Differential equations (D.Es)

You should have done integration

Definition

These are equations involving differential coefficients (derived functions) like $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots \frac{d^ny}{dx^n}$.

The order of a differential equation

The order of a differential equation is the order n of which the highest derivative $\frac{d^ny}{dx^n}$ contained in the differential equation. E.g.

(a) $\frac{dy}{dx} + 2 = y$ (1st order D.E)

(b) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2 = 0$ (2nd order D.E)

(c) $\frac{d^4y}{dx^4} + 3\frac{dy}{dx} = 0$ (4th order D.E)

Solution to differential equations

This involves the elimination of all the differential coefficients in the given equation. This is normally done by integrating

Example 1

Solve the differential equations

(a) $\frac{dy}{dx} - 5 = 0$

Solution

$\frac{dy}{dx} - 5 = 0$

$\int dy = \int 5dx$

y = 5x +c

(b) $\frac{dy}{dx} + 3 = 0$

Solution

$\frac{dy}{dx} + 3 = 0$

$\int dy = -\int 3dx$

y = -3x +c

Types of solutions of differential equations

There are two types of solutions to differential equations, i.e.

- General solutions remain with arbitrary constant c unsolved, e.g.

  Solve $\frac{dy}{dx} - 1 = 0$

  $\int dy = \int dx$

  y = x + c (general solution)

- Specific solutions where the arbitrary constant c is eliminated and replaced with specific value, e.g.

  $ydy = xdx$, given that y =1 when x = 0

  $\int ydy = \int xdx$

  $\frac{1}{2}y^2 = \frac{1}{2}x^2 + c$

  Substituting for x = 0 and y = 1, c = $\frac{1}{2}$

  $\therefore \frac{1}{2}y^2 = \frac{1}{2}x^2 + \frac{1}{2}$

  => $y^2 = x^2 + 1$ (specific solution)

Forming differential equations

This involves forming equations with differential coefficients by eliminating the arbitrary constants.

If the equation has got only one constant of integration, then the differential equation formed will be that of first order.

On the other hand, if the equation has got two constants of integration, the differential equation formed will be that of a second order.

Example 3

Form differential equations from the following equation:

(a) y = x + $\frac{A}{x}$

Solution

y = x + $\frac{A}{x}$

⇨ A = x(y – x)

$\frac{dy}{dx} = 1 - \frac{A}{x^2}$

Substituting for A:

$\frac{dy}{dx} = 1 - \frac{x(y-x)}{x^2} = 1 - \frac{y-x}{x}$

$x\frac{dy}{dx} - 2x - y$ (1st order D.E)

(b) $y = 4x^2 - A$

Solution

$\frac{dy}{dx} = 8x$

Or

$\frac{dy}{dx} - 8x = 0$

Exercise 1

Form first order differential equations from each of the following equations

(a) y =3x^2 + Ax $\left[\frac{dy}{dx} = 6x + A\right]$

(b) $y = \frac{A}{x}$ $\left[\frac{dy}{dx} - \frac{A}{x^2} = 0\right]$

(c) $y = \frac{1}{x^2}$ $\left[\frac{\delta y}{\delta x} - \frac{-2}{x^3} = 0\right]$

(d) $x^2 + 4y^2 = A$ $\left[\frac{dy}{dx} + \frac{x}{4y} = 0\right]$

(e) $y^2 = xy + A$ $\left[(2y - x)\frac{dy}{dx} - y = 0\right]$

Solving 1st order differential equation

There are three basic methods employed to solve first order differential equations i.e.

(a) Separable differential equations.
(b) Differential equations with no separable variables.
These may either be exact or Non-exact (Inexact)
(c) Homogenous differential equations.

Separable differential equations

These are solved by separating variables

Suppose that the given differential equation is in the form of $f(y)\frac{dy}{dx} = g(x)$, we separate the variables in such a way that $f(y)dy = g(x)dx$. Then we integrate both sides

Example 4

(a) Solve the differential equation 8y$\frac{dy}{dx}$ = 9x^2

Solution

8ydy = 9x^{2}dx

$\int 8ydy = \int 9x^2$dx

$4y^2 = 3x^3 + c$

Hence find the solution given that y = 2 when x = 1

Substituting for y = 2 and x = 1

4 x 2^2 = 3 x 1^3 + c

c = 16 – 3 = 13

Substituting c in the equation

=> $4y^2 = 3x^3 + 13$

(b) Solve the differential equation $\frac{dy}{dx}$ = 2x + 5, given that y = -1 when x = 3

Solution

dy = (2x +5)dx

$\int dy = \int(2x + 5)dx$

y = x^2 + 5x + c

substituting for y = -1 and x =3

-1 = 9 + 15 + c

c = -25

Hence equation: y = x^2 + 5x - 25

(c) Solve the differential equation

3y$\frac{dy}{dx} = \frac{1}{x^2}$ given that y = 2 when x = 1

Solution

3ydy = x^{-2}dx

$\int$ 3ydy = $\int x^{-2}$dx

$\frac{3y^2}{2} = -\frac{1}{x} + c$................(i)

Substituting for y = 2 and x = 1 in equation (i)

$\frac{3(2)^2}{2}$= -$\frac{1}{1} + c$ => =7

Substituting for c = 7 in equation (i)

$\frac{3y^2}{2} = -\frac{1}{x} + 7$

Application of differential equation

A. Rates of decay, decomposition and cooling

Example 5

(i) The rate of decay of a radioactive material is proportional to the amount x grams of the material present at any time t. Initially there was 60 grams of the material. After 8 years the material reduced to 15 grams.

(a) Form a differential equation for the rat of decay of the material. (03marks)

$\frac{dx}{dt} = -kx$

(b) Solve the differential equation formed in (a) above. (10 marks)

$\int \frac{dx}{x} = -k \int dt$

Inx = -kt + c

At t = 0, c = In60

t = 8, x = 15

In$\frac{60}{15}$ = 8k

k =$\frac{1}{8} In \frac{60}{15}$ = 0.173 year^{-1}

Thus, $In \frac{60}{x} = 0.173t$

(c) Find the time taken for the material to reduce to 10 grams. (02 marks)

$In \frac{60}{10} = 0.173t$

t = 10.36 years

(ii) Chemical A is converted into another chemical by a chemical reaction. The rate at which a chemical A is being converted is directly proportional to the amount present at any time. Initially 100g of chemical A was present. After 5 minutes, 90g of A is present.

(a) Form a differential equation for chemical reaction. (03marks)

Let the amount present at time t = x

$\frac{dx}{dt} = -kx$

(b) By solving the differential equation formed in (a), determine the

(i) amount of chemical A present after 20 minutes.

$\int \frac{dx}{x} = -k \int dt$

$Inx = -kt + c$

At t =0, x = 100 => c = In100

At t = 5

In90 = -k x 5 + In100

k = $\frac{1}{5} In \frac{100}{90}$ = 0.021min^{-1}

⇨ In$\frac{100}{x} = 0.021t$

At t = 20

In$\frac{100}{x} = 0.021x\ 20 = 0.42$

$\frac{100}{x} = 1.522$; = 65.7

Hence the amount present after 20 minutes = 65.7g

(ii) time taken for the amount of chemical A to be reduced to 20 g (12 marks)

From In$\frac{100}{x} = 0.021t$

⇨ $\text{In}\frac{100}{20} = 0.021t$

t = 76.64minutes

(iii) The rate of cooling of a body is proportional to the excess temperature above the surrounding air. Given that the surrounding air temperature is 20^0C and the body cools from 100^0C to 60^0C in 5 minutes.

(a) Determine the temperature of the body after another 20 minutes

(b) How long does it take for the body to cool to 30^0C.

Solution

Let θ be the temperature of the body.

⇨ $\frac{d\theta}{dt} \propto (\theta - 20)$

$\frac{d\theta}{dt} = -k(\theta - 20)$

By separating variable

$\int \frac{d\theta}{(\theta-20)} = -\int kdt$

$In(\theta - 20) = -kt + c$

When t = 0, θ = 100 => c = In80

Substituting for c

$In(\theta - 20) = -kt + In80$

When t = 5, θ = 60 => In40 = -5k + In80

k = $\frac{In2}{5} min^{-1}$

$In(\theta - 20) = -\frac{In2}{5}t + In80$

Or

$In\frac{80}{(\theta-20)} = \frac{In2}{5}t$

(i) After another 20minutes, t = 25

$In\frac{80}{(\theta-20)} = \frac{In2}{5}x25 = 5In2$

$In\frac{(\theta-20)}{80} = -5In2$

$\theta = 22.5^0$

∴after another 20 minutes the temperature will be 22.5^0.

(ii) When θ = 30^0C

$In\frac{(30-20)}{80} = -\frac{1}{5}In2t$

t = 15minutes.

(iv) The rate of cooling of a body is proportional to the temperature of the body at that time. Given that the body cools from 72^0C to 32^0C in 10 minutes, determine how much longer it will take the body to cool to 27^0C.

$\frac{d\theta}{dt} \propto \theta$

$\frac{d\theta}{dt} = -k\theta$

By separating variables

$\frac{d\theta}{\theta} = -kdt$

$\int \frac{d\theta}{\theta} = -\int kdt$

$In\theta = -kt + c$

At t = 0, θ = 72^0C

By substitution

c = In72

$In\theta = -kt + In72$

At θ = 32^0C, t = 10

$In32 = -10k + In72$,

=> k = 0.0811min^{-1}

When θ = 270C

$In27 = -0.0811t + In72$; t = 12.1min

∴ it will take more 12.1 – 10 = 2.1minutes for the temperature to fall to 27^0C

(v) The rate of decay at any instant of a radioactive substance is proportional to the amount of substance remaining at that instant. If the initial amount of substance is A and the amount remaining after time t is x.

(a) Prove that $x = Ae^{-kt}$, where k is a constant.

(b) If the amount remaining is reduced from $\frac{1}{2}A$ to $\frac{1}{3}A$ in 8hours, prove that the initial amount of substance was halved in about 13.7hours.

Solution

(a) $\frac{dx}{dt} \propto x$

$\frac{dx}{dt} = -kx$

By separating variables

$\frac{dx}{x} = -kdt$

$\int \frac{dx}{x} = -\int kdt$

$Inx = -kt + c$

At time t = 0, x = A

By substituting InA = C

$Inx = -kt + InA$

$In\frac{x}{A} = -kt$

$\frac{x}{A} = e^{-kt}$

$x = Ae^{-kt}$

(b) $Inx = -kt + c$

At t = 0, x = $\frac{1}{2}A$; => c = $In\left(\frac{1}{2}\right)$

When t = 8, x = $\frac{1}{2}A$;

$In\left(\frac{1}{3}\right) = -8k + In\left(\frac{1}{2}\right)$

$k = \frac{1}{8}In\frac{2}{3}$

$Inx = -\left(\frac{1}{8}In\frac{2}{3}\right)t + In\left(\frac{1}{2}\right)$

When x is halved to $\frac{1}{4}A$

$In\left(\frac{1}{4}A\right) = -\left(\frac{1}{8}In\frac{2}{3}\right)t + In\left(\frac{1}{2}\right)$

$t = 13.676\text{hours} \approx 13.7 hours$

(vi) A substance loses mass at a rate which is proportional to the amount M present at time t.

(a) Form a differential equation connecting M, t and proportionality constant k. (02marks)

$-\frac{dM}{dt} \propto M$

$-\frac{dM}{dt} = kM$

$\frac{dM}{dt} = -kM$

(b) If the initial mass of the substance is M_0, show that M = M_0e^{-kt}. (05marks)

$\frac{dM}{M} = -kdt$

$\int \frac{dM}{M} = \int -kdt$

$InM = -\text{kt} + \text{C}$

At t= 0; M = M_0

c = InM_0

⇨ $InM = -\text{kt} + \text{InM}_0$

$\text{InM} - \text{InM}_0 = \text{-kt}$

$In\frac{M}{M_0} - kt$ or M = M_0e^{-kt}

(c) Given that half of the substance is lost in 1600years, determine the number of years 15g of the substance would take to reduce to 13.6g

From $In\frac{M}{M_0} = -kt$

$In\frac{\frac{M_0}{2}}{M_0} = -k\ x\ 1600$

; $In\frac{1}{2} = -k\ x\ 1600$

-k = $\frac{1}{1600}In\frac{1}{2}$

Let the required time be t

$\text{In}\frac{13.6}{15} = \frac{1}{1600}In\frac{1}{2}t$

t = 226.17years

Rates of formation, growth and spreading

Example 6

(a) The rate of growth of a substance is proportional to the original amount. Find the equation of the amount present at any time t.

Solution

Let x be the amount present

$\frac{dx}{dt} \propto x$

$\frac{dx}{dt} = kx$

By separating variables

$\frac{dx}{x} = kdt$

$\int \frac{dx}{x} = \int kdt$

$Inx = kt + c$

Suppose that at time t = 0, x = x_0(initial amount)

c = In x_0

by substituting for c.

$Inx = kt + Inx_0$

$Inx - Inx_0 = kt$

$In\frac{x}{x_0} = kt$

$\frac{x}{x_0} = e^{kt}$

$x = x_0e^{kt}$

(b) The rate of growth of population in a country is proportional to the number of people living at that time. In 1980, the population was 18m and in 1990 it was 22m. Estimate

(i) The number of population in 2005

Solution

Let P be the amount present

$\frac{dP}{dt} \propto P$

$\frac{dP}{dt} = kP$

By separating variables

$\frac{dP}{P} = kdt$

$\int \frac{dP}{P} = \int kdt$

$InP = kt + c$

At t = 0 (In 1980), P = 18 => c = In18

$InP = kt + In18$

At t = 10 (1990), P = 22

$In22 = 10k + In18$

$k = \frac{1}{10} In\left(\frac{22}{18}\right)$

$InP = \frac{1}{10} In\left(\frac{22}{18}\right)\ t + In18$

In 2005, t = 25

$InP = \frac{1}{10} In\left(\frac{22}{18}\right) \text{x}25 + In18$

⇨ P = 29.73m

(ii) How long will it take the population to reach 36m.

From

$InP = \frac{1}{10} In\left(\frac{22}{18}\right)\ t + In18$

$In36 = \frac{1}{10} In\left(\frac{22}{18}\right)\ t + In18$

t = 34.54 years

It will take 34.54 years for the population to reach 36m

(c) The rate at which bush fire spreads is proportional to the unburnt area of the bush. Initially when observation was made, $\frac{1}{5}$ of the bush area had been burnt. Two hours later, $\frac{1}{3}$ of the bush area had been burnt. Find the fraction of the bush area that will remain unburnt after 5 hours.

Solution

Let the fraction of unbush area be x

$\frac{dx}{dt} = -kx$

(-ve because the unburnt area decreases with time

$\int \frac{dx}{x} = \int kdt$

$Inx = -kt + c$

At t = 0, x = $\left(1 - \frac{1}{5}\right) = \frac{4}{5}$ => c = $In\left(\frac{4}{5}\right)$

$Inx = -kt + In\left(\frac{4}{5}\right)$

At t = 2, x =$\left(1 - \frac{1}{3}\right) = \frac{2}{3}$

$In\left(\frac{2}{3}\right) = -2k + In\left(\frac{4}{5}\right)$

$k = \frac{1}{2} In\left(\frac{6}{5}\right)$

$Inx = -\frac{1}{2} In\left(\frac{6}{5}\right) t + In\left(\frac{4}{5}\right)$

At t = 5

$Inx = -In(1.2)x5 + In\left(\frac{4}{5}\right)$

$x = 0.32$

∴The fraction remaining after 5hours is 0.32

Linear motion

Example 11

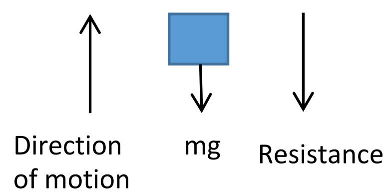
(a) A body of mass m is projected upwards with initial velocity u. Given that it experiences a resistance proportional to the velocity, v on its upward path, determine the

(i) Velocity v after time t,

(ii) Displacement, s, after time t.

(iii) Limiting speed of the body.

Solution



Resistance, R = kv

Using Newton's 2^{nd} law of motion

(i) Resultant force = m$\frac{dv}{dt}$

$0-(mg+kv)=m\frac{dv}{dt}$

$\frac{mdv}{mg+kv}=-dt$

⇨ $\int\frac{mdv}{mg+kv}=-\int dt$

$\frac{m}{k}In(mg+kv)=-t+c$

Initially (t = 0), v = v_0 => $\frac{m}{k}In(mg+ku)=c$

$\therefore\frac{m}{k}In(mg+kv)=-t+\frac{m}{k}In(mg+ku)$

or

$\frac{m}{k}In\left(\frac{mg+ku}{mg+kv}\right)=t$

$mg+kv=(mg+ku)e^{-\frac{k}{m}t}$

$v=\frac{1}{k}(mg+ku)e^{-\frac{k}{m}t}-\frac{mg}{k}$

(ii) $v=\frac{1}{k}(mg+ku)e^{-\frac{k}{m}t}-\frac{mg}{k}$

$$\int dsdt=\int\left[\frac{1}{k}(mg+ku)e^{-\frac{k}{m}t}-\frac{mg}{k}\right]dt$$

$$s=\left[-\frac{1}{k^2}(mg+ku)e^{-\frac{k}{m}t}-\frac{mgt}{k}\right]+c$$

At t = 0, s = 0

0 =$-\frac{1}{k^2}(mg+ku)+c$

c = $\frac{1}{k^2}(mg+ku)$

$$s=\left[-\frac{1}{k^2}(mg+ku)e^{-\frac{k}{m}t}-\frac{mgt}{k}\right]+\frac{1}{k^2}(mg+ku)$$

$$\therefore\ s=\left[-\frac{1}{k^2}(mg+ku)\left(1-e^{-\frac{k}{m}t}\right)-\frac{mgt}{k}\right]$$

(iii) From $v=\frac{1}{k}(mg+ku)e^{-\frac{k}{m}t}-\frac{mg}{k}$

As t →∞, $e^{-\frac{k}{m}t}\to 0$

∴limiting velocity =$-\frac{mg}{k}$

(b) The rate of change of atmospheric pressure P with respect to altitude, h in kilometres is proportional to the pressure. If the pressure at 2km is ¼ of the pressure P_0 at sea level. Find the formula for the pressure at any height.

Solution

$\frac{dP}{dh}=-kP$ (-ve because pressure decreases with altitude)

Separation of variables

$\int\frac{dP}{P}=-\int kdh$

$InP=kh+c$

At sea level (h = 0), p = P_0 => c = InP_0

$\therefore\ InP=kh+InP_0$

When h = 2km, P = $\frac{1}{4}P_0$

$In\left(\frac{1}{4}P\right)=-2k+InP_0$

$k=\frac{1}{2}In4$

$InP=\frac{h}{2}In4+InP_0$

$P=P_0e^{-hIn2}$

(c) The differential equation $\frac{dp}{dt}=\ kp(c-p)$ shows the rate at which information flows in a student population c. p represents the number who have heard the information in t days and k is a constant.

(i) Solve the differential equation.

$\frac{dp}{dt}=\ kp(c-p)$

Separating variables

$\frac{dp}{p(c-p)}$ = kdt

$\int\frac{dp}{p(c-p)}$ = $\int kdt$

$\int\frac{dp}{p(c-p)}$ = kt + a where a is a constant

By partial fractions

$\frac{1}{p(c-p)}\equiv\frac{A}{p}+\frac{B}{c-p}$

$1\equiv A(c-p)+B(p)$

$1\equiv Ac-Ap+Bp$

$1\equiv Ac+(B-A)p$

Equating constants

1 =Ac

A = $\frac{1}{c}$

Equating coefficient of p

0 = B – A

A = B = $\frac{1}{c}$

⇨ $\int\frac{dp}{p(c-p)}=\frac{1}{c}\int\frac{1}{p}dp+\frac{1}{c}\int\frac{1}{c-p}dp$

$=\frac{1}{c}Inp-\frac{1}{c}In(c-p)$

$=\frac{1}{c}In\ \frac{p}{(c-p)}$

$\therefore\frac{1}{c}In\ \frac{p}{(c-p)}=kt+a$

(ii) A school has a population of 1000 students. Initially 20 students had heard the information. A day later, 50 students had heard the information. How many students heard the information by the tenth day?

Solution

Given c =1000, at t=0, p= 20

By substitution, we have

$\frac{1}{1000} In \frac{20}{1000-20} = 0 + a$

$a = \frac{1}{1000} In \frac{20}{980} = \frac{1}{1000} In \frac{1}{49}$

⇨ $\frac{1}{1000} In \frac{p}{(1000-p)} = kt + \frac{1}{1000} In \frac{1}{49}$

After t = 1, p = 50; by substitution, we have

$\frac{1}{1000} In \frac{50}{(1000-50)} = k(1) + \frac{1}{1000} In \frac{1}{49}$

k = $\frac{1}{1000} In \frac{50}{950} - \frac{1}{1000} In \frac{1}{49}$

= $\frac{1}{1000} In \frac{1}{19} \div \frac{1}{49}$

= $\frac{1}{1000} In \frac{49}{19}$

⇨ $\frac{1}{1000} In \frac{p}{(1000-p)}$

$= \left(\frac{1}{1000} In \frac{49}{19}\right) t + \frac{1}{1000} In \frac{1}{49}$

$In \frac{p}{(1000-p)} = \left[In \left(\frac{49}{19}\right)\right] t + In \frac{1}{49}$

Note: by 10th day is the same as after 9 days

Substituting for t = 9

$In \frac{p}{(1000-p)} = \left[In \left(\frac{49}{19}\right)\right] (9) + In \frac{1}{49}$

$p = 990.3835$

Number of students who heard the information by the 10th day is 990

Exercise 5

1. The rate, cm^3s^{-1}, at which air is escaping from a balloon at time t seconds, is proportional to the volume of air, Vcm^3 in the balloon at the instant. Initial volume is $1000cm^3$.
   (a) Show that V = $1000e^{-kt}$, where k is a positive constant.
   (b) Given that V = $500cm^3$ when t = 6; show that k = $\frac{1}{6}In2$.
   (c) Calculate the value of V when t = 12s. [250 cm^3]
2. At time t minutes the rate of cooling of a liquid is proportional to the temperature, T^0C of the liquid at that time. Initially T = 80
   (a) Show that T = $80e^{-kt}$, where k is a positive constant.
   (b) Given that T = 20 when t = 6; show that k = $\frac{1}{3}Ih2$.
   (c) Calculate the time at which the temperature will reach 10^0C. [9]
3. The value of a certain product depreciates in such a way that when it is t years old, the rate of decrease in value is proportional to the value, x, of the product at that time. The product costs 12000 when new.
   (a) Show that x = $12000e^{-kt}$
   (b) Given that after 3 years the value dropped to 400;
      (i) show that k = $\frac{1}{3}In3$.
      (ii) Calculate to the nearest month, the time taken for the value to drop to 2000. [4years 11mnths]
4. A lump of a radioactive substance is decaying is proportional to the mass M in grams at time t. Initially M = 72 and decreases to 50 in 2 hours. Show that M = $72^{In\left(\frac{6}{5}\right)}$.
5. The rate at which a bacteria is reduced by a chemical is proportional to the number of bacterial present. Given that the population of the bacteria is reduced to half in six days; show that the population will be reduced to 1% of the original population in about 40days.

Topical revision exercise

1. Solve the equations
   (a) $\frac{dy}{dx} + ycotx = 3sinxcosx$
   $[y = sin^2x + c(cosx)]$
   (b) $x^2\frac{dy}{dx} + y - x^2e^{\frac{1}{x}} = 0$; given y = 2 when x = 0 $\left[y = xe^{\frac{1}{x}}\right]$
   (c) $(x^2+1)\frac{dy}{dx} - xy = x$; x = 0, y = 1.
   $\left[y = 2(x^2+1)^{\frac{1}{2}} - 1\right]$
   (d) $(x^2+1)\frac{dy}{dx} + y^2 + 1 = 0$; x = 0, y = 1.
   $\left[y = \left(\frac{1-x}{1+x}\right)\right]$

(e) $\frac{dt}{d\theta} = tcot\theta = 2cos\theta$;
t = 3 when θ = $\frac{\pi}{2}$

$\left[t = \frac{1}{2}cosec\theta(5 - cos2\theta)\right]$

(f) $y\frac{dy}{dx} = 2x - y$ (use substitution y =vx)

$\left[\left(\frac{y}{x} + 2\right)\left(\frac{y}{x} - 1\right) = k\right]$

(g) $\frac{dy}{dx} - ytanx = cos^2x$

$\left[y = secx\left(sinx - \frac{1}{3}sin^3x + k\right)\right]$

(h) $\frac{dy}{dx} = e^{2x} + x$ give R(0) = 3

$\left[y = \frac{1}{2}e^{2x} + \frac{1}{2}x^2 + \frac{5}{2}\right]$

(i) $tanx\frac{dy}{dx} - y = sin^2x$

$[y = sin^2x + csinx]$

(j) $\frac{1}{x}\frac{dy}{dx} = sinxsec^23y$

$\left[\frac{y}{2} + \frac{1}{12}sin6y = sinx - xcosx + c\right]$

(k) $\frac{dy}{dx} + 3y = e^{2x}$;z x=0, y= 1

$\left[y = \frac{1}{5}(e^{2x} + 4e^{-3x})\right]$

(l) $x\frac{dy}{dx} - y = x^3e^{x^2}$ $\left[y = \frac{x}{2}(e^{x^2} + A)\right]$

(m) $x\frac{dy}{dx} + y = e^x$ $\left[y = \frac{1}{x}e^x + \frac{c}{x}\right]$

(n) $\frac{dy}{dx} = \frac{sin^2x}{y^2}$; x = 0, y = 1

$[4y^3 = 6x - 3sin2x + 4]$

(o) $(1 - x^2)\frac{dy}{dx} - xy^2 = 0$, y = 1 when x = 0.

$\left[\frac{1}{y} = In(1 - x^2)^{\frac{1}{2}} + 1\right]$

(p) $\frac{dy}{dx} = 1 + y^2$; y = 1 when x = 0.

$\left[y = tan\left(x + \frac{\pi}{4}\right)\right]$

2. The rate of cooling of a body is proportional to the difference θ between the temperature of the body and that of the surrounding air.

(a) Write down a differential equation involving θ for this process

$\left[\frac{d\theta}{dt} = -kt\right]$

(b) If the surrounding air temperature is stable at 20^0C and the body cools from 80^0C to 70^0C in 5 minutes; find the temperature after 15minutes [54.72^0C

3. A metallic teapot losses heat due to a steady breeze at a rate proportional to its temperature θ and gains temperature from a directed beam at a rate proportional to time t. Show that at any time t, $\theta = At + B + ce^{-kt}$

4. The rate of change of atmospheric pressure, p, with respect to altitude, h, in km is proportional to pressure. If the pressure at 6000m is half the pressure P0 at sea level. Find the formula for the pressure at any height

$\left[P = P_0 2^{-\frac{h}{6}} \text{ or } P = P_0 e^{\frac{-hIn2}{6}}\right]$

5. The mass of a man together with his parachute is 70kg. when the parachute is fully open, the system experiences an upward force proportional to the velocity of the system. If the constant of proportionality is 1/10 when the system is descending at 10 ms^{-1}, determine the speed of the parachute three minutes later [7.74ms^{-1}]

6. A vessel in a shape of an inverted right circular cone contains a liquid. The rate of evaporation of the liquid is proportional to the surface exposed to the atmosphere. The radius of the base of the cone is 9cm and the height of the cone is 15cm. if it takes 1 minute for the radius of the surface of the liquid to decrease from 9cm to 4.5cm, how long will it take for the liquid to evaporate completely. [6.88minute]

7. A rumour spreads through town at a rate which is proportional to the product of the number of the people who have heard it and that of those who have not heard. Given that x is the fraction of the population who have heard the rumour after time t.

(a) Form a differential equation connecting x, t and constant k.

$\left[\frac{dy}{dx} = k(1 - x)\right]$

(b) If initially a fraction C of the population had heard the rumour, deduce that $x = \frac{C}{C+(1-C)e^{-kt}}$

(c) Given that 15% had heard the rumour at 9.00am and another 15%

by noon, find what fraction of the population would have heard the rumour by 3.00pm [0.21]

8. A research to investigate the effect of a certain chemical on a crop virus infection revealed that the rate at which the virus population is destroyed is directly proportional to the population at that time. Initially the population was P_0 at time t months later, it was found to be P.
   (a) Form a differential equation connecting P and t.
   $\left[\frac{dP}{dt} = -kP\right]$
   (b) Given that the virus population reduced to one third of the initial population in 4 months, solve the equations in (a)
   $\left[P = 3^{-\frac{t}{4}}P_0\right]$
   (c) Find
   (i) How long it will take for 5% of the original population to remain. [10.907months]
   (ii) What percentage of the original virus population will be left after 2 ½ months.[50.33%]
9. In a culture of bacteria, the rate of growth is proportional to the population present at time t. the population doubles every day. Given that the initial population P_0 is one million, determine the day when the population will be 100 million.[7^{th} day]
10. (i) The volume of a water reservoir is generated by rotating the curve y =kx^2 about the y-axis. Show that when the central depth of the water in the reservoir is h meters, the surface area, A is proportional to h and the volume is proportional to h^2.
    (ii) If the rate of loss of water from the reservoir due to evaporation is λA m^2 per day, obtain a differential equation for h per day.
    $\left[\frac{dh}{dt} = -\lambda\right]$
    (iii) Given that λ = ½, determine how long it will take for the depth of water to decrease from 20m to 2m [36days]
11. The acceleration of a particle after time t seconds is given by α = 5 + cos$\frac{1}{2}t$. If initially the particle was moving at 1 ms-1, find its velocity after 2π second and the distance it would have covered by then.
    $[v = 10\pi + 1; s = 10\pi^2 + 2\pi + 4]$
12. An athlete runs at a speed proportional to square root of the distance he still has to cover. If the athlete starts running at $10ms^{-1}$ and has a distance of 1600m to cover, find how long he will take to cover the distance. [320s]
13. A hot body at a temperature of 100^0C is placed in a room of temperature 20^0C. Ten minutes later, its temperature is 60^0C. determine the temperature of the body after another 10 minutes. [40^0C]
14. The number of car accidents x in a year on a highway was found to approximate the differential equation $\frac{dy}{dx} = kx$, where t is time in years and k is a constant. At the beginning of 2000 the number of recorded accidents is 50. If the number of accidents increased to 60 at at the beginning of 2002, estimate the number that was expected at the beginning of 2005. [79]
15. In a certain process the rate of production of yeast is kx grams per minute, where x gram is the amount produced and k= 0.003.
    (a) Show that the amount of yeast is doubled in in about 230 minutes
    (b) If in addition yeast is removed at a constant rate of m grams per minute, find the
       (i) Amount of yeast at time, t minutes, given that when t = 0; x = p grams
       $[x = m + (p - m)e^{0.003t}]$
       (ii) Value of m if p =20,000g and the supply of yeast is exhausted in 100minutes. [77166g]

16. Bacteria in a culture increase at a rate proportional to the number of bacteria present. If the number increases from 1000 to 2000 in an hour,
    (a) Find how many bacteria will be present after 1 ½ hour. [2829]
    (b) How long will it take for the number of bacteria in the culture to become 4000? [2hours]
17. It is observed that the rate at which a body cools is proportional to the amount by which its temperature exceeds that of its surroundings. A body at 78^{0}C is placed in a room at 20^{0}C and after 5 minutes the body had cooled to 65^{0}C. what will be its temperature after further 5minutes.
18. At 3.00pm, the temperature of a hot metal was 80^{0}C and that of the surrounding was 20^{0}C. At 3.03 ppm the temperature of the metal had dropped to 42^{0}C. The rate of cooling of the metal was directly proportional to the difference between its temperature θ and that of the surroundings. Find the temperature of the metal at 3.05pm. [31.27^{0}C]
19. Solve the differential equation $\frac{dy}{dx} = (xy)^{\frac{1}{2}}Inx$, given that y = 1 when x = 1.
    $\left[\sqrt{y} = \frac{1}{3}x\sqrt{x}Inx - \frac{2}{9}x\sqrt{x} + \frac{11}{9}\right]$
    Hence find the value of y when x = 4
    $[y = 9.8673]$
20. The rate at which the temperature of a body falls is proportional to the difference between the temperature of the body and that of its surrounding. The temperature of the body is initially 60^{0}C. After 15 minutes the temperature of the body is 50^{0}C. The temperature of the surrounding is 10^{0}C.
    (a) Form a differential equation for the temperature of the body. (09marks)
    $\frac{d\theta}{dt} \propto (\theta - 10)$
    $\frac{d\theta}{dt} = k(\theta - 10)$
    $\frac{d\theta}{(\theta-10)}$ = -kdt
    $\int \frac{d\theta}{(\theta-10)} = -k\int dt$
    $In\,(\theta - 10) = -kt + c$
    When t = 0, θ = 60
    c = $In\,(60 - 10)$ =In50
    when t = 15, θ = 50
    $In\,40 = -k\,x\,15 + In50$
    15k = In$\left(\frac{50}{40}\right)$
    k = $\frac{1}{15}In\left(\frac{5}{4}\right)$
    Hence the differential equation is
    $\frac{d\theta}{dt} = \frac{1}{15}In\left(\frac{5}{4}\right)(\theta - 10)$

    (b) Determine the time it takes for the temperature of the body to reach 30^{0}C. (03marks)
    In (θ – 10) = -$\frac{1}{15}In\left(\frac{5}{4}\right)t$ + In 50
    When t = T and θ = 30
    -$\frac{1}{15}In\left(\frac{5}{4}\right)T = In\,50 - In\,20$
    T = $\frac{15In\left(\frac{5}{2}\right)}{In\left(\frac{4}{4}\right)}$ = 61.5943minutes
21. Solve the differential equation $\frac{dy}{dx} + ycot\,x = x$, given that y = 1 when x = $\frac{\pi}{2}$. (08marks)
    $\frac{dy}{dx} + ycot\,x = x$
    Integrating factor,
    I.F = $e^{\int \cot x\,dx}$
    = $e^{\int \frac{\cos x}{\sin x}dx}$
    = $e^{In\sin x}$
    = sinx
    Multiplying all terms by integrating factor
    sinx$\frac{dy}{dx}$ + ycotxsinx = xsinx
    sinx$\frac{dy}{dx}$ + y$\frac{\cos x}{\sin x}$sinx = xsinx
    sinx$\frac{dy}{dx}$ + ycosx = xsinx
    $\frac{d(ysin\,x)}{dx}dx$ = xsinx
    Integrating with respect to x
    $\int \frac{d(ysin\,x)}{dx}dx = \int xsinxdx$
    Ysin x = $\int xsinxdx$

Let u = x, $\frac{du}{dx} = 1$

$\frac{dv}{dx} = sinx, v = \int sinx\ dx = -\cos x$

Using integration by parts on RHS

ysinx = -xcosx + $\int \cos x\ dx$

ysinx = -xcosx + sin x + c

But y = 1, x = $\frac{\pi}{2}$

1sin$\left(\frac{\pi}{2}\right)$ = sin$\left(\frac{\pi}{2}\right)$ -$\left(\frac{\pi}{2}\right) cos \left(\frac{\pi}{2}\right)$ +C

C = 0

By substitution,

∴ ysinx = sinx - xcosx

Thank you

Dr. Bbosa Science