Differential equations (D.Es)

Definition

These ae equations involving differential coefficients (derived functions) like $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}$.

The order of a differential equation

The order of a differential equation is the order n of which the highest derivative $\frac{d^ny}{dx^n}$ contained in the differential equation. E.g.

(a) $\frac{dy}{dx} + 2 = y$ (1st order D.E)

(b) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2 = 0$ (2nd order D.E)

(c) $\frac{d^4y}{dx^4} + 3\frac{dy}{dx} = 0$ (4th order D.E)

Solution to differential equations

This involves the elimination of all the differential coefficients in the given equation. This is normally done by integrating

Example 1

Solve the differential equations

(a) $\frac{dy}{dx} - 5 = 0$

Solution

$\frac{dy}{dx} - 5 = 0$

$\int dy = \int 5dx$

y = 5x +c

(b) $\frac{dy}{dx} + 3 = 0$

Solution

$\frac{dy}{dx} + 3 = 0$

$\int dy = -\int 3dx$

y = -3x +c

Types of solutions of differential equations

There are two types of solutions to differential equations, i.e.

- General solutions remain with arbitrary constant c unsolved, e.g.
  Solve $\frac{dy}{dx} - 1 = 0$
  $\int dy = \int dx$
  y = x + c (general solution)

- Specific solutions where the arbitrary constant c is eliminated and replaced with specific value, e.g.
  $ydy = xdx$, given that y =1 when x = 0
  $\int ydy = \int xdx$
  $\frac{1}{2}y^2 = \frac{1}{2}x^2 + c$
  Substituting for x = 0 and y = 1, c = $\frac{1}{2}$
  $\therefore \frac{1}{2}y^2 = \frac{1}{2}x^2 + \frac{1}{2}$
  => $y^2 = x^2 + 1$ (specific solution)

Forming differential equations

This involves forming equations with differential coefficients by eliminating the arbitrary constants.

If the equation has got only one constant of integration, then the differential equation formed will be that of first order.

On the other hand, if the equation has got two constants of integration, the differential equation formed will be that of a second order.

Example 3

Form differential equations from the following equation:

(a) y = x + $\frac{A}{x}$

Solution

y = x + $\frac{A}{x}$

⇨ A = x(y – x)

$\frac{dy}{dx} = 1 - \frac{A}{x^2}$

Substituting for A:

$\frac{dy}{dx} = 1 - \frac{x(y-x)}{x^2} = 1 - \frac{y-x}{x}$

$x\frac{dy}{dx} - 2x - y$ (1st order D.E)

(b) y = Acosx

Solution

A = $\frac{y}{cosx}$

$\frac{dy}{dx} = -Asinx$

Substituting for A

$\frac{dy}{dx} = -\frac{y}{cosx}.sinx$

$\frac{dy}{dx} + ytanx = 0$ (1st orderD.E)

(c) y = Ax^2 + Bx

Solution

Let y = Ax^2 + Bx …………….. (i)

$\frac{dy}{dx} = 2Ax + B$ ……………… (ii)

$\frac{d^2y}{dx^2} = 2A$ ……………………… (iii)

Substituting for A in euation (ii)

$\frac{dy}{dx} = x\frac{d^2y}{dx^2} + B$

B = $\frac{dy}{dx} - x\frac{d^2y}{dx^2}$

Substituting for A and B in equation (i)

$y = \frac{1}{2}x^2 + x\left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)$

$\therefore y + \frac{1}{2}x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$ (2nd order D.E)

(d) x = tan(Ay)

Solution

Let x = tan(Ay) …………………….(i)

1 = Asec^2(Ay)$\frac{dy}{dx}$ =A[1 + tan^2(Ay)] $\frac{dy}{dx}$

Substituting for tan(Ay)

$1 = A(1+x^2)\frac{dy}{dx}$

$A = \frac{1}{(1+x^2)}\frac{dx}{dy}$

Substituting for A into eqn. (i)

$x = tan\left(\frac{y}{(1+x^2)}\frac{dx}{dy}\right)$

$tan^{-1}x = \frac{y}{(1+x^2)}\frac{dx}{dy}$

$(1+x^2)tan^{-1}x\frac{dy}{dx} - y = 0$ (1st order D.E)

Exercise 1

Form first order differential equations from each of the following equations

(a) y =3x^2 + Ax $\left[\frac{dy}{dx} = 6x + A\right]$

(b) $y = \frac{A}{x}$ $\left[\frac{dy}{dx} - \frac{A}{x^2} = 0\right]$

(c) $y = \frac{1}{x^2}$ $\left[\frac{\delta y}{\delta x} - \frac{-2}{x^3} = 0\right]$

(d) $y = 4x^2 - A$ $\left[\frac{dy}{dx} - 8x = 0\right]$

(e) $y = Ae^{x^2}$ $\left[\frac{dy}{dx} - 2Axe^{x^2} = 0\right]$

(f) $y = e^{x^3}$ $\left[\frac{dy}{dx} = 3x^2e^{x^3}\right]$

(g) $y = \sqrt{\frac{1+sinx}{1-sinx}}$ $\left[\frac{dy}{dx} = \frac{1}{1-sinx}\right]$

(h) $y = AInx$ $\left[\frac{dy}{dx} - \frac{A}{x} = 0\right]$

(i) $6sin\sqrt{x}$ $\left[\frac{dy}{dx} - \frac{3cos\sqrt{x}}{\sqrt{x}} = 0\right]$

(j) $x^2 + 4y^2 = A$ $\left[\frac{dy}{dx} + \frac{x}{4y} = 0\right]$

(k) $y^2 = xy + A$ $\left[(2y-x)\frac{dy}{dx} - y = 0\right]$

(l) y = x + sinx $\left[\frac{dy}{dx} = 1 + cosx\right]$

(m) y= sin^{-1}x $\left[\frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2}}\right]$

Solving 1st order differential equation

There are three basic methods employed to solve first order differential equations i.e.

(a) Separable differential equations.
(b) Differential equations with no separable variables.

These may either be exact or Non-exact (Inexact)

(c) Homogenous differential equations.

Separable differential equations

These are solved by separating variables

Suppose that the given differential equation is in the form of $f(y)\frac{dy}{dx} = g(x)$, we separate the variables in such a way that $f(y)dy = g(x)dx$. Then we integrate both sides

Example 4

(a) Solve the equation (1+cos2θ)$\frac{dy}{d\theta} = 2$. Hence find the particular equation given that, y= 1 when $\theta = \frac{1}{4}\pi$.

Solution

(1+cos2θ)$\frac{dy}{d\theta} = 2$

Separating variables

$\int dy = \int \frac{2}{1+cos2\theta} d\theta$

$= \int \frac{2}{2cos^2\theta} d\theta$

$= \int sec^2\theta d\theta$

$y = tan\theta + c$

Hence:

Substituting for y = 1 and $\theta = \frac{1}{4}\pi$

1 = tan$\frac{1}{4}\pi$ +c => c = 0

$\therefore y = tan\theta$

(b) Solve the equation $\frac{dy}{dx} = \sqrt{1-y^2}$, (y = 0 when x = $\frac{1}{6}\pi$.

Solution

By separating variables

$\int \frac{dy}{\sqrt{1-y^2}} = \int dx$

$sin^{-1}y = x + c$

Substituting for y = 0 and x $= \frac{1}{6}\pi$

$sin^{-1}0 = \frac{1}{6}\pi + c$

0 =$\frac{1}{6}\pi + c$ => c = $-\frac{1}{6}\pi$

$\therefore sin^{-1}y = x - \frac{1}{6}\pi$

$y = \sin\left(x - \frac{1}{6}\pi\right)$

(c) Solve the equation $\frac{dy}{dx} = 2y + 3$, given that y = 0 when x = 0

Solution

$\int \frac{dy}{2y+3} = \int dx$

$\frac{1}{2}In(2y + 3) = x + c$

When y = 0 and x = 0;c = $\frac{1}{2}$In3

By substitution

$\frac{1}{2}In(2y + 3) = x + \frac{1}{2}In3$

$\frac{1}{2}In(2y + 3) - \frac{1}{2}In3 = x$

$\left(\frac{2y+3}{3}\right) = e^{2x}$

$\therefore y = \frac{3}{2}(e^{2x} - 1)$

(d) Given that $\frac{dy}{dx} + xy^3 = 2xy$ and that x = 0 when y = 0, show that

y = $\sqrt{2e^2/e^2 + 1}$ when x = 1

Solution

$\frac{dy}{dx} + xy^3 = 2xy$

$\frac{dy}{dx} = 2xy - xy^3 = xy(2 - y^2)$

$= xy\left(\left(\sqrt{2}\right)^2 - y^2\right) = xy(\sqrt{2} - y)(\sqrt{2} + y)$

$\int \frac{1}{y(\sqrt{2}-y)(\sqrt{2}+y)} dy = \int dx$

Let $\frac{1}{y(\sqrt{2}-y)(\sqrt{2}+y)} = \frac{A}{y} + \frac{B}{\sqrt{2}-y} + \frac{C}{\sqrt{2}+y}$

⇨ 1=A(2 – y^2)+ By($\sqrt{2}$ – y) + Cy($\sqrt{2}$ + y)

Putting y = 0; A= $\frac{1}{2}$

Putting y = -$\sqrt{2}$ B =$-\frac{1}{4}$

Putting y = $\sqrt{2}$ C =$-\frac{1}{4}$

$\therefore \frac{1}{2}\int \frac{dy}{y} - \frac{1}{4}\int \frac{dy}{\sqrt{2}-y} - \frac{1}{4}\int \frac{dy}{\sqrt{2}+y} = \int xdx$

$\frac{1}{2}Iny - \frac{1}{4}\left[In\left(\sqrt{2} - y\right) + In\left(\sqrt{2} + y\right)\right] = \frac{1}{2}x^2 + c$

$\frac{1}{2}\text{Iny} - \frac{1}{4}In\left[\left(\sqrt{2} - y\right)\left(\sqrt{2} + y\right)\right] = \frac{1}{2}x^2 + c$

$\frac{1}{4}[2\text{Iny} - In(2 - y^2)] = \frac{1}{2}x^2 + c$

$\frac{1}{4}\text{In}\frac{y^2}{(2-y^2)} = \frac{1}{2}x^2 + c$

When x =0, y = 1 => c = 0

$\therefore \frac{1}{4}\text{In}\frac{y^2}{(2-y^2)} = \frac{1}{2}x^2$

⇨ $\text{In}\frac{y^2}{(2-y^2)} = 2x^2$

$\therefore y^2 = (2-y^2)e^{2x^2}$

$y^2(1+e^{2x^2}) = 2e^{2x^2}$

$y^2 = \frac{2e^{2x^2}}{(1+e^{2x^2})}$

$y = \sqrt{\frac{2e^{2x^2}}{(1+e^{2x^2})}}$

When x = 1; $y = \sqrt{\frac{2e^2}{(e^2+1)}}$

(e) Given that x = 0 when y = 2, solve
$y\frac{dy}{dx} = 2x(1+y)$
Solution
$\int \frac{y}{1+y}dy = 2xdx$
Using long division
$\frac{y}{1+y} = 1 - \frac{1}{1+y}$
$\int \left(1 - \frac{1}{1+y}\right)dy = 2xdx$
$y - In(1+y) = x^2 + c$
Substituting for x = 0 and y = 2
$2 - In(1+2) = 0 + c$ => c = 2 – In3
$\therefore y - In(1+y) = x^2 + 2 - In3$
$y + In\left(\frac{3}{1+y}\right) = x^2 + 2$

(f) Solve the equation
$(x^2+1)\frac{dy}{dx} + y^2 + 1 = 0$ given that y = 1 when x = 0
Solution
$\int(x^2+1)dx = -\int(1+y^2)dy$
$tan^{-1}y = -tan^{-1}x + c$
Substituting y = 1 and x = 0
$tan^{-1}1 = tan^{-1}0 + c$ => c = $\tan^{-1}1$
⇨ $tan^{-1}y = -tan^{-1}x + tan^{-1}1$
$tan^{-1}y = tan^{-1}1 - tan^{-1}x$
$tan^{-1}y = tan^{-1}\left(\frac{1-x}{1+x}\right)$
$y = \frac{1-x}{1+x}$

Exercise 2

1. Find the general solution for each of the following differential equation.

(a) $\frac{dy}{dx} = \frac{x+1}{y}$
$[y^2 = x^2 + 2x + c]$

(b) $\frac{dy}{dx} = \frac{3x^2-2}{2y}$
$[4y^2 = 9x^2 - 2x + c]$

(c) $y^2\frac{dy}{dx} = 2x + 3$
$[y^3 = 12x^2 + 9x + c]$

(d) $\frac{dy}{dx} = 3x^2\sqrt{y}$
$[2\sqrt{y} = x^3 + c]$

(e) $\frac{dy}{dx} = \frac{2x}{18y^2-1}$
$[x^2 - 6y^3 + y = A]$

(f) $\frac{dy}{dx} = \frac{-(2xy^3+sinx)}{(\cos y+2y^2x^2-2)}$
$[Siny + x^2y^3 - cosx - 2y = 0]$

(g) $\frac{dy}{dx} = 3x^2y$ $[y = e^{x^3+c}]$

(h) $\sqrt{x}\frac{dy}{dx} - sin\sqrt{x} = 0$
$[y = -2cos\sqrt{x} + c]$

(i) $\frac{dy}{dx} = e^{3x}sin2x$
$\left[y = \frac{1}{13}e^{2x}(3sin2x - 2cos3x) + c\right]$

(j) $\frac{dy}{dx} = x^3e^{x^2}$ $\left[y = \frac{e^{x^2}}{2}(x^2-1) + c\right]$

(k) $\frac{dy}{dx} = \frac{7}{(3-7x)^2}$ $\left[y = \frac{1}{3-7x} + c\right]$

(l) $\frac{dy}{dx} = 2\text{sinxcosx}$ $[y = sin^2x]$

(m) $\frac{dy}{dx} = \frac{1}{1-sinx}$ $\left[y = \sqrt{\frac{1+sinx}{1-sinx}},\right]$

(n) $\frac{dy}{dx} = tan^3x$
$\left[y = \frac{1}{2}tan^2x - Incosx + c\right]$

(o) $\frac{dy}{dx} = \frac{4x^2}{\sqrt{1-x^6}}$
$\left[\frac{4}{3}sin^{-1}(x^3) + c\right]$

2. Find the particular solution for each of the following differential equations

(a) $\frac{dy}{dx} = 4 - 3x^2$, y = 5 at x= 1
$[y = 4x - x^3 + 2]$

(b) $\frac{dy}{dx} = \frac{x+1}{y}$, y = 3 at x= -2
$[y^2 = x^2 + 2x + 9]$

(c) $(y-3)\frac{dy}{dx} = x + 3$, y = 4 at x= 0
$[y^2 - 6y - x^2 - 6x + 8 = 0]$

(d) $3y^2\frac{dy}{dx} + 2x = 1$, y = 2 at x= 4
$[y^3 - x + x^2 = 20]$

$\frac{dy}{dx} = sec^3xtanx$ y = $\frac{1}{3}$ when x = 0
$\left[y = \frac{1}{3}sec^3x\right]$

(e) $\frac{dy}{dx} = \frac{\sin^2 x}{1+\cos^2 x}$, y = 0.027π when x = $\frac{\pi}{4}$
$\left[\sqrt{2}tan^{-1}\left(\frac{\sqrt{2}}{2}tanx\right) - x\right]$

Exact differential equations

An exact first order differential equation (O.D.E) is an equation written in form A(x, y)dx + B(x, y)dy = 0 such that there exist a function f of two variables x and y which have continuous partial derivative such that $f_x = A(x,y)$ and $f_y = B(x,y)$

Testing for exactness

This is done by using partial derivatives, such as $\frac{\delta A}{\delta y} = \frac{\delta B}{dx}$

Example 5

Show whether the following equation are exact or not.

(a) $y + x\frac{dy}{dx} = 5$
Solution
Expressing $y + x\frac{dy}{dx} = 5$
in form A(x,y)dx + B(x,y)dy = 0, we have
$y = 5 + x\frac{dy}{dx} = 0$
$(y-5)dx + xdy = 0$
⇨ A(x,y) = y – 5 and B(x,y) = x
$\frac{\delta A}{\delta y} = 1$ and $\frac{\delta B}{\delta x} = 1$
Since $\frac{\delta A}{\delta y} = \frac{\delta B}{\delta x} = 1$, therefore the equation is exact.

(b) $2xy^3 + 3x^2y^2\frac{dy}{dx} = cosx$
Expressing $2xy^3 + 3x^2y^2\frac{dy}{dx} = cosx$
in form A(x,y)dx + B(x,y)dy = 0, we have
$2xy^3 - cosx + 3x^2y^2\frac{dy}{dx} = 0$
⇨ A(x,y) = $2xy^3 - cosx$ and B(x,y) = $3x^2y^2$
$\frac{\delta A}{\delta y} = 6xy^2$ and $\frac{\delta B}{\delta x} = 6xy^2$
Since $\frac{\delta A}{\delta y} = \frac{\delta B}{\delta x} = 6xy^2$, therefore the equation is exact.

(c) $x^2\frac{dy}{dx} + xy^2 = 5x$
Expressing $x^2\frac{dy}{dx} + xy^2 = 5x$
in form A(x,y)dx + B(x,y)dy = 0, we have
$x^2\frac{dy}{dx} + xy^2 = 5x$
⇨ A(x,y) = $xy^2 - 5x$ and B(x,y) = x^2
$\frac{\delta A}{\delta y} = 2xy$ and $\frac{\delta B}{\delta x} = 2x$
Since $\frac{\delta A}{\delta y} \neq \frac{\delta B}{\delta x}$, therefore the equation is not exact.

Solving exact differential equation

After testing that a given differential equation is exact, this means that, the terms on the LHS are a result of the derivatives of the main function.

Note: the main function is equal to the integral of B (the coefficient of dy) with respect to y.

Example 6

(a) Solve the following equations
(i) y + x$\frac{dy}{dx}$ = 5
Solution
Since the equation is exact; the main function = xy (integral of x with respect to y)
⇨ $\frac{d}{dx}(xy) = 5$
$\int \frac{d}{dx}(xy)dx = \int 5dx$
$xy = 5x + c$
(ii) $2xy^2 + 3x^2y^2\frac{dy}{dx} = \cos x$
Solution
Since the equation is exact; the main function =$\int 3x^2y^2dy =$ x^2y^3
⇨ $\frac{d}{dx}(x^2y^3)dx = \int \cos x dx$
$x^2y^3 = sinx + c$

(b) Solve the following equations
(i) $x^2\frac{dy}{dx} + 2xy = 1$
Solution
Since the equation is exact, the main function = $\int x^2dy = x^2y$
⇨ $\frac{d}{dx}(x^2y) = 1$
$\int \frac{d}{dx}(x^2y)dx = \int 1dx$
$x^2y = x + c$

(ii) $\frac{u^2}{x}\frac{dx}{du} + 2tInx = 3cosu$

Solution

Since the equation is exact, the main function = $\int \frac{u^2}{x}dx = u^2Inx$

⇨ $\frac{d}{dx}(u^2Inx) = 3cosu$

$\int \frac{d}{dx}(u^2Inx)du = \int 3cosu\, du$

$u^2Inx = 3sinu + c$

Or x = $e^{\left(\frac{3sinu+c}{u^2}\right)}$

(iii) $x^2 \cos t \frac{dt}{dx}$ + 2xsint = $\frac{1}{x}$

Solution

Since the equation is exact, the main function = $\int x^2 \cos t\, dt = x^2sint$

⇨ $\frac{d}{dx}(x^2sint)dx = \frac{1}{x}$

$\int \frac{d}{dx}(x^2sint)dx = \int \frac{1}{x}dx$

$x^2sint = Inx + c$

Or t =$\sin^{-1}\left(\frac{Inx+c}{x^2}\right)$

(iv) $e^y + xe^y\frac{dy}{dx} = 2$

Solution

Since the equation is exact, the main function = $\int xe^ydy = xe^y$

⇨ $\frac{d}{dx}(xe^y) = 2$

$\int \frac{d}{dx}(xe^y)dx = \int 2\,dx$

$xe^y = 2x + c$

$e^y = 2 + \frac{c}{x}$

Inexact differential equation

If we have ordinary differential equation in the form A(x, y)dx + B(x, y)dy = 0, where $\frac{dA}{dy} \neq \frac{dB}{dx}$, the equation is said to be inexact.

An inexact ordinary differential equation is solved by first converting it into an exact differential equation. This is done by multiplying the given equation by an integrating factor.

Examples of inexact ordinary differential equations are the linear ordinary differential equations.

Linear ordinary differential equations

A linear ordinary differential equation is an equation expressed in the form $\frac{dy}{dx} + py = Q$ where P and Q are function of x.

Converting inexact ordinary equation into and exact ordinary equation

This is done by multiplying through by a factor known as an integrating factor.

Suppose λ = integrating factor, then

$$\lambda\frac{dy}{dx} + \lambda py = \lambda Q \quad \text{......................... (i)}$$

⇨ $\frac{d}{dx}(x,y) = \lambda\frac{dy}{dx} + y\frac{d\lambda}{dx}$ ....(ii)

Comparing (i) and (ii)

$\lambda Py = y\frac{d\lambda}{dx}$

$Py = \frac{d\lambda}{dx}$

By separating variables

$Pdx = \frac{d\lambda}{\lambda}$

$\int Pdx = \int\frac{d\lambda}{\lambda} = In\lambda$

$\lambda = e^{\int Pdx}$

∴ Integrating factor is $\lambda = e^{\int Pdx}$

Note: multiplying the original equation with integrating factor an exact equation is formed, i.e.

$e^{\int Pdx}\frac{dy}{dx} + e^{\int Pdx}py = e^{\int Pdx}Q$

Main function=$\int e^{\int Pdx}dy = y\,e^{\int Pdx}$

⇨ $\frac{d}{dx}\left(ye^{\int Pdx}\right) = e^{\int Pdx}Q$

$\int\frac{d}{dx}\left(ye^{\int Pdx}\right)dx = \int e^{\int Pdx}Qdx$

$\therefore ye^{\int Pdx} = \int e^{\int Pdx}Qdx$

Example 6

a. Solve the differential equation $\frac{dy}{dx}+\frac{3}{x}y=5x+4$, given that y = 3 when x = 1.

Solution

Comparing the given equation with $\frac{dy}{dx}+Py=0$

⇨ P= $\frac{3}{x}$ and Q = 5x + 4

∴ the integrating factor,

$\lambda=e^{\int Pdx}=e^{\int\frac{3}{x}dx}=e^{3Inx}=x^3$

Multiplying the equation by integrating factor.

$x^3\frac{dy}{dx}+3x^2y=5x^4+4x^3$

Main function = $\int x^3dy=x^3y$

$\frac{d}{dx}(x^3y)dx=\int 5x^4+4x^3\,dx$

$x^3y=x^5+x^4+c$

When x = 1 and y = 3

⇨ 3 = 1 + 1 + c; c = 1

$\therefore x^3y=x^5+x^4+1$

Or

$y\ =\ x^2+x+\frac{1}{x}$

b. Find the general solution of the equation $\frac{dy}{dx}+ycotx=3\sin xcosx$

Solution

$\lambda=e^{\int cotxdx}=e^{Insinx}\ =sinx$

Multiplying the equation by integrating factor.

$sinx\frac{dy}{dx}+ycosx=3\sin^2 xcosx$

Main function = $\int sinxdy=ysinx$

⇨ $\frac{d}{dx}(ysinx)=\ 3\sin^2 xcosx$

$\int\frac{d}{dx}(ysinx)dx=\ \int 3\sin^2 xcosx\,dx$

$ysinx=\ sin^3x+c$

Or

$y=\ sin^2x+c(cosecx)$

c. Solve the equation $x^2\frac{dy}{dx}+y=x^2e^{\frac{1}{x}}$ given that y = 2 when x = 0.

Solution

Dividing through by x^2

$\frac{dy}{dx}+\frac{1}{x^2}y=e^{\frac{1}{x}}$

$\lambda=e^{\int\frac{1}{x^2}dx}=e^{-\frac{1}{x}}$

Multiplying through by λ

$e^{-\frac{1}{x}}\frac{dy}{dx}+\frac{1}{x^2}e^{-\frac{1}{x}}y=e^{\frac{1}{x}}.e^{-\frac{1}{x}}$ =1

Main function = $e^{-\frac{1}{x}}y$

⇨ $\frac{d}{dx}\left(e^{-\frac{1}{x}}y=1\right)$

$\int\frac{d}{dx}\left(e^{-\frac{1}{x}}y\right)dx=\int 1\,dx$

$ye^{-\frac{1}{x}}=x+c$

When x = 0; y = 3 => c = 0

since $e^{-\frac{1}{0}}=e^{\infty}=0$

$\therefore ye^{-\frac{1}{x}}=x$

Or

$ye^{-\frac{1}{x}}=xe^{\frac{1}{x}}$

d. Solve the equation $\frac{dt}{dx}+tcotx=2cosx$ given that t = 3 when x = $\frac{\pi}{2}$.

Solution

$\lambda=e^{\int cotxdx}=sinx$

Multiplying through by λ.

$sinx\frac{dt}{dx}+tcosx=2sinxcosx$

Main function = $\int sinxdt=tsinx$

⇨ $\frac{d}{dx}(tsinx)=\ 3\sin^2 xcosx$

$\int\frac{d}{dx}(tsinx)dx=\ \int 3\sin^2 xcosx\,dx$

$tsinx=\ sin^3x+c$

Given that t = 3 when x = $\frac{\pi}{2}$;

$3sin90=\ sin^390+c$ => c = 2

$\therefore\ t=\ sin^2x+2cosecx$

Exercise 3

Show that each of the following differential equations is exact and use that property to find the general solution

1. $\frac{1}{x}dy-\frac{y}{x^2}dx=o\ [y=Ax]$
2. $2xy\frac{dy}{dx}+y^2-2x$=0 $[y^2x-x^2=A]$
3. $2(y1)e^xdx+2(e^x-2y)dy=0$ $[(y+1)e^x-y^2=A]$
4. $(2xy+6x)dx+(x^2+4y^3)dy=0$ $[x^2y+3x^2+y^3]$
5. $(8y-x^2y)\frac{dy}{dx}+x-xy^2=0$

$$\left[\frac{1}{2}x^2(1-y^2)+4y^2=A\right]$$

6. $(e^{4x}+2xy^2)dx+(cosy+2x^2y)dy=0$
$$\left[\frac{1}{4}e^{4x}+x^2y^2+siny=A\right]$$
7. $(3x^2+ycosx)dx+(sinx-4y^3)dy=0$
$$[x^3+ysinx-y^4]$$
8. $x\tan^{-1}y.dx+\frac{x^2}{2(1+y^2)}.dx=0$
$$\left[\frac{x^2}{2}\tan^{-1}y=A\right]$$
9. $(2x+x^2y^3)dx+(x^3y^2+4y^3dy=0)$
$$\left[x^2+\frac{x^3y^3}{3}+y^4\right]$$
10. $(2x^3 - 3x^2y + y^2)\frac{dy}{dx} = 2x^3 - 6x^2y + 3xy^2$
$$\left[\frac{x^4}{2}-\frac{3}{2}x^2y^2-\frac{y^4}{4}\right]$$
11. $(y^2cosx – sinx)dx + (2ysinx + 2)dy = 0$
$$[y^2sinx+cosx+2y=A]$$
12. $\frac{dy}{dx}+2y=2e^x\left[y=\frac{2}{3}e^x+Ae^{-3x}\right]$
13. $x\frac{dy}{dx}=2y=x^2\ [y=x^2Inx+Ax^2]$
14. $x\frac{dy}{dx}y=\sqrt{x}\ \left[y=\frac{2}{3}\sqrt{x}+\frac{A}{x}\right]$
15. $(1+t)\frac{du}{dt}+u=1+t,t>0\left[u=\frac{t^22t+2c}{2(t+1)}\right]$
16. $\frac{dy}{dx}+2y=e^{-2x}cosx\ [y=e^{-2x}(sinx+A)]$
17. $2x\frac{dy}{dx}=x-y+3\left[y=\frac{x}{3}+3+Ax^{-1}\right]$
18. $x\frac{dy}{dx}-y=\frac{x}{x-1}\ \left[y=xIn\frac{A(x+1)}{x}\right]$
19. $x\frac{dy}{dx}+2y=x^{-1}cosx\ [y=x^{-2}(sinx+A)]$
20. $sinx\frac{dy}{dx}+y=sin^2x$
$$\left[y=(x-sinx+A)cot\frac{1}{2}x\right]$$
21. $3y+(x-2)\frac{dy}{dx}=\frac{2}{x-2}$
$$[y=(x-2)^{-1}+A(x-2)^{-3}]$$

Homogenous differential equations

If the degree of the individual terms in the differential equation is the same and constant, then the equation is said to be homogeneous.

Or: A differential equation is said to homogeneous if there is no isolated constant term in the equation i.e. each term in the differential equation for y in each term.

Examples of homogeneous equation are

(a) $x\frac{dy}{dx}+y=x$ 1st degree
(b) $x^2\frac{dy}{dx}+y^2=xy$ 2nd degree
(c) $x^2\frac{dy}{dx}-x^2-y^2=xy$ 2nd degree
(d) $t^3\frac{d\theta}{dt}+t^2\theta=t\theta^2+t^3$ 3rd degree

The above equations are solved by use of substitution mainly y = vx or y =ux which transform them into separable differential equations.

Example 7

(a) Use the substitution y = vx solve the equation $x^2\frac{dy}{dx}=x^2+y^2+xy$, given that y = 0 when x = $\frac{\pi}{4}$.

Solution

Using y = vx

$\frac{dy}{dx}=x\frac{dv}{dx}+v$ [the product rule]

Substituting in the given equation

$x^2\left(x\frac{dv}{dx}+v\right)=x^2+v^2x^2+vx^2$

$x\frac{dv}{dx}+v=1+v^2+v$

Separating variables

$\int\frac{dv}{1+v^2}=\int\frac{dx}{x}$

$\tan^{-1}v=Inx+c$

$\tan^{-1}\left(\frac{y}{x}\right)=Inx+c$

When y = 0, x = $\frac{\pi}{4}$

⇨ $\tan^{-1}0=In\left(\frac{\pi}{4}\right)+c$

c = $-In\left(\frac{\pi}{4}\right)$

Substituting for c

$\tan^{-1}\left(\frac{y}{x}\right)=Inx-In\left(\frac{\pi}{4}\right)$

$\frac{y}{x}=tan\left(In\frac{4x}{\pi}\right)$

$y=xtan\left(In\frac{4x}{\pi}\right)$

(b) Solve the equation$x\frac{dy}{dx}=2y$

Solution

Either by separation of variables

$\int\frac{dy}{y}=2\int\frac{dx}{x}$

Iny =2Inx+ c

Iny = Inx^2+ InA

Iny = InAx^2

y = Ax^2

Or

Let y = vx

$\frac{dy}{dx} = x\frac{dv}{dx} + v$

$x\left(x\frac{dv}{dx} + v\right) = 2vx$

$v + x\frac{dv}{dx} = 2v$

$x\frac{dv}{dx} = v$

$\int \frac{dv}{v} = \int \frac{dx}{x}$

Inv = Inx + c

Inv = Inx + InA

Inv = In(Ax)

v = AX

$\frac{y}{x} = Ax$

$\therefore y = Ax^2$

(c) Solve the equation $y\frac{dy}{dx} = 2x - y$ using the substitution y = vx.

Solution

$\frac{dy}{dx} = x\frac{dv}{dx} + v$

vx$\left(x\frac{dv}{dx} + v\right) = 2x - vx$

$v^2 + vx\frac{dv}{dx} = 2 - v$

$vx\frac{dv}{dx} = 2 - v - v^2 = (1-v)(1+v)$

$\frac{vdv}{(1-v)(1+v)} = \frac{dx}{x}$

Let $\frac{v}{(1-v)(1+v)} = \frac{A}{(1-v)} + \frac{B}{(1+v)}$

$v = A(1+v) + B(1-v)$

Putting v = -2; -2 = B i.e. B = $\frac{-2}{3}$

Putting v = 1, 1 = 3A 1.e. A = $\frac{1}{3}$

$\therefore \frac{v}{(1-v)(1+v)} = \frac{1}{3(1-v)} - \frac{2}{3(1+v)}$

$\int \frac{vdv}{(1-v)(1+v)} = \frac{1}{3}\int \frac{dv}{3(1-v)} - \frac{2}{3}\int \frac{dv}{3(1+v)} = \int \frac{dx}{x}$

$-\frac{1}{3}In(1-v) - \frac{2}{3}In(1+v) = Inx + c$

$-\frac{1}{3}In(1-v) - \frac{1}{3}In(1+v)^2 = Inx + c$

$In(1-v)(1+v)^2 = InAx^{-3}$ where InA = c

$x^3(1-v)(1+v)^2 = A$

Substituting for v = $\frac{y}{x}$

$(x-y)(y+2x)^2 = A$

⇨ $4x^3 - 3xy^2 - y^3 = A$

(d) Using the substitution y = vx or otherwise show that the solution of the equation $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$ is given by $\frac{(y-x)^2}{xy^2} = A$ where A is a constant

Solution

Let y = vx

$\frac{dy}{dx} = x\frac{dv}{dx} + v$

The given equation becomes

$2\left(x\frac{dv}{dx} + v\right) = v + v^2$

$2x\frac{dv}{dx} = v(v-1)$

$\frac{2dv}{v(v-1)} = \frac{dx}{x}$

Let $\frac{2}{v(v-1)} = \frac{A}{v} + \frac{B}{(v-1)}$

2 = Bv + A(v – 1)

When v = 1; B = 2 and when v = 0; A = -2

$\therefore \frac{2}{v(v-1)} = \frac{-2}{v} + \frac{2}{(v-1)}$

⇨ $\int \frac{2}{v(v-1)}dv = -2\int \frac{dv}{v} + 2\int \frac{dv}{(v-1)} = \int \frac{dx}{x}$

$-2Inv + 2In(v-1) = Inx + c$

$In\left(\frac{v-1}{v}\right)^2 = Incx$ or $\left(\frac{v-1}{v}\right)^2 = cx$

$\left(\frac{\frac{y}{x}-1}{\frac{y}{x}}\right)^2 = cx$

$\frac{(y-x)^2}{y^2} = cx$

$\frac{(y-x)^2}{xy^2} = c$

(e) Solve the differential equation $xy\frac{dy}{dx} = y^2 + x\sqrt{x^2+y^2}$

Solution

Let y = vx

$\frac{dy}{dx} = x\frac{dv}{dx} + v$ (product rule)

Substituting in the given equation

$$vx^2\left(x\frac{dv}{dx} + v\right) = v^2x^2 + x\sqrt{x^2 + v^2x^2}$$

$$v\left(x\frac{dv}{dx} + v\right) = v^2 + \sqrt{1 + v^2}$$

$$vx\frac{dv}{dx} + v^2 = v^2 + \sqrt{1 + v^2}$$

$$\frac{vdv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

$$\int\frac{vdv}{\sqrt{1+v^2}} = \int\frac{dx}{x}$$

$$In(1 + v^2)^{\frac{1}{2}} = Inx + c$$

$$In(1 + v^2)^{\frac{1}{2}} = Inx + InA$$

$$In(1 + v^2)^{\frac{1}{2}} = InAx$$

$$(1 + v^2)^{\frac{1}{2}} = Ax$$

Substituting v = $\frac{y}{x}$

$$\left(1 + \left(\frac{y}{x}\right)^2\right)^{\frac{1}{2}} = Ax$$

$$\frac{\sqrt{x^2+y^2}}{x} = Ax$$

or

$$\sqrt{x^2 + y^2} = Ax^2$$

Exercise 4

Using the substitution y = vx or otherwise solve the following differential equations

1. $(x^2 - y^2)dx + xydy = 0$ $\left[\frac{y^2}{x^2} = In\frac{A}{x}\right]$
2. $(x - y)dx + xdy = 0$ $\quad x = e^{-\frac{y}{x}} + c$
3. 2(x+2y)dx + (y – x)dy = 0; y(1) = 0
   $\left(\frac{y}{x} + 2\right)^3 = \frac{8}{x}\left(\frac{y}{x} + 1\right)^2$
4. $x^2\frac{dy}{dx} = 3x^2 + xy$
   $\left[x^3 = Ae^{\frac{y}{x}}\right]$
5. xy$dy = (x^2 - y^2)dx$
   $[x^2(x^2 - 2y^2) = A]$
6. $x^2\frac{dy}{dx} - y^2 = x^2 + xy$
   $\left[\tan^{-1}\frac{y}{x} = InAx\right]$
7. $3\frac{dy}{dx} = \frac{y^2}{x^2}$ $\quad [3x - y + Axy = 0]$
8. $(x^2 + y^2)dy = xydx$ $\quad \left[y = Ae^{\frac{x^2}{2y^2}}\right]$
9. $\frac{dy}{dx} = \frac{4x}{4x-y}$ $\quad [2x = (2x - y)InA(2x - y)]$
10. $x\frac{dy}{dx} - y = (x^2 - y^2)^{\frac{1}{2}}$
    $\left[\sin^{-1}\frac{y}{x} = In(Ax)\right]$
11. $x\frac{dy}{dx} - 2y = x$ $\quad [y = x(Ax - 1)]$
12. $\frac{dy}{dx} = \frac{2x+y}{y}$ $\quad [(x + y)(2x - y)^2 = A]$
13. $\frac{dy}{dx} - 1 = x^{-2}y^2$
    $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2y-x}{x\sqrt{3}}\right) = In(Ax)$

Applications of differential equations

Differential equations are applied in the following fields

1. Gradient of tangents
2. Rates of decay, decomposition, cooling etc.
3. Rates of formation, growth, spreading etc.
4. Displacement, velocity and acceleration

Note that in any of the above fields, an increase is a positive change/variation while a decrease is a negative change or variation.

Gradients of tangents

The gradient at any given point on a gradient in terms of x and y is $\frac{dy}{dx}$.

Example 8

a. The gradient of the tangent at any point on a curve is $x - \frac{y}{x}$. Given tat the curve passes through the point (2, 4), find its equation

Solution

$\frac{dy}{dx} = x - \frac{2y}{x}$

$\frac{dy}{dx} + \frac{2y}{x} = x$

The integrating factor, $\lambda = e^{\int \frac{dx}{x}} = x^2$

Multiplying the given equation by λ

$x^2\frac{dy}{dx} + 2xy = x^3$

Main factor = x^2y

⇨ $\frac{dy}{dx}(2xy) = x^3$

$\int \frac{dy}{dx}(2xy)dx = \int x^3\,dx$

$2xy = \frac{1}{4}x^4 + c$

At (2,4); 16 = 4 + c => c = 12

$\therefore\ 2xy = \frac{1}{4}x^4 + 12$

b. A curve in x-y plane has the property that the slope of the tangent to a curve in x – y plane at point (x, y) is equal to y + cosx. Given that the line passes through (0, 1), find the equation.

Solution

$\frac{dy}{dx} = y + cosx$

$\frac{dy}{dx} - y = cosx$

I.F, $\lambda = e^{-\int dx} = e^{-x}$

Multiplying the given equation by λ

$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}cosx$

Main function $e^{-x}y$

⇨ $\frac{dy}{dx}(e^{-x}y) = e^{-x}cosx$

$\int \frac{dy}{dx}(e^{-x}y)dx = \int e^{-x}cosx\,dx$

$e^{-x}y = \int e^{-x}cosx\,dx$

sign	differentiate	integrate
+	e^{-x}	$cosx$
-	$-e^{-x}$	sinx
+	e^{-x}	-cosx

Let I = $\int e^{-x}cosx\,dx$

I = $e^{-x}sinx - e^{-x}cosx - \int e^{-x}cosx$

I = $e^{-x}sinx - e^{-x}cosx - I$

2I = $e^{-x}sinx - e^{-x}cosx$

I = $\frac{1}{2}e^{-x}sinx - \frac{1}{2}e^{-x}cosx$

$\therefore\ e^{-x}y = \int e^{-x}cosx\,dx$

$= \frac{1}{2}e^{-x}(sinx - cosx) + c$

When x = 0, y = 1; 1 $= -\frac{1}{2} + c$ => c =$\frac{3}{2}$

$e^{-x}y = \frac{1}{2}e^{-x}(sinx - cosx) + \frac{3}{2}$

$y = \frac{1}{2}(sinx - cosx) + \frac{3}{2}e^{x}$

$\therefore\ y = \frac{1}{2}(sinx - cosx + 3e^{x})$

c. A curve passes through the point (0, 1) and its slope at a point (x, y) = $\frac{2xy}{x^2-y^2}$. Find the equation of the curve.

Solution

Given $\frac{dy}{dx} = \frac{2xy}{x^2-y^2}$ (a homogeneous equation)

Let y = vx => $\frac{dy}{dx} = v + x\frac{dv}{dx}$

By substitution;

$v + x\frac{dv}{dx} = \frac{2vx^2}{x^2-v^2x^2} = \frac{2v}{1-v^2}$

$x\frac{dv}{dx} = \frac{2v}{1-v^2} - v = \frac{v+v^3}{1-v^2}$

$\int \frac{1-v^2}{v+v^3}dv = \int \frac{1}{x}dx$

$Let\ \frac{1-v^2}{v+v^3} = \frac{A}{v} + \frac{Bv+c}{1+v^2}$

$1 - v^2 = A(1+v^2) + (Bv+c)v$

For v = 0; A = 1

Expanding

$1 - v^2 = (A+B)v^2 + cv + A$

Equating coefficients:

For v^2; A+B = -1 i.e. B = -2 since A = 1

For v; c = 0

⇨ $\frac{1-v^2}{v(1+v^2)} = \frac{1}{v} - \frac{2v}{1+v^2}$

$\int \frac{1-v^2}{v(1+v^2)} dv = \int \frac{dv}{v} - \int \frac{2v}{1+v^2} dv = \int \frac{dx}{x}$
$Inv - In(1 + v^2) = Inx + c$
$Inv - In(1 + v^2) = Inx + InA$
$In \frac{v}{1+v^2} = InAx$
$\frac{v}{1+v^2} = Ax$
Substituting for v
$\frac{\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = Ax$ or $y = A(x^2 + y^2)$
When y = 1, x = 0 => A = 1
$\therefore y = (x^2 + y^2)$

Rates of decay, decomposition and cooling

Example 9

a. The rate of cooling of a body is proportional to the excess temperature above the surrounding air. Given that the surrounding air temperature is 20^0C and the body cools from 100^0C to 60^0C in 5 minutes.
   (i) Determine the temperature of the body after another 20 minutes
   (ii) How long does it take for the body to cool to 30^0C.

Solution

Let θ be the temperature of the body.

⇨ $\frac{d\theta}{dt} \propto (\theta - 20)$
$\frac{d\theta}{dt} = -k(\theta - 20)$

By separating variable

$\int \frac{d\theta}{(\theta-20)} = -\int kdt$

$In(\theta - 20) = -kt + c$

When t = 0, θ = 100 => c = In80

Substituting for c

$In(\theta - 20) = -kt + In80$

When t = 5, θ = 60 => In40 = -5k + In80

k = $\frac{In2}{5} min^{-1}$

$In(\theta - 20) = -\frac{In2}{5}t + In80$

Or

$In \frac{80}{(\theta-20)} = \frac{In2}{5}t$

(i) After another 20minutes, t = 25

$In \frac{80}{(\theta-20)} = \frac{In2}{5}x25 = 5In2$

$In \frac{(\theta-20)}{80} = -5In2$

$\theta = 22.5^0$

∴after another 20 minutes the temperature will be 22.5^0.

(ii) When θ = 30^0C
$In \frac{(30-20)}{80} = -\frac{1}{5}In2t$
t = 15minutes.

b. The rate of cooling of a body is proportional to the temperature of the body at that time. Given that the body cools from 72^0C to 32^0C in 10 minutes, determine how much longer it will take the body to cool to 27^0C.
$\frac{d\theta}{dt} \propto \theta$
$\frac{d\theta}{dt} = -k\theta$
By separating variables
$\frac{d\theta}{\theta} = -kdt$
$\int \frac{d\theta}{\theta} = -\int kdt$
$In\theta = -kt + c$
At t = 0, θ = 72^0C
By substitution
c = In72
$In\theta = -kt + In72$
At θ = 32^0C, t = 10
$In32 = -10k + In72$, => k = 0.0811min^{-1}
When θ = 270C

$In27 = -0.0811t + In72$; t = 12.1min
∴ it will take more 12.1 – 10 = 2.1minute for the temperature to fall to 27^0C

c. The rate of decay at any instant of a radioactive substance is proportional to the amount of substance remaining at that instant. If the initial amount of substance is A and the amount remaining after time t is x.

(i) Prove that $x = Ae^{-kt}$, where k is a constant.

(ii) If the amount remaining is reduced from $\frac{1}{2}A$ to $\frac{1}{3}A$ in 8hours, prove that the initial amount of substance was halved in about 13.7hours.

Solution

(i) $\frac{dx}{dt} \propto x$

$\frac{dx}{dt} = -kx$

By separating variables

$\frac{dx}{x} = -kdt$

$\int \frac{dx}{x} = -\int kdt$

$Inx = -kt + c$

At time t = 0, x = A

By substituting InA = C

$Inx = -kt + InA$

$In\frac{x}{A} = -kt$

$\frac{x}{A} = e^{-kt}$

$x = Ae^{-kt}$

(ii) $Inx = -kt + c$

At t = 0, x = $\frac{1}{2}A$; => c = $In\left(\frac{1}{2}\right)$

When t = 8, x = $\frac{1}{2}A$;

$In\left(\frac{1}{3}\right) = -8k + In\left(\frac{1}{2}\right)$

$k = \frac{1}{8}In\frac{2}{3}$

$Inx = -\left(\frac{1}{8}In\frac{2}{3}\right)t + In\left(\frac{1}{2}\right)$

When x is halved to $\frac{1}{4}A$

$In\left(\frac{1}{4}A\right) = -\left(\frac{1}{8}In\frac{2}{3}\right)t + In\left(\frac{1}{2}\right)$

$t = 13.676\text{hours} \approx 13.7hours$

d. A substance loses mass at a rate which is proportional to the amount M present at time t.

(i) Form a differential equation connecting M, t and proportionality constant k. (02marks)

$-\frac{dM}{dt} \propto M$

$-\frac{dM}{dt} = kM$

$\frac{dM}{dt} = -kM$

(ii) If the initial mass of the substance is M_0, show that M = M_0e^{-kt}. (05marks)

$\frac{dM}{M} = -kdt$

$\int \frac{dM}{M} = \int -kdt$

$InM = -kt + C$

At t= 0; M = M_0

c = InM_0

⇨ $InM = -kt + InM_0$

$InM - InM_0 = -kt$

$In\frac{M}{M_0} - kt$ or M = M_0e^{-kt}

(iii) Given that half of the substance is lost in 1600years, determine the number of years 15g of the substance would take to reduce to 13.6g

From $In\frac{M}{M_0} = -kt$

$In\frac{\frac{M_0}{2}}{M_0} = -k\ x\ 1600$

; $In\frac{1}{2} = -k\ x\ 1600$

-k = $\frac{1}{1600}In\frac{1}{2}$

Let the required time be t

$In\frac{13.6}{15} = \frac{1}{1600}In\frac{1}{2}t$

t = 226.17years

Rates of formation, growth and spreading

Example 10

(a) The rate of growth of a substance is proportional to the original amount. Find the equation of the amount present at any time t.

Solution

Let x be the amount present

$\frac{dx}{dt} \propto x$

$\frac{dx}{dt} = kx$

By separating variables

$\frac{dx}{x} = kdt$

$\int \frac{dx}{x} = \int kdt$

$Inx = kt + c$

Suppose that at time t = 0, x = x_0(initial amount)

c = ln x_0

by substituting for c.

$Inx = kt + Inx_0$

$Inx - Inx_0 = kt$

$In\frac{x}{x_0} = kt$

$\frac{x}{x_0} = e^{kt}$

$x = x_0e^{kt}$

(b) The rate of growth of population in a country is proportional to the number of people living at that time. In 1980, the population was 18m and in 1990 it was 22m. Estimate

(i) The number of population in 2005

Solution

Let P be the amount present

$\frac{dP}{dt} \propto P$

$\frac{dP}{dt} = kP$

By separating variables

$\frac{dP}{P} = kdt$

$\int \frac{dP}{P} = \int kdt$

$InP = kt + c$

At t = 0 (In 1980), P = 18 => c = In18

$InP = kt + In18$

At t = 10 (1990), P = 22

$In22 = 10k + In18$

$k = \frac{1}{10}In\left(\frac{22}{18}\right)$

$InP = \frac{1}{10}In\left(\frac{22}{18}\right)\ t + In18$

In 2005, t = 25

$InP = \frac{1}{10}In\left(\frac{22}{18}\right)\text{x}25 + In18$

⇨ P = 29.73m

(ii) How long will it take the population to reach 36m.

From

$InP = \frac{1}{10}In\left(\frac{22}{18}\right)\ t + In18$

$In36 = \frac{1}{10}In\left(\frac{22}{18}\right)\ t + In18$

t = 34.54 years

It will take 34.54 years for the population to reach 36m

(c) The rate at which bush fire spreads is proportional to the unburnt area of the bush. Initially when observation was made, $\frac{1}{5}$ of the bush area had been burnt. Two hours later, $\frac{1}{3}$ of the bush area had been burnt. Find the fraction of the bush area that will remain unburnt after 5 hours.

Solution

Let the fraction of unbush area be x

$\frac{dx}{dt} = -kx$

(-ve because the unburnt area decreases with time

$\int \frac{dx}{x} = \int kdt$

$Inx = -kt + c$

At t = 0, x = $\left(1 - \frac{1}{5}\right) = \frac{4}{5}$ => c = $In\left(\frac{4}{5}\right)$

$Inx = -kt + In\left(\frac{4}{5}\right)$

At t = 2, x =$\left(1 - \frac{1}{3}\right) = \frac{2}{3}$

$In\left(\frac{2}{3}\right) = -2k + In\left(\frac{4}{5}\right)$

$k = \frac{1}{2}In\left(\frac{6}{5}\right)$

$Inx = -\frac{1}{2}In\left(\frac{6}{5}\right)t + In\left(\frac{4}{5}\right)$

At t = 5

$Inx = -In(1.2)x5 + In\left(\frac{4}{5}\right)$

$x = 0.32$

∴The fraction remaining after 5hours is 0.32

Linear motion

Example 11

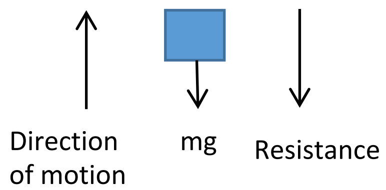
(a) A body of mass m is projected upwards with initial velocity u. Given that it experiences a resistance proportional to the velocity, v on its upward path, determine the

(i) Velocity v after time t,

(ii) Displacement, s, after time t.

(iii) Limiting speed of the body.

Solution



Resistance, R = kv

Using Newton's 2nd law of motion

(i) Resultant force = m$\frac{dv}{dt}$

$0-(mg+kv)=m\frac{dv}{dt}$

$\frac{mdv}{mg+kv}=-dt$

⇨ $\int\frac{mdv}{mg+kv}=-\int dt$

$\frac{m}{k}In(mg+kv)=-t+c$

Initially (t = 0), v = v_0 => $\frac{m}{k}In(mg+ku)=c$

∴ $\frac{m}{k}In(mg+kv)=-t+\frac{m}{k}In(mg+ku)$

or

$\frac{m}{k}In\left(\frac{mg+ku}{mg+kv}\right)=t$

$mg+kv=(mg+ku)e^{-\frac{k}{m}t}$

$v=\frac{1}{k}(mg+ku)e^{-\frac{k}{m}t}-\frac{mg}{k}$

(ii) $v=\frac{1}{k}(mg+ku)e^{-\frac{k}{m}t}-\frac{mg}{k}$

$$\int dsdt=\int\left[\frac{1}{k}(mg+ku)e^{-\frac{k}{m}t}-\frac{mg}{k}\right]dt$$

$$s=\left[-\frac{1}{k^2}(mg+ku)e^{-\frac{k}{m}t}-\frac{mgt}{k}\right]+c$$

At t = 0, s = 0

0 =$-\frac{1}{k^2}(mg+ku)+c$

c = $\frac{1}{k^2}(mg+ku)$

$$s=\left[-\frac{1}{k^2}(mg+ku)e^{-\frac{k}{m}t}-\frac{mgt}{k}\right]+\frac{1}{k^2}(mg+ku)$$

$$\therefore\ s=\left[-\frac{1}{k^2}(mg+ku)\left(1-e^{-\frac{k}{m}t}\right)-\frac{mgt}{k}\right]$$

(iii) From $v=\frac{1}{k}(mg+ku)e^{-\frac{k}{m}t}-\frac{mg}{k}$

As t →∞, $e^{-\frac{k}{m}t}\to 0$

∴limiting velocity =$-\frac{mg}{k}$

(b) The rate of change of atmospheric pressure P with respect to altitude, h in kilometres is proportional to the pressure. If the pressure at 2km is ¼ of the pressure P_0 at sea level. Find the formula for the pressure at any height.

Solution

$\frac{dP}{dh}=-kP$ (-ve because pressure decreases with altitude)

Separation of variables

$\int\frac{dP}{P}=-\int kdh$

$InP=kh+c$

At sea level (h = 0), p = P_0 => c = InP_0

$\therefore\ InP=kh+InP_0$

When h = 2km, P = $\frac{1}{4}P_0$

$In\left(\frac{1}{4}P\right)=-2k+InP_0$

$k=\frac{1}{2}In4$

$InP=\frac{h}{2}In4+InP_0$

$P=P_0e^{-hIn2}$

(c) The differential equation $\frac{dp}{dt}=\ kp(c-p)$ shows the rate at which information flows in a student population c. p represents the number who have heard the information in t days and k is a constant.

(i) Solve the differential equation.

$\frac{dp}{dt}=\ kp(c-p)$

Separating variables

$\frac{dp}{p(c-p)}$ = kdt

$\int\frac{dp}{p(c-p)}$ = $\int kdt$

$\int\frac{dp}{p(c-p)}$ = kt + a where a is a constant

By partial fractions

$\frac{1}{p(c-p)}\equiv\frac{A}{p}+\frac{B}{c-p}$

1≡ $A(c-p)+B(p)$

1≡ $Ac-Ap+Bp$

1≡ $Ac+(B-A)p$

Equating constants

1 =Ac

A = $\frac{1}{c}$

Equating coefficient of p

0 = B – A

A = B = $\frac{1}{c}$

⇨ $\int \frac{dp}{p(c-p)} = \frac{1}{c}\int \frac{1}{p}dp + \frac{1}{c}\int \frac{1}{c-p}dp$

$= \frac{1}{c}Inp - \frac{1}{c}In(c-p)$

$= \frac{1}{c}In\ \frac{p}{(c-p)}$

$\therefore \frac{1}{c}In\ \frac{p}{(c-p)} = kt + a$

(ii) A school has a population of 1000 students. Initially 20 students had heard the information. A day later, 50 students had heard the information. How many students heard the information by the tenth day?

Solution

Given c =1000, at t=0, p= 20

By substitution, we have

$\frac{1}{1000}In\ \frac{20}{1000-20} = 0 + a$

$a = \frac{1}{1000}In\ \frac{20}{980} = \frac{1}{1000}In\ \frac{1}{49}$

⇨ $\frac{1}{1000}In\ \frac{p}{(1000-p)} = kt + \frac{1}{1000}In\ \frac{1}{49}$

After t = 1, p = 50; by substitution, we have

$\frac{1}{1000}In\ \frac{50}{(1000-50)} = k(1) + \frac{1}{1000}In\ \frac{1}{49}$

k = $\frac{1}{1000}In\ \frac{50}{950} - \frac{1}{1000}In\ \frac{1}{49}$

= $\frac{1}{1000}In\ \frac{1}{19} \div \frac{1}{49}$

= $\frac{1}{1000}In\ \frac{49}{19}$

⇨ $\frac{1}{1000}In\ \frac{p}{(1000-p)}$

$= \left(\frac{1}{1000}In\ \frac{49}{19}\right)t + \frac{1}{1000}In\ \frac{1}{49}$

$In\ \frac{p}{(1000-p)} = \left[In\ \left(\frac{49}{19}\right)\right]t + In\ \frac{1}{49}$

Note: by 10th day is the same as after 9 days

Substituting for t = 9

$In\ \frac{p}{(1000-p)} = \left[In\ \left(\frac{49}{19}\right)\right](9) + In\ \frac{1}{49}$

$p = 990.3835$

Number of students who heard the information by the 10th day is 990

Exercise 5

1. The rate, cm^3s^{-1}, at which air is escaping from a balloon at time t seconds, is proportional to the volume of air, Vcm^3 in the balloon at the instant. Initial volume is $1000cm^3$.
   (a) Show that V = $1000e^{-kt}$, where k is a positive constant.
   (b) Given that V = $500cm^3$ when t = 6; show that k = $\frac{1}{6}In2$.
   (c) Calculate the value of V when t = 12s. [250 cm^3]
2. At time t minutes the rate of cooling of a liquid is proportional to the temperature, T^0C of the liquid at that time. Initially T = 80
   (a) Show that T = $80e^{-kt}$, where k is a positive constant.
   (b) Given that T = 20 when t = 6; show that k = $\frac{1}{3}Ih2$.
   (c) Calculate the time at which the temperature will reach 10^0C. [9]
3. The value of a certain product depreciates in such a way that when it is t years old, the rate of decrease in value is proportional to the value, x, of the product at that time. The product costs 12000 when new.
   (a) Show that x = $12000e^{-kt}$
   (b) Given that after 3 years the value dropped to 400;
      (i) show that k = $\frac{1}{3}In3$.
      (ii) Calculate to the nearest month, the time taken for the value to drop to 2000. [4years 11mnths]
4. A lump of a radioactive substance is decaying is proportional to the mass M in grams at time t. Initially M = 72 and decreases to 50 in 2 hours. Show that M = $72^{In\left(\frac{6}{5}\right)}$.
5. The rate at which a bacteria is reduced by a chemical is proportional to the number of bacterial present. Given that the population of the bacteria is reduced to half in six days; show that the population will be reduced to 1% of the original population in about 40days.

Topical revision exercise

1. Solve the equations

(a) $\frac{dy}{dx} + ycotx = 3sinxcosx$
$[y = sin^2x + c(cosx)]$

(b) $x^2\frac{dy}{dx} + y - x^2e^{\frac{1}{x}} = 0$; given y = 2 when x = 0 $\left[y = xe^{\frac{1}{x}}\right]$

(c) $(x^2 + 1)\frac{dy}{dx} - xy = x$; x = 0, y = 1.
$\left[y = 2(x^2 + 1)^{\frac{1}{2}} - 1\right]$

(d) $(x^2 + 1)\frac{dy}{dx} + y^2 + 1 = 0$;
x = 0, y = 1.
$$\left[y = \left(\frac{1 - x}{1 + x}\right)\right]$$

(e) $\frac{dt}{d\theta} = tcot\theta = 2cos\theta$;
t = 3 when θ = $\frac{\pi}{2}$
$$\left[t = \frac{1}{2}cosec\theta(5 - cos2\theta)\right]$$

(f) $y\frac{dy}{dx} = 2x - y$ (use substitution y =vx)
$\left[\left(\frac{y}{x} + 2\right)\left(\frac{y}{x} - 1\right) = k\right]$

(g) $\frac{dy}{dx} - ytanx = cos^2x$
$$\left[y = secx\left(sinx - \frac{1}{3}sin^3x + k\right)\right]$$

(h) $\frac{dy}{dx} = e^{2x} + x$ give R(0) = 3
$\left[y = \frac{1}{2}e^{2x} + \frac{1}{2}x^2 + \frac{5}{2}\right]$

(i) $tanx\frac{dy}{dx} - y = sin^2x$
$[y = sin^2x + csinx]$

(j) $\frac{1}{x}\frac{dy}{dx} = sinxsec^23y$
$\left[\frac{y}{2} + \frac{1}{12}sin6y = sinx - xcosx + c\right]$

(k) $\frac{dy}{dx} + 3y = e^{2x}$;z x=0, y= 1
$\left[y = \frac{1}{5}(e^{2x} + 4e^{-3x})\right]$

(l) $x\frac{dy}{dx} - y = x^3e^{x^2}$ $\left[y = \frac{x}{2}(e^{x^2} + A)\right]$

(m) $x\frac{dy}{dx} + y = e^x$ $\left[y = \frac{1}{x}e^x + \frac{c}{x}\right]$

(n) $\frac{dy}{dx} = \frac{sin^2x}{y^2}$; x = 0, y = 1
$$[4y^3 = 6x - 3sin2x + 4]$$

(o) $(1 – x^2)\frac{dy}{dx} - xy^2$ =0, y = 1 when x = 0.
$\left[\frac{1}{y} = In(1 - x^2)^{\frac{1}{2}} + 1\right]$

(p) $\frac{dy}{dx} = 1 + y^2$; y = 1 when x = 0.
$\left[y = tan\left(x + \frac{\pi}{4}\right)\right]$

2. The rate of cooling of a body is proportional to the difference θ between the temperature of the body and that of the surrounding air.
(a) Write down a differential equation involving θ for this process
$\left[\frac{d\theta}{dt} = -kt\right]$
(b) If the surrounding air temperature is stable at 20^0C and the body cools from 80^0C to 70^0C in 5 minutes; find the temperature after 15minutes [54.72^{0}C

3. A metallic teapot losses heat due to a steady breeze at a rate proportional to its temperature θ and gains temperature from a directed beam at a rate proportional to time t. Show that at any time t, $\theta = At + B + ce^{-kt}$

4. The rate of change of atmospheric pressure, p, with respect to altitude, h, in km is proportional to pressure. If the pressure at 6000m is half the pressure P0 at sea level. Find the formula for the pressure at any height
$\left[P = P_02^{-\frac{h}{6}} or\ P = P_0e^{\frac{-hIn2}{6}}\right]$

5. The mass of a man together with his parachute is 70kg. when the parachute is fully open, the system experiences an upward force proportional to the velocity of the system. If the constant of proportionality is 1/10 when the system is descending at 10 ms^{-1}, determine the speed of the parachute three minutes later [7.74ms^{-1}]

6. A vessel in a shape of an inverted right circular cone contains a liquid. The rate of evaporation of the liquid is proportional to the surface exposed to the atmosphere. The radius of the base of the cone is 9cm and the height of the cone is 15cm. if it takes 1 minute for the radius of the surface of the liquid to decrease from 9cm to 4.5cm, how long will it take for the liquid to evaporate completely. [6.88minute]

7. A rumour spreads through town at a rate which is proportional to the product

of the number of the people who have heard it and that of those who have not heard. Given that x is the fraction of the population who have heard the rumour after time t.

(a) Form a differential equation connecting x, t and constant k. $\left[\frac{dy}{dx} = k(1-x)\right]$

(b) If initially a fraction C of the population had heard the rumour, deduce that $x = \frac{C}{C+(1-C)e^{-kt}}$

(c) Given that 15% had heard the rumour at 9.00am and another 15% by noon, find what fraction of the population would have heard the rumour by 3.00pm [0.21]

8. A research to investigate the effect of a certain chemical on a crop virus infection revealed that the rate at which the virus population is destroyed is directly proportional to the population at that time. Initially the population was P_0 at time t months later, it was found to be P.

(a) Form a differential equation connecting P and t. $\left[\frac{dP}{dt} = -kP\right]$

(b) Given that the virus population reduced to one third of the initial population in 4 months, solve the equations in (a) $\left[P = 3^{-\frac{t}{4}}P_0\right]$

(c) Find

(i) How long it will take for 5% of the original population to remain. [10.907months]

(ii) What percentage of the original virus population will be left after 2 ½ months.[50.33%]

9. In a culture of bacteria, the rate of growth is proportional to the population present at time t. the population doubles every day. Given that the initial population P_0 is one million, determine the day when the population will be 100 million.[7th day]

10. (i) The volume of a water reservoir is generated by rotating the curve y =kx^2 about the y-axis. Show that when the central depth of the water in the reservoir is h meters, the surface area, A is proportional to h and the volume is proportional to h^2.

(ii) If the rate of loss of water from the reservoir due to evaporation is λA m^2 per day, obtain a differential equation for h per day. $\left[\frac{dh}{dt} = -\lambda\right]$

(iii) Given that λ = ½, determine how long it will take for the depth of water to decrease from 20m to 2m [36days]

11. The acceleration of a particle after time t seconds is given by α = 5 + cos$\frac{1}{2}t$. If initially the particle was moving at 1 ms-1, find its velocity after 2π second and the distance it would have covered by then. $[v = 10\pi + 1; s = 10\pi^2 + 2\pi + 4]$

12. An athlete runs at a speed proportional to square root of the distance he still has to cover. If the athlete starts running at 10ms^{-1} and has a distance of 1600m to cover, find how long he will take to cover the distance. [320s]

13. A hot body at a temperature of 100^{0}C is placed in a room of temperature 20^{0}C. Ten minutes later, its temperature is 60^{0}C. determine the temperature of the body after another 10 minutes. [40^{0}C]

14. The number of car accidents x in a year on a highway was found to approximate the differential equation $\frac{dy}{dx} = kx$, where t is time in years and k is a constant. At the beginning of 2000 the number of recorded accidents is 50. If the number of accidents increased to 60 at at the beginning of 2002, estimate the number that was expected at the beginning of 2005. [79]

15. In a certain process the rate of production of yeast is kx grams per minute, where x gram is the amount produced and k= 0.003.

(a) Show that the amount of yeast is doubled in in about 230 minutes

(b) If in addition yeast is removed at a constant rate of m grams per minute, find the

(i) Amount of yeast at time, t minutes, given that when t = 0; x = p grams $[x = m + (p - m)e^{0.003t}]$

(ii) Value of m if p =20,000g and the supply of yeast is exhausted in 100minutes. [77166g]

16. Bacteria in a culture increase at a rate proportional to the number of bacteria present. If the number increases from 1000 to 2000 in an hour,

(a) Find how many bacteria will be present after 1 ½ hour. [2829]

(b) How long will it take for the number of bacteria in the culture to become 4000? [2hours]

17. It is observed that the rate at which a body cools is proportional to the amount by which its temperature exceeds that of its surroundings. A body at 78^{0}C is placed in a room at 20^{0}C and after 5 minutes the body had cooled to 65^{0}C. what will be its temperature after further 5minutes.

18. At 3.00pm, the temperature of a hot metal was 80^{0}C and that of the surrounding was 20^{0}C. At 3.03 ppm the temperature of the metal had dropped to 42^{0}C. The rate of cooling of the metal was directly proportional to the difference between its temperature θ and that of the surroundings. Find the temperature of the metal at 3.05pm. [31.27^{0}C]

19. Solve the differential equation $\frac{dy}{dx} = (xy)^{\frac{1}{2}}Inx$, given that y = 1 when x = 1.

$\left[\sqrt{y} = \frac{1}{3}x\sqrt{x}Inx - \frac{2}{9}x\sqrt{x} + \frac{11}{9}\right]$

Hence find the value of y when x = 4 $[y = 9.8673]$

20. The rate at which the temperature of a body falls is proportional to the difference between the temperature of the body and that of its surrounding. The temperature of the body is initially 60^{0}C. After 15 minutes the temperature of the body is 50^{0}C. The temperature of the surrounding is 10^{0}C.

(a) Form a differential equation for the temperature of the body. (09marks)

$\frac{d\theta}{dt} \propto (\theta - 10)$

$\frac{d\theta}{dt} = k(\theta - 10)$

$\frac{d\theta}{(\theta-10)}$ = -kdt

$\int \frac{d\theta}{(\theta-10)} = -k\int dt$

$In\ (\theta - 10) = -kt + c$

When t = 0, θ = 60

c = $In\ (60 - 10)$ =In50

when t = 15, θ = 50

$In\ 40 = -k\ x\ 15 + In50$

15k = In$\left(\frac{50}{40}\right)$

k = $\frac{1}{15}In\left(\frac{5}{4}\right)$

Hence the differential equation is

$\frac{d\theta}{dt} = \frac{1}{15}In\left(\frac{5}{4}\right)(\theta - 10)$

(b) Determine the time it takes for the temperature of the body to reach 30^{0}C. (03marks)

In (θ – 10) = -$\frac{1}{15}In\left(\frac{5}{4}\right)t$ + In 50

When t = T and θ = 30

-$\frac{1}{15}In\left(\frac{5}{4}\right)T = In\ 50 - In\ 20$

T = $\frac{15In\left(\frac{5}{2}\right)}{In\left(\frac{4}{4}\right)}$ = 61.5943minutes

21. Solve the differential equation $\frac{dy}{dx} + ycot\ x = x$, given that y = 1 when x = $\frac{\pi}{2}$. (08marks)

$\frac{dy}{dx} + ycot\ x = x$

Integrating factor,

I.F = $e^{\int \cot x\ dx}$

= $e^{\int \frac{\cos x}{\sin x}dx}$

= $e^{In \sin x}$

= sinx

Multiplying all terms by integrating factor

sinx$\frac{dy}{dx}$ + ycotxsinx = xsinx

sinx$\frac{dy}{dx}$ + y$\frac{\cos x}{\sin x}$sinx = xsinx

sinx$\frac{dy}{dx}$ + ycosx = xsinx

$\frac{d(ysin\ x)}{dx}dx$ = xsinx

Integrating with respect to x

$$\int \frac{d(ysin\ x)}{dx}dx = \int xsinxdx$$

Ysin x = $\int xsinxdx$

Let u = x, $\frac{du}{dx} = 1$

$$\frac{dv}{dx} = sinx, v = \int sinx\ dx = -\cos x$$

Using integration by parts on RHS

ysinx = -xcosx + $\int \cos x\ dx$

ysinx = -xcosx + sin x + c

But y = 1, x = $\frac{\pi}{2}$

1sin$\left(\frac{\pi}{2}\right)$ = sin$\left(\frac{\pi}{2}\right)$ -$\left(\frac{\pi}{2}\right) cos\left(\frac{\pi}{2}\right)$ +C

C = 0

By substitution,

∴ ysinx = sinx - xcosx

Thank you

Dr. Bbosa Science