - 1\right) - (-12 - 6 + 8)$

$= 13.5 sq.units$

Example 34

Find the area enclosed between the curve y = x^2 – x – 3 and the line 2x +1

Solution

Finding the points of intersection

x^2 – x – 3 = 2x +1

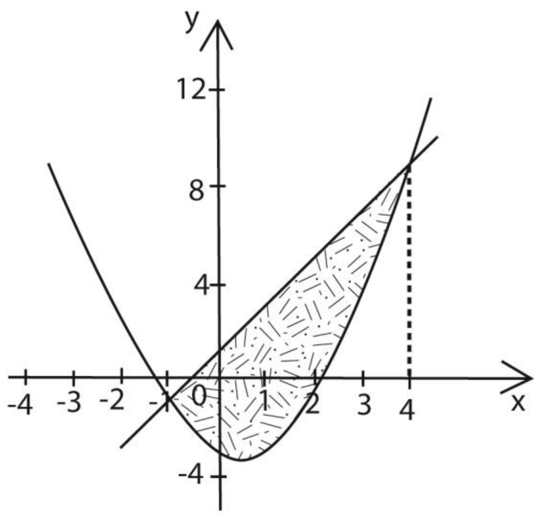
x^2 – 3x – 4 = 0

(x+1)(x-4) = 0

X = -1 or x = 4

When x = -1, y = -1

When x= 4, y = 9



Area required

$= \int_{-1}^{4}[(2x+1)-(x^2-x-3)]\,dx$

$=\int_{-1}^{4}(4+3x-x^2)$dx

$=\left[4x + \frac{3}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^{4}$

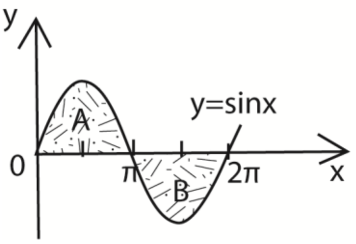
$=\left(16+24-\frac{64}{3}\right)-\left(-4+\frac{3}{3}+\frac{1}{2}\right)$

=20.83 sq.units

Example 35

Find the area enclosed by the curve y = sinx and the x-axis between x = 0 and x = 2π.

Solution



Required area = A + B

$=\int_0^{\pi} sinxdx + \int_{\pi}^{2\pi} sinxdx$

$=[-cosx]_0^{\pi} + [-cosx]_{\pi}^{2\pi}$

$=-(-cos\pi - cos0) - (-cos2\pi - cos\pi)$

=-(-1 – 1)- (-1 – 1)

= 2 +2 = 4sq. units

Volume of a solid of revolution

A solid of revolution is formed when a given area rotates about a fixed axix. Due to the way in which it is formed, it is referred to as solid of revolution.

These bodies have always got axes of symmetry.

The solids formed is subdivided into small cylindrical disks of thickness δx and height y.

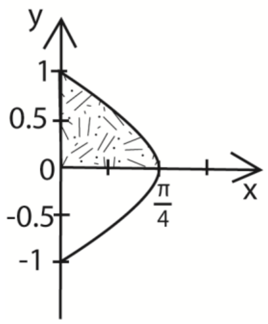
Volume of each disk = $\pi y^2 dx$

Thereforethe colume of the whole solid of revolution is obtained by rotating through one revolution about the x-axis, the region bounded by the curve y = f(x) and the linex x = a and x =b is given by v = $\int_a^b \pi y^2 dx$

If the rotation is about the y-axis, the volume is given by v =$\int_a^b \pi x^2 dy$

Example 36

(a) Find the volume of revolution when the portion of the curve y = cos2x for 0≤x≤$\frac{\pi}{4}$ is rotated through four right angles about the x-axis.
Solution



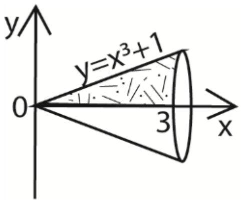
$V=\pi\int_0^{\frac{\pi}{2}} y^2dx = \pi\int_0^{\frac{\pi}{2}} cos^2 2xdx$

$=\pi\int_0^{\frac{\pi}{2}}(1+cos4x)dx$

$=\frac{\pi}{2}\left[x+\frac{1}{4}sin4x\right]_0^{\frac{\pi}{2}}$

$=\frac{1}{8}\pi^2$ cubic units

(b) Find the volume of the area bounded by the curve y = x^3 + 1, the x-axis and limits x = 0 and x = 3 when rotated through four right angles about the x-axis.



$V=\pi\int_0^3 y^2dx = \pi\int_0^3 (x^3+1)^2dx$

$=\pi\int_0^3 (x^6+2x^3+1)^2dx$

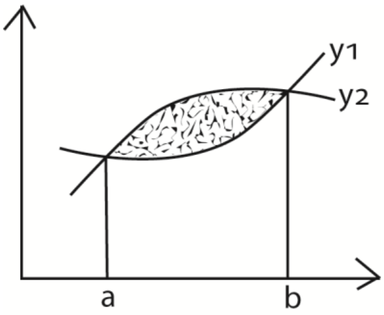
$=\pi\left[\frac{x^7}{7}+\frac{x^4}{2}+x\right]_0^3$

$=\pi\left(\frac{3^7}{7}+\frac{3^4}{2}+3\right)-(0)$

$=1118.25$ cubic units.

Rotation the area enclosed between two curves

If we have two curves y_1 and y_2 that enclose some area between a and b as shown below



Now if we rotate this area about the x-axis the volume of the solid formed is given by

$v = \pi \int_a^b [(y_2)^2 - (y_1)^2]\, dx$

Example 36

(a)A cup is madeby rotating the area between $y= x^2$ and $y = x+1$ with $x \geq 0$ about the x-axis. Find the volume of the material needed to make the cup.

Solution

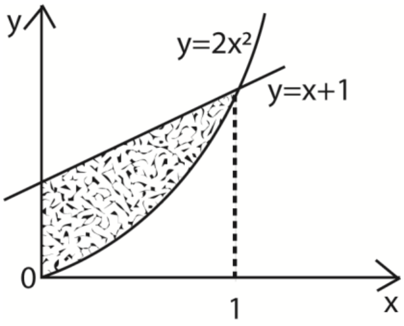
Finding the points of intersection

$2x^2 = x + 1$

$2x^2 – x + 1 = 0$

$(2x+1)(x – 1) = 0$

x =1 since we only need to consider x≥ 0.



V = $\pi \int_0^1 [(y+1)^2 - (2x^2)^2] dx$

=$\pi \int_0^1 (x^2 + 2x + 1 - 4x^4) dx$

=$\pi \left[\frac{x^3}{3} + x^2 + x - \frac{4x^5}{5}\right]_0^1$

= $\pi \left(\frac{1}{3} + 1 + 1 - \frac{4}{5}\right) - 0$

=$\frac{23}{15}\pi$ units cubed

Example 37

Find the volume of revolution when the portion of the area between the curves $y = x^2$ and $x = y^2$ is rotated through 360^0 about the x-axis.

Solution

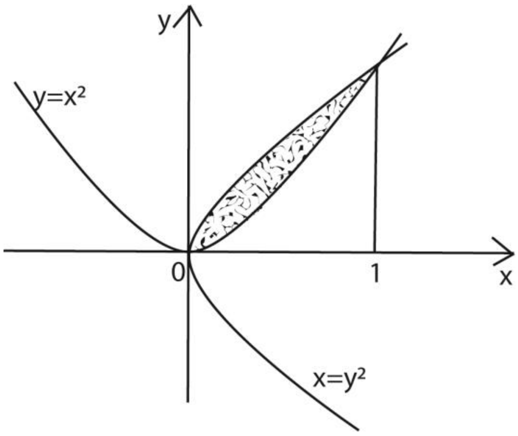
Points of intersection

$x^2 = x^{\frac{1}{2}} => x^4 = x$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

Either x = 0 or x = 1



The volumeof revolution

= $\pi \int_0^1 \left[\left(\sqrt{x}\right)^2 - (x^2)^2\right] dx$

=$\pi \int_0^1 (x - x^4) dx$

=$\pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$

=$\pi \left[\left(\frac{1}{2} - \frac{1}{5}\right) - 0\right]$

=$\frac{3}{10}\pi$

Example 38

Find the volume generated when the area enclosed by the curve $y = 4 – x^2$ and the line $y = 4 – 2x$ is rotated through 2π.

Solution

Finding the points of intersection

$4 – 2x = 4 – x^2$

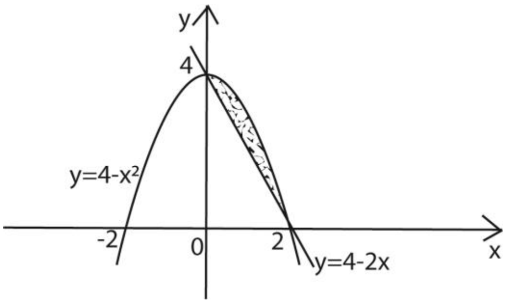
$x^2 - 2x = 0$

$x(x - 2) = 0$

Either x = 0 or x = 2

When x = 0, y = 4

When x = 2, y = 0



Required volume

$=\pi \int_0^2 [(4 - x^2)^2 - (4 - 2x)^2] dx$

$= \pi \int_0^2 [(16 - 8x^2 + x^4) - (16 - 8x - 4x^2)] dx$

$=\pi \int_0^2 (x^4 - 4x^2 + 8x)\, dx$

$=\pi \left[\frac{x^5}{5} - \frac{4x^3}{3} + 4x^2\right]_0^2$

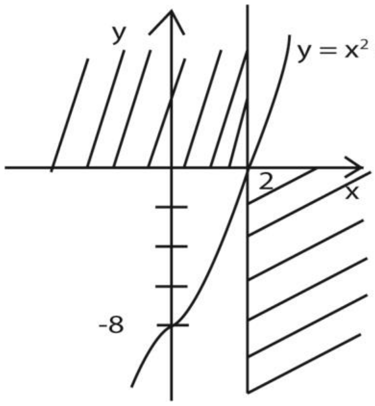
$=\frac{176}{15}\pi = 36.86$cubic units

Example 39

(a) Sketch the curve y = $x^3 - 8$ (08marks)

y = $x^3 - 8$
Intercepts
When x = 0, y = -8
When y = 0, x = 2
(x, y) = (2, 0)
Turning point: $\frac{dy}{dx} = 3x^2$
3x2 = 0
x = 0
$\frac{d^2y}{dx^2} = 6x$
$\frac{d^2y}{dx^2} = 0, x = 0$
Point of reflection= (0, 8)

	x < 2	x > 2
y	-	+



(b) The area enclosed by the curve in (a), the y-axis and x-axis is rotate about the line y = 0 through 360^0. Determine the volume of the solid generated. (04 marks)

V = $\pi \int_0^2 y^2 dx$

$= \pi \int_0^2 (x^3 - 8)^2 dx$

$= \pi \int_0^2 (x^6 - 16x^3 + 64) dx$

$=\pi \left[\frac{x^7}{7} - 4x^4 + 64x\right]_0^2$

$= \pi \left(\frac{128}{7} - 64 + 128\right)$

$= \frac{576\pi}{7}$ =250.5082 units^3

The mean value theorem for integrals

If f(x) is a continuous function on the closed interval [a, b], then there exist a number c in the closed interval such that

Area of the rectangle = f(c).(b-a)

But area under the curve between a and b $=\int_a^b f(x)dx$

Equating the two

$\int_a^b f(x)dx - f(c).(b - a)$

Dividing both sides by (b – a)

$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$

Where f(c) is the height of the rectangle

This height is the average value of the function over the interval in the question.

Hence the mean value of f(x) over a closed interval (a, b) is given by

M.V =$\frac{1}{b-a}\int_a^b f(x)dx$

Example 40

Find the mean value of y= x^2 + 2 for x = 1 and x = 4.

Solution

M.V =$\frac{1}{4-1}\int_1^4 (x^2+2)dx$

=$\frac{1}{3}\int_1^4 (x^2+2)dx$

=$\frac{1}{3}\left[\frac{x^3}{3}+2x\right]_1^4$

=$\frac{1}{3}\left[\left(\frac{64}{3}+8\right)-\left(\frac{1}{3}+2\right)\right]=9$

Example 41

Find the mean value of

y = $\frac{1}{1+sin^2\theta}$ for 0 ≤ θ≤ $\frac{\pi}{4}$

Solution

M.V =$\frac{1}{\frac{\pi}{4}-0}\int_0^{\frac{\pi}{4}}\frac{1}{1+sin^2\theta}dx$

=$\frac{4}{\pi}\int_0^{\frac{\pi}{4}}\frac{sec^2\theta}{sec^2\theta+tan^2\theta}dx$

=$\frac{4}{\pi}\int_0^{\frac{\pi}{4}}\frac{1+tan^2\theta}{1+2tan^2\theta}dx$

Let t = tan θ => dt = $sec^2\theta d\theta$= $(1+t^2)d\theta$

$d\theta=\frac{dt}{1+t^2}$

Changing limits

When θ = 0, t = 0 and when θ=$\frac{\pi}{4}$, t = 1

$\therefore M.V=\frac{\pi}{4}\int_0^1\frac{1+t^2}{1+2t^2}.\frac{dt}{1+t^2}$

$=\frac{\pi}{4}\int_0^1\frac{1}{1+2t^2}dt$

$=\frac{\pi}{4}\left[\frac{1}{\sqrt{2}}tan^{-1}\sqrt{2}t\right]_0^1$

$=\frac{2\sqrt{2}}{\pi}tan^{-1}\sqrt{2}$

= 0.86

Example 42

Find the mean value of y = x(4 – x)in the interval where y ≥ 0.

Solution

Given y ≥ 0 => x(4 –x)≥0 (positive)

The solution is 0 ≤ x ≤ 4

M.V =$\frac{1}{4-0}\int_0^4 x(4-x)dx=\frac{1}{4}\int_0^4(4x-x^2)dx$

=$\frac{1}{4}\left[2x^2-\frac{x^3}{3}\right]_0^4$=$\frac{8}{3}$

Revision exercise 17

1. Find the volume generate in each case whenthe area enclosed by the curve y = x^2 – 6x + 18 and the line y = 10 is rotated about
   (i) Y = 10 [1541π $units^3$]
   (ii) x –axix, [256π $units^3$]
2. Find the volume generated when the area enclosed by the curve y = x4 from y = 3 and y = 6 is rotated about the y-axis [6.33π $units^3$]
3. The displacement x of a particle at time t is given by x = sint. Find the mean value of its velocity over the interval 0 < t < $\frac{\pi}{2}$
   (i) with respect to t [0.637ms^{-1}]
   (ii) with respect to displacement x [0.785ms^{-1}]
4. (a) Determine the equation of the normal to the curve y = $\frac{1}{x}$ and the point x = 2. Find the coordinates of the other point where the normal meets the curve again.
   [2y – 8x+15 =0; $\left(-\frac{1}{8},-8\right)$
   (b) Find the area of the region bounded by the curve y =$\frac{1}{x(2x+1)}$, the x-axis and the lines x = 1 and x = 2. $\left(In\left(\frac{6}{5}\right)\right)$
5. A shell is formed by rotating the portion of the parabola y2= 4x for which 0 ≤ x ≤ 1 through two right angles about its axis. Find
   (i) the volume of the solid formed [2π]
   (ii) the area of the base of the solid formed [4π $units^2$]

6. Show that the tangents at (-1,3) and (1,5)on the curve y = $2x^2 + x + 2$ passes through the origin. Find the area enclosed between the curve and these two tangents $\left[\frac{4}{3}\right]$
7. Sketch the curve $y = x - \frac{8}{x^2}$ for x > 0, showing any a symptotes. Find the area enclosed by the x-axis, the line x = 4 and the curve $x - \frac{8}{x^2}$. [10 sq. units]
   If this area is now rotated about the x-axis through 3600, determine the volume of the solid generated, correct to 3 significant figures. [42.1 cubic units]
8. Show that the tangents to the curve $4 - 2x - 2x^2$ at points 9-1, 4) and $\left(\frac{1}{2}, 2\frac{1}{2}\right)$respectively passes through the point $\left(-\frac{1}{4}, 5\frac{1}{2}\right)$. Calculate the area of the curve enclosed between the curve and the x-axis. [9sq.units]
9. (i) find the Cartesian equation of the curve given parametrically by
   x = $\frac{1+t}{1-t}$, y = $\frac{2t^2}{1-t}$ $\left[y = \frac{(x-1)^2}{x+1}\right]$
   (ii) sketch the curve
   (iii) find the area enclosed between the curve and the line y = 1 [1.955sq.units]
10. Given the curve y = sin3x, find the
    (a)(i) the value of $\frac{dy}{dx}$ at the point $\left(\frac{\pi}{2}, 0\right)$
    (ii) equation of the tangent to the curve at this point [y= 3x+π)

    (b) (i) sketch the curve y = sin3x

    (ii) Calculate the area bounded by the tangent in (a)(i) above, the curve and y-axis

    [0.9783sq. units]
11. A hemisphericalbowl of internal radius r is fixed with its rim horizontal and contains a liquid to the depth h. show by integration that the volume of the liquid in the bowl is $\frac{1}{3}\pi h^2(3r - h)$
12. Find the volume of the solid of revolution formed by rotating the area enclosed by the curve y = x(1+x), the x-axis, the lines x = 2 and x = 3 through four right anglesabout the x-axis. [31.033π cubic units]

Thank you

Dr. Bbosa Science