UACE MATHEMATICS PAPER 1 2012 and marking guide

Section A

1. Solve the simultaneous equations
   $3x - y + z = 3$
   $x - 2y + 4z = 3$
   $2x + 3y - z = 4$
2. (a) Prove that $\frac{tan\theta}{1+tan^2\theta} = sin2\theta$
   (b) Solve $\sin2\theta = \cos\theta$ for $0^0 \le \theta \le 90^0$.
3. Differentiate $\frac{3x-1}{\sqrt{x^2+1}}$ with respect to x.
4. A line passes through the points A(4, 6, 3 and B(1, 3,3 3)
   (a) Find the vector equation of the line
   (b) Show that the point (2, 4, 3) lies on the line above
5. The sum of the first n terms of Geometric Progression (G.P) is $\frac{4}{3}(x^n - 1)$. Find the nth term as an integral power of 2.
6. The line y = mx + c is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{y^2} = 1$ when c= $\pm\sqrt{a^2m^2 + b^2}$ find the equations of the tangents to the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$ from the point $(0, \sqrt{5})$
7. Using a suitable substitution, find $\int \frac{\sin^{-1}2x}{\sqrt{1-4x^2}}$.
8. Find the equation of the normal to the curve $x^2y + 3y^2 - 4x - 12 = 0$ at the point (0, 2).

Section B

9. If $z = \frac{(2-i)(5+12i)}{(1+2i)^2}$
   (a) Find
      (i) Modulus of z
      (ii) Argument of z
   (b) Represent z on a complex plane
   (c) Write z in the polar form
10. (a) Solve the equation $8\cos^4x - 10\cos^2x + 3 = 0$
    (b) Prove that cos4A – cos4B –ncos4C = 4sin2Bsin2Ccos2A – 1 given that A, B and C are angles of a triangle.
11. Find
    (a) the derivative with respect to x of the following.
       (i) $\frac{cos2x}{1+sin2x}$

(ii) In(secx + tanx)

(b) $\int_0^{\frac{\pi}{2}} x^2 sinx dx$

12. Triangle OAB has OA = a and OB = b. C is a point on OA such that OC = $\frac{2a}{3}$. D is the midpoint of AB. When CD is produced it meets OB at E., such that DE = nCD and BE = kb.
Express DE in terms of
(a) n, a and b
(b) k, a and b
Hence find the values of n and k.

13. (a) Find the equation of the locus of a point which moves such that its distance from D(4, 5) is twice its distance from origin.
(b) The line y= mx intersects the curve y= $2x^2$ – x at points A and B. Find the equation of locus of the point P which divides AB in the ratio 2:5.

14. (a) On the same axis, sketch the curves y = x(x+ 2) and y = x(4x –x)
(b) Find the area enclosed by the two curve in (a)
(c) Determine the volume of the solid generated when the area in (a) is rotated about the x-axis.

15. Solve for x in the following equation
(a) $9^x – 3^{x+1}$ = 10
(b) $\log_4 x^2 - 6\log_x 4 - 1 = 0$

16. At 3.00pm, the temperature of a hot metal was 80^0C and that of the surrounding is 20^0C. At 3.03pm the temperature of the metal had dropped to 42^0C. The rate of cooling of the metal was directly proportional to the difference between its temperature θ and that of the surroundings.
(a) (i) Write a differential equation to represent the rate of cooling of the metal.
(ii) Solve the differential equation using the given conditions.
(b) Find the temperature of the metal at 3.05pm.

Marking guides

1. Solve the simultaneous equations

   3x – y + z = 3

   x – 2y + 4z = 3

   2x + 3y – z = 4

   Solution

   Method 1

   3x – y + z = 3 ……………………………… (i)

   x – 2y + 4z = 3……………………………(ii)

   2x + 3y – z = 4 ……………………………(iii)

   2Eqn. (i) – eqn. (ii)

   5x – 2z = 3 …………………………………(iv)

   3eqn. (i) + eqn. (iii)

   11x + 2z = 13 ……………………………..(v)

   Eqn. (iv) + eqn. (v)

   16x = 16

   x = 1

   Substituting x into eqn. (iv)

   5 – 2z = 3 => z = 1

   Substituting x and z into eqn. (i)

   3 – y + 1 = 3 => y = 1

   ∴x = 1, y = 1 and z = 1

   Method 2

   By using row reduction to echelon form

   Expressing the equation in matrix form

   $$\begin{pmatrix}3 & -1 & 1\\ 1 & -2 & 4\\ 2 & 3 & -1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}3\\ 3\\ 4\end{pmatrix}$$

   The augmented matrix

   $$\begin{pmatrix}3 & -1 & 1:3\\ 1 & -2 & 4:3\\ 2 & 3 & -1:4\end{pmatrix}$$

   Transforming augmented matrix a unity triangular matrix

$$\begin{matrix}R_1\\ R_2\\ R_3\end{matrix}\begin{pmatrix}3 & -1 & 1:3\\ 1 & -2 & 4:3\\ 2 & 3 & -1:4\end{pmatrix}\begin{matrix}\to \frac{1}{3}R_1 = R_1\\ \to R_1 - 3R_2 = R_2\\ \to 2R_1 - 3R_2 = R_3\end{matrix}\begin{pmatrix}1 & -\frac{1}{3} & \frac{1}{3} & :1\\ 0 & 5 & -11 & -6\\ 0 & -11 & 5 & -6\end{pmatrix}$$

$$\begin{matrix}R_1\\ R_2\\ R_3\end{matrix}\begin{pmatrix}1 & -\frac{1}{3} & \frac{1}{3} & :1\\ 0 & 5 & -11 & -6\\ 0 & -11 & 5 & -6\end{pmatrix}\begin{matrix}\to R_1 = R_1\\ \to \frac{1}{5}R_2 = R_2\\ \to 11R_2 - 5R_2 = R_3\end{matrix}\begin{pmatrix}1 & -\frac{1}{3} & \frac{1}{3} & :1\\ 0 & 1 & \frac{-11}{5} & \frac{-6}{5}\\ 0 & 0 & -96 & -96\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1 & -\frac{1}{3} & \frac{1}{3} & :1\\ 0 & 1 & \frac{-11}{5} & \frac{-6}{5}\\ 0 & 0 & -96 & -6\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}:1\\ \frac{-6}{5}\\ -96\end{pmatrix}$$

   -96z = -96 => z = 1

   $y - \frac{11}{5}z = \frac{-6}{5}$ => y = 1

   $x - \frac{1}{3} + \frac{1}{3} = 1$ => x = 1

   ∴x = 1, y = 1 and z = 1

2. (a) Prove that $\frac{tan\theta}{1+tan^2\theta} = sin2\theta$

   Solution

   Considering LHS

   $$\frac{tan\theta}{1+tan^2\theta} = \frac{2sin\theta}{cos\theta} + \left(1 + \frac{sin^2\theta}{cos^2\theta}\right)$$

   $$= \frac{2sin\theta}{cos\theta} x \frac{cos^2\theta}{cos^2\theta+sin^2\theta}$$

   $$= \frac{2sin\theta}{cos\theta} x \frac{cos^2\theta}{1}$$

   $$= 2sin\theta cos\theta$$

   $$= sin2\theta$$

   (b) Solve sin2θ = cosθ for $0^0 \leq \theta \leq 90^0$.

   Solution

   2sinθcosθ = cosθ

   2sinθcosθ – cosθ= 0

   Cosθ(2sinθ – 1) = 0

   Either cosθ = 0 => θ = $cos^{-1}0 = 90^0$

   Or sinθ = ½ => θ= sin^{-1} (½) = 30^0

   Hence θ = 30^0 or 90^0.

3. Differentiate $\frac{3x-1}{\sqrt{x^2+1}}$ with respect to x.

   Solution

   Method 1

   Let y = $\frac{3x-1}{\sqrt{x^2+1}}$

   Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

   $$\frac{dy}{dx} = \frac{(x^2+1)^{\frac{1}{2}}\frac{d}{dx}(3x-1) - (3x-1)\frac{d}{dx}(x^2+1)^{\frac{1}{2}}}{\left(\sqrt{x^2+1}\right)^2}$$

   $$= \frac{3(x^2+1)^{\frac{1}{2}} - \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(3x-1).2x}{x^2+1}$$

   $$= \frac{3(x^2+1)^{\frac{1}{2}} - \frac{x(3x-1)}{(x^2+1)^{\frac{1}{2}}}}{x^2+1}$$

   $$= \frac{3(x^2+1) - x(3x-1)}{x^2+1(x^2+1)^{\frac{1}{2}}}$$

   $$= \frac{x+3}{(x^2+1)^{\frac{3}{2}}}$$

Method 2

   Let y = $\frac{3x-1}{\sqrt{x^2+1}}$

   Iny = $In\left(\frac{3x-1}{\sqrt{x^2+1}}\right)$ = In(3x – 1) - In$(x^2 + 1)^{\frac{1}{2}}$

$= \ln(3x-1) - \frac{1}{2}\ln(x^2+1)$

$\frac{1}{y}\frac{dy}{dx} = \frac{3}{3x-1} - \frac{x}{x^2+1}$

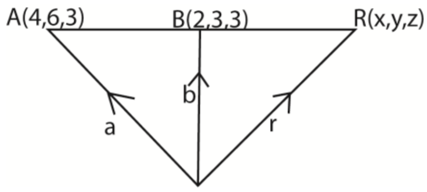
$= \frac{3(x^2+1) - x(3x-1)}{(3x-1)(x^2+1)}$

$= \frac{x+3}{(3x-1)(x^2+1)}$

$\frac{dy}{dx} = \frac{x+3}{(3x-1)(x^2+1)} \cdot \frac{3x-1}{\sqrt{x^2+1}}$

$= \frac{x+3}{(x^2+1)^{\frac{3}{2}}}$

4. A line passes through the points A(4, 6, 3 and B(1, 3,3 3)
   (a) Find the vector equation of the line
   Solution
   Let point R(x, y, z) lie on the same line



AR is parallel to AB

OP – OA = λ(OB – OA)

r = $\begin{pmatrix}4\\6\\3\end{pmatrix} + \lambda\left[\begin{pmatrix}1\\3\\3\end{pmatrix} - \begin{pmatrix}4\\6\\3\end{pmatrix}\right]$

$= \begin{pmatrix}4\\6\\3\end{pmatrix} + \lambda\begin{pmatrix}-3\\-3\\0\end{pmatrix}$

   (b) Show that the point (2, 4, 3) lies on the line above
   Solution
   If the point C(2, 4, 3) lies on the line, then this point must satisfy the above equation. So the value of λ must be same throughout

$\begin{pmatrix}2\\4\\3\end{pmatrix} = \begin{pmatrix}4\\6\\3\end{pmatrix} + \lambda\begin{pmatrix}-3\\-3\\0\end{pmatrix}$

$\begin{pmatrix}2\\4\\3\end{pmatrix} = \begin{pmatrix}4-3\lambda\\6-3\lambda\\3\end{pmatrix}$

For i

2 = 4 - 3λ

λ = $\frac{2}{3}$

For j

4 = 6 - 3λ

λ = $\frac{2}{3}$

For k

3 = 3

Since the value of λ = $\frac{2}{3}$ is constant, then the point C(2, 4, 3) lies on the line in (a) above

5. The sum of the first n terms of Geometric Progression (G.P) is $\frac{4}{3}(x^n - 1)$. Find the n^{th} term as an integral power of 2.
   Solution

$s_n = \frac{a(r^n - 1)}{r-1}$

Comparing with $s_n = \frac{4}{3}(x^n - 1)$.

⇨ a = 4

r – 1 = 3; r = 4

The n^{th} term, $U_n = ar^n - 1$

$= 4 \times {}^{4n-1}$

$= 4 \times 4^{2(n-1)}$

$= 2^{2n}$

OR

Given $s_n = \frac{4}{3}(x^n - 1)$

For n = 1

First term a = $\frac{4}{3}(4^1 - 1)$

$= \frac{4}{3}(3) = 4$

For n = 2

First term a = $\frac{4}{3}(4^2 - 1)$

$= \frac{4}{3}((16-1) - 4) = 4$

$= \frac{4}{3}(15 - 4)$

$= 20 - 4 = 16$

4r = 16

r = 4

$U_n = ar^n - 1$

$= 4 \times {}^{4n-1}$

$= 4 \times 4^{2(n-1)}$

$= 2^{2n}$

6. The line y = mx + c is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{y^2} = 1$ when c= $\pm\sqrt{a^2m^2 + b^2}$ find the equations of the tangents to the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$ from the point (0, $\sqrt{5}$)
   Solution
   y = mx + c

Substituting for c= $\pm\sqrt{a^2m^2+b^2}$

y =mx $\pm\sqrt{a^2m^2+b^2}$

For ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$

a^2 = 2 and b^2 = 1

Substituting of a^2 and b^2

y =mx $\pm\sqrt{4m^2+1}$

For point (0,$\sqrt{5}$) lies on the tangent

$\sqrt{5}=\pm\sqrt{4m^2+1}$

Squaring both sides

5 = $4m^2$ + 1

$4m^2$ = 4

m^2 = 1

m =±1

When m = 1

y = x$\pm\sqrt{5}$

Testing for the correct equation by substitution using (x, y) = (0, $\sqrt{5}$)

$\sqrt{5}=0\pm\sqrt{5}$

Hence the equation of the tangent is

y = x +$\sqrt{5}$

Testing for the correct equation by substitution using (x, y) = (0, $\sqrt{5}$)

$\sqrt{5}=0\pm\sqrt{5}$

Hence the equation of the tangent is

y = x +$\sqrt{5}$

When m = -1

y = -x ±$\sqrt{5}$

Testing for the correct equation by substitution using (x, y) = (0, $\sqrt{5}$)

$\sqrt{5}=0\pm\sqrt{5}$

Hence the equation of the tangent is

y = -x +$\sqrt{5}$

∴the equation of the tangent are

y = ±x+$\sqrt{5}$

7. Using a suitable substitution,

find$\int\frac{\sin^{-1}2x}{\sqrt{1-4x^2}}$.

Solution

Let 2x = sinu

2dx = cosudu

dx = $\frac{1}{2}$cosudu

⇨ $\int\frac{\sin^{-1}2x}{\sqrt{1-4x^2}}=\int\frac{u}{\sqrt{1-sin^2u}}.\frac{1}{2}cosudu$

$=\int\frac{u}{cosu}.\frac{1}{2}cosudu$

$=\frac{1}{2}udu=\frac{1}{4}u^2+c$

$=\left(\frac{\sin^{-1}2x}{2}\right)^2$

$\therefore\int\frac{\sin^{-1}2x}{\sqrt{1-4x^2}}=\left(\frac{\sin^{-1}2x}{2}\right)^2$

8. Find the equation of the normal to the curve $x^2y+3y^2-4x-12=0$ at the point (0, 2).

Solution

$x^2y+3y^2-4x-12=0$

$x^2\frac{dy}{dx}+2xy+6y\frac{dy}{dx}-4=0$

$(x^2+6y)\frac{dy}{dx}=4-2xy$

$\frac{dy}{dx}=\frac{4-2xy}{(x^2+6y)}$

At point (0, 2)

$\frac{dy}{dx}=\frac{4-0}{(0+12)}=\frac{1}{3}$

Gradient of the normal = -3

Let a point (x, y) lie on the normal

⇨ $\frac{y-2}{x-0}=-3$

$y=-3x+2$

Section B

9. If $z=\frac{(2-i)(5+12i)}{(1+2i)^2}$

(a) Find

(i) Modulus of z

Solution

Method 1

$z=\frac{(2-i)(5+12i)}{(1+2i)^2}$

$=\frac{10+24i-5i+12}{1+4i-4}$

$=\frac{22+19i}{-3+4i}$

$=\frac{(22+19i)(-3-4i)}{(-3+4i)(-3-4i)}$

$=\frac{-66-88i-57i+76}{(-3)^2+(4i)^2}$

$=\frac{10-145i}{9+16}=\frac{10-145i}{25}$

z= $\frac{2}{5}-\frac{29}{5}i$

$|z|=\sqrt{\left(\frac{2}{5}\right)^2+\left(\frac{29}{5}\right)^2}=5.814$

Method 2

$|z| = \left|\frac{(2-i)(5+12i)}{|(1+2i)|^2}\right|$

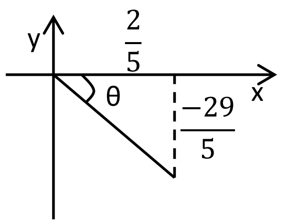
$= \frac{|(2-i)||(5+12i)|}{|(1+2i)|^2}$

$= \frac{|(2-i)||(5+12i)|}{|(1+2i)|^2}$

$= \frac{\sqrt{(2^2+(-1)^2)}.\sqrt{(5^2+12^2)}}{\left(\sqrt{(1^2+2^2)}\right)^2}$

$= \frac{\sqrt{5}.\sqrt{169}}{(\sqrt{2})^2} 5.814$

(ii) Argument of z



$\tan\theta = \frac{-29}{5} x \frac{5}{2} = \frac{-29}{2}$

$\theta = \tan^{-1}\left(\frac{-29}{2}\right) = -86.055$

Arg(z) = −86.055

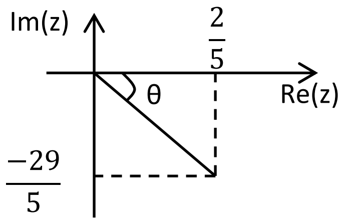
Method 2

Arg(z) = arg(2 - i) + arg(5+ 12i) - 2arg(1+ 2i)

= $-26.565^0 + 67.38^0 - 126.87^0$

= -86.055^0

(b) Represent z on a complex plane



(c) Write z in the polar form

z = 5.814(cos(-86.055^0) - isin(-86.055^0))

= 5.814(cos 0.478π – isin 0.478π

10. (a) Solve the equation

$8\cos^4x - 10\cos^2x + 3 = 0$

Solution

Let cos2x = y

⇨ $8y^2 - 6y - 4y + 3 = 0$

(2y – 1)(4y – 3) = 0

Either y = $\frac{1}{2}$ or y = $\frac{3}{4}$

If y = $\frac{1}{2}$ => $\cos^2x = \frac{1}{2}$

cosx = $\pm\frac{1}{\sqrt{2}}$

Taking cos x =$\frac{1}{\sqrt{2}}$

x = $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^0$

Taking cos x =$-\frac{1}{\sqrt{2}}$

x = $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = 135^0$

If y = $\frac{3}{4}$ => $\cos^2x = \frac{3}{4}$

cosx = $\pm\frac{\sqrt{3}}{2}$

Taking cos x =$\frac{\sqrt{3}}{2}$

x = $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^0$

Taking cos x =$-\frac{1}{\sqrt{2}}$

x = $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = 150^0$

∴x = $30^0, 45^0, 135^0, 150^0$

(b) Prove that cos4A – cos4B –cos4C = 4sin2Bsin2Ccos2A – 1 given that A, B and C are angles of a triangle

LHS = cos4A – cos4B –cos4C

= cos4A – [cos4B + cos4C]

= cos4A $-\left[2cos\left(\frac{4B+4C}{2}\right)cos\left(\frac{4B-4C}{2}\right)\right]$

= cos4A – 2cos(2B + 2C)cos(2B – 2C)

= $2\cos^2 2A$ – 1 - 2cos(2B + 2C)cos(2B – 2C)

= $2\cos^2 2A$- 2cos(2B + 2C) cos(2B – 2C) – 1

Now, A + B + C = 180^0

2A + 2B + 2C = 360^0

2B + 2C = (360^0 – 2A)

cos (2B + 2C) = cos (360^0 – 2A)

cos (2B + 2C) = cos 2A

By substitution, we have

= $2\cos^2 2A$- 2cos2Acos(2B – 2C) – 1

=2cos2A[cos2A – cos(2B – 2C)] – 1

=2cos2A$\left[2sin\left(\frac{2B+2C+2B-2C}{2}\right)sin\left(\frac{2B+2C-2B+2C}{2}\right)\right] - 1$

=2cos2A[sin2Bsin2C] – 1

= 4cos2Asin2Bsin2C as required.

11. Find

(b) the derivative with respect to x of the following.

(i) $\frac{cos2x}{1+sin2x}$

Method 1

Let y = $\frac{cos2x}{1+sin2x}$

$\frac{dy}{dx} = \frac{(1+sin2x)\frac{d}{dx}cos2x - cos2x\frac{d}{dx}(1+sin2x)}{(1+sin2x)^2}$

$= \frac{-2sin2x-2sin^2 2x-2cos^2 2x}{(1+sin2x)^2}$

$= \frac{-2sin2x(1+sin^2 2x)}{(1+sin2x)^2}$

$= \frac{-2}{1+sin2x}$

Method 2

Let y = $\frac{cos2x}{1+sin2x}$

Iny =Incos2x – In(1+sin2x)

$\frac{1}{y}\frac{dy}{dx} = \frac{-2sin2x}{cos2x} - \frac{2cos2x}{1+sin2x}$

= $\frac{-2sin2x(1+sin2x)-2cos^2 2x}{cos2x(1+sin2x)}$

= $\frac{-2[sin2x+sin^2 2x+ 2cos^2 2x]}{cos2x(1+sin2x)}$

= $\frac{-2}{cos2x}$

$\frac{dy}{dx} = \frac{-2}{cos2x}$ y =$\frac{-2}{cos2x}.\frac{cos2x}{1+sin2x}$

$\frac{dy}{dx} = \frac{-2}{1+sin2x}$

(ii) In(secx + tanx)

Solution

Let y = In(secx + tanx)

$\frac{dy}{dx} = \frac{secxtanx+sec^2x}{secx+tanx}$

$= \frac{secx(tanx+secx}{secx+tanx}$

$= secx$

(b) $\int_0^{\frac{\pi}{2}} x^2 sinxdx$

Solution

Method 1

Using integration by parts

$\int u\left(\frac{dv}{dx}\right)dx = uv - \int v\left(\frac{du}{dx}\right)dx$

Let u = x^2, $\frac{du}{dx} = sinx$

$\frac{du}{dx} = 2x$, v = -cosx

$\int_0^{\frac{\pi}{2}} x^2 sinxdx$= -$x^2$cosx-2$\int(-cox)xdx$

= -x^2cosx+2$\int xcosxdx$

Let u = x, $\frac{dy}{dx} = cosx$

$\frac{du}{dx} = 1$, v = sinx

$\int_0^{\frac{\pi}{2}} x^2 sinxdx$

= -x^2cosx+2$[sinx - \int(sinx).1dx]$

= -x^2cosx+2xsinx−2 $\int sinxdx$

= -x^2cosx + 2xsinx + 2cosx + c

$\int_0^{\frac{\pi}{2}} x^2 sinxdx$= $[-x^2cosx + 2xsinx + 2cosx]_0^{\frac{\pi}{2}}$

$= \left[-\left(\frac{\pi}{2}\right)^2 cos\frac{\pi}{2} + 2.\frac{\pi}{2}sin\frac{\pi}{2} + 2co\frac{\pi}{2}\right] - 2cos\frac{\pi}{2}$

= π - 2

Method 2

Using simplified form (special case) of integration by parts

	Differentiate	integrate
+	x^2	sinx
-	2x	-cosx
+	2	-sinx
-	0	cosx

$\int_0^{\frac{\pi}{2}} x^2 sinxdx$= $[-x^2cosx + 2xsinx + 2cosx]_0^{\frac{\pi}{2}}$

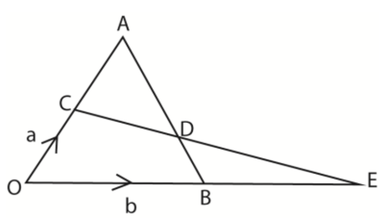
$= \left[-\left(\frac{\pi}{2}\right)^2 cos\frac{\pi}{2} + 2.\frac{\pi}{2}sin\frac{\pi}{2} + 2co\frac{\pi}{2}\right] - 2cos\frac{\pi}{2}$

= π - 2

12. Triangle OAB has OA = a and OB = b. C is a point on OA such that OC = $\frac{2a}{3}$. D is the midpoint of AB. When CD is produced it meets OB at E., such that DE = nCD and BE = kb. Express DE in terms of

(a) n, a and b

Solution



DE = nCD

=n(CA + AD)

= n$\left[\frac{1}{3}a + \frac{1}{2}(BO + OA\right]$

$= n\left[\frac{1}{3}a + \frac{1}{2}(-b + a)\right]$

$= n\left[\frac{1}{3}a - \frac{1}{2}b + \frac{1}{2}a\right]$

$= n\left[\frac{5}{6}a - \frac{1}{2}b\right]$

$= \frac{5n}{6}a - \frac{n}{2}b$

(b) k, a and b

Solution

DE = DB + BE

$= \frac{1}{2}AB + kb$

$= \frac{1}{2}(-b + a) + kb$

$= -\frac{1}{2}b + \frac{1}{2}a + kb$

$= \frac{1}{2}a + \frac{1}{2}(2k - 1)b$

Hence find the values of n and k.

Equating DE in (a) to DE in (b)

$\frac{5n}{6}a - \frac{n}{2}b = \frac{1}{2}a + \frac{1}{2}(2k - 1)b$

For a

$\frac{5n}{6}a = \frac{1}{2}a$ => $n = \frac{3}{5}$

For b

$-\frac{n}{2} = \frac{1}{2}(2k - 1)$

-n =2k – 1

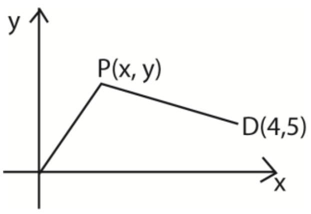
$-\frac{3}{5}$ = 2k – 1

2k = $\frac{2}{5}$

k = $\frac{1}{5}$

13. (a) Find the equation of the locus of a point which moves such that its distance from D(4, 5) is twice its distance from origin.

Solution



OP= $\sqrt{x^2 + y^2}$

PD= $\sqrt{(x - 4)^2 + (y - 5)^2}$

PD = 3OP

$\sqrt{(x - 4)^2 + (y - 5)^2} = \sqrt{x^2 + y^2}$

$(x - 4)^2 + (y - 5)^2 = 9(x^2 + y^2)$

$8x^2 + 8y^2 + 8x + 10y - 41 = 0$

(b) The line y= mx intersects the curve $y= 2x^2 - x$ at points A and B. Find the equation of locus of the point P which divides AB in the ratio 2:5.

Solution

Substituting for y = mx into y = $2x^2$ –x

mx = 2x2 –x

2x2 – x – mx = 0

x(2x – 1 – m) = 0

Either x = 0

Or

2x = m+1

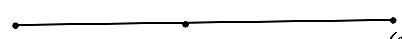
x = $\frac{m+1}{2}$

If x = 0, y = 0

(x, y) = (0, 0)

If x = $\frac{m+1}{2}$, y= $\frac{m(m+1)}{2}$

(x, y) = $\left(\frac{m+1}{2}, \frac{m(m+1)}{2}\right)$



A(0,0) P(x,y) B$\left(\frac{m+1}{2}, \frac{m(m+1)}{2}\right)$

AP =$\frac{2}{7}AB$

$$\begin{pmatrix} x - 0 \\ y - 0 \end{pmatrix} = \frac{2}{7}\begin{pmatrix} \frac{m+1}{2} - 0 \\ \frac{m(m+1)}{2} - 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{2}{7}\begin{pmatrix} \frac{m+1}{2} \\ \frac{m(m+1)}{2} \end{pmatrix} = \begin{pmatrix} \frac{m+1}{7} \\ \frac{m(m+1)}{7} \end{pmatrix}$$

$x = \frac{m+1}{7}$ ………………………………………… (i)

m =7x -1

y =$\frac{m(m+1)}{7}$……………………………………(ii)

Substituting m into equation (ii)

y =$\frac{(7x - 1)(7x - 1 + 1)}{7}$

y = $7x^2$ - x

14. (a) On the same axis, sketch the curves y = x(x+ 2) and y = x(4x –x)

Solution

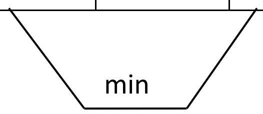
Considering y = x(x+2)

If x = 0, y = 0

If y = 0

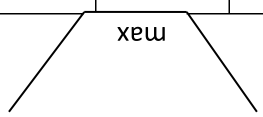
x(x + 2) = 0
x = 0 or x = -2
(x, y) = (0,0) and (-2, 0)
Finding the turning point
$y = x^2 + 2x$
$\frac{dy}{dx} = 2x + 2$
⇨ 2x + 2 = 0
x = -1; y = 1 – 2 = -1
The turning point = (-1, -1)
Finding the nature of the turning point;

x	-2	-1	0
$\frac{dy}{dx}$	-2	0	2



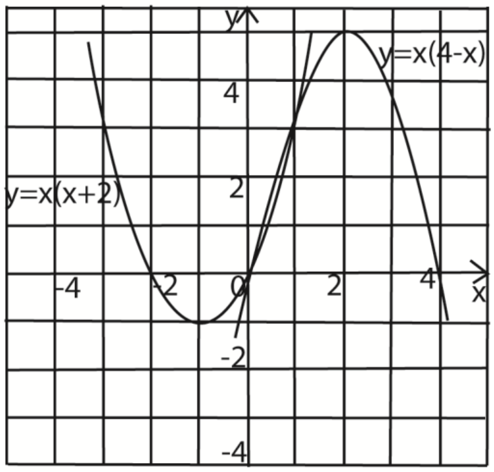
OR
By finding the second derivative
$\frac{dy}{dx} = 2x + 2$
$\frac{d^2y}{dx^2} = 2$ (min)
Hence the turning point is minimum
Considering y = x(x – 4)
Finding intercept
If x = 0, y = 0
(x, y) = (0, 0)
If y = 0
x(x – 4) = 0
Either x =0 or x = 4
(x,y) =(0, 0) and (0, 4)
Finding the turning point
$y = 4x – x^2$
$\frac{dy}{dx} = 4 - 2x$
⇨ $0 = 4 - 2x$
x = 2, y = 4
The turning point = (2, 4)
Finding the nature of the turning point

x	1	2	3
$\frac{dy}{dx}$	3	0	-2

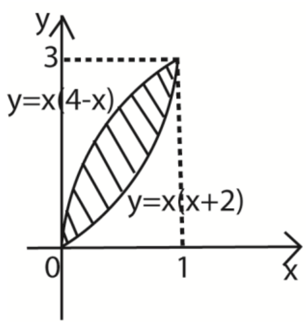


Or
By finding the second derivative;
$\frac{dy}{dx} = 4 - 2x$
$\frac{d^2y}{dx^2} = -2$ (max)
Hence the turning point is maximum

Finding the point of intersection of the curves
$x^2 + 2x = 4x – x^2$
$2x^2 – 2x = 0$
2x(x – 1) = 0
Either x = 0 or x = 1;
If x = 0, y = 0; (x, y) = (0, 0)
If x = 1, y = 3; (x, y) = (1, 3)



(b) Find the area enclosed by the two curve in (a)



Area of shaded region
$=\int_0^1 (4x - x^2)dx - \int_0^1 (x^2 + 2x)dx$
$=\left[2x^2 - \frac{x^3}{3}\right]_0^1 - \left[\frac{x^3}{3} + x^2\right]_0^1$
$=\left[2 - \frac{1}{3}\right]_0^1 - \left[\frac{1}{3} + 1\right]_0^1 = \frac{1}{3}$sq. units

(c) Determine the volume of the solid generated when the area in (a) is rotated about the x-axis.
Solution
$V = \pi \int_0^1 [(4x - x^2)^2 - (x^2 + 2x)^2]$
$= \pi \int_0^1 (12x^2 - 12x^3)$
$= \pi [4x^3 + 3x^4]_0^1$
$= \pi(4 - 3)$
$= \pi \; cubic \; units$

15. Solve for x in the following equation
(a) $9^x - 3^{x+1} = 10$
Solution
$9^x - 3^{x+1} = 10$
$3^{2x} - 3^x.3^1 = 10$
Let $y = 3^x$
⇨ $y^2 - 3y = 10$
$(y - 5)(y + 2) = 0$
Either y = 5 or y = -2
If y = 5
$3^x = 5$
xlog3 = log 5
$x = \frac{log5}{log3} = 1.465$
If y = -2
$3^x = -2$
xlog3 = log -2
$x = \frac{log-2}{log3}(invalid)$

(b) $\log_4 x^2 - 6\log_x 4 - 1 = 0$
Solution
Expressing the logs to base 4
$2\log_4 x - \frac{\log_4 4^6}{\log_4 x} = 1$
$2\log_4 x - \frac{6}{\log_4 x} = 1$
Let $\log_4 x = y$
$2y^2 - y - 6 = 0$
$(y - 2)(2y + 3) = 0$
Either y = 2 or $y = -\frac{3}{2}$
If y = 2
$\log_4 x = 2$
$4^2 = x$
x = 16
If y = -1.5
$\log_4 x = -1.5$
$4^{-1.5} = x$
$x = \frac{1}{8}$

16. At 3.00pm, the temperature of a hot metal was 80^0C and that of the surrounding is 20^0C. At 3.03pm the temperature of the metal had dropped to 42^0C. The rate of cooling of the metal was directly proportional to the difference between its temperature θ and that of the surroundings.

(a) (i) Write a differential equation to represent the rate of cooling of the metal
Solution
$-\frac{d\theta}{dt} \propto (\theta - 20)$
$\frac{d\theta}{dt} = -k(\theta - 20)$
By separating variable
$\frac{d\theta}{\theta-20} = -kdt$
$\int \frac{d\theta}{\theta-20} = -k\int dt$
$In(\theta - 20) = -kt + c$
$\theta - 20 = e^{-kt+c}$
$\theta - 20 = e^{-kt}.e^c$
Let $e^c = A$
$\theta = 20 + Ae^{-kt}$

(ii) Solve the differential equation using the given conditions.
Solution
At time t = 0, $\theta = 80^0$
A = 80 – 20 = 60
$\theta = 20 + 60e^{-kt}$
At time t = 3min, θ = 42^0C
$42 - 20 = 60e^{-kt}$
$22\ \ 60e^{-kt}$
$k = \frac{1}{3}In\left(\frac{60}{22}\right)$

(b) Find the temperature of the metal at 3.05pm

Solution

After 5 min

$\theta = 20 + 60e^{-\frac{5}{3}In\left(\frac{60}{22}\right)}$ =31.27^{0}C

Thank you
Dr. Bbosa Science