Logarithms

A logarithm is an exponent, an index or power

The logarithm of a positive quantity p to a given base q is defined as the index or power to which the bases q must be raised to make it equal to P. i.e. $\log_q p = x \ means\ that\ q^x = p$ or x s the logarithm of p to base q

- x is the power (index, logarithm or exponent)
- q is the base
- p is the number (which must be positive)

Example 1

Find the values of x in the following

(a) $\log_2 8 = x$

Solution

$8 = 2^3$

$\therefore \log_2 8 = 3$; x = 3

(b) $\log_x 25 = 2$

Solution

$25 = 5^2$

⇨ $x^2 = 5^2$

$\therefore x = 5$

Example 2

Evaluate

(a) $\log_{27} 9\sqrt{3}$

Solution

Let $\log_{27} 9\sqrt{3} = x$

$27^x = 9\sqrt{3}$

$3^{3x} = 3^2.3^{\frac{1}{2}} = 3^{\frac{5}{2}}$

Equating powers

$3x = \frac{5}{2}$

$x = \frac{5}{6}$

$\therefore \log_{27} 9\sqrt{3} = \frac{5}{6}$

(b) $\log_{\frac{1}{2}} \frac{1}{4}$

Solution

Let $\log_{\frac{1}{2}} \frac{1}{4} = x$

$\left(\frac{1}{2}\right)^x = \frac{1}{4}$

$= \left(\frac{1}{2}\right)^2$

Equating powers x = 2

$\therefore \log_{\frac{1}{2}} \frac{1}{4} = 2$

Example 3

Given that $\log_3 x = 2\log_3 4 - \log_3 5 + \log_3 9$, find the value of x. (05marks)

$\log_3 x = \log_3 \frac{4^2 x9}{5}$

$= \log_3 \frac{16x9}{5}$

$= \log_3 28.8$

Comparing both sides

x = 28.8

Example 4

Given that p = $\log_a (a^3 y^{-2})$ and q = $\log_a ay^2$, find the value of p + q. (05 marks)

Solution

P + Q = $\log_a (a^3 y^{-2}) + \log_a ay^2$

$=\log_a(a^3y^{-2}x\,ay^2)$

$=\log_a(a^4)$

$= 4\log_a a$

= 4 x 1 = 4

Example 5

Evaluate $\frac{\log_6 216+\log_2 64}{\log_3 243-\log_{10} 0.1}$

Solution

$\frac{\text{Log}_6 216+\log_2 64}{\log_3 243-\log_{10} 0.1} = \frac{\log_6 6^3+\log_2 2^6}{\log_3 3^5-\log_{10} 10^{-1}}$

$= \frac{3\log_6 6+6\log_2 2}{5\log_3 3- -1\log_{10} 10}$

$= \frac{3x1+6x1}{5x1+1x1}$

$= \frac{9}{6}$

$= 1.5$

Rules of logarithms

(a) (i) $\log_a a$ = 1

Proof

Let $\log_a a = x$

$a^x = a^1$

x = 1

$\therefore \log_a a = 1$

(ii) $\log_a 1$ = 0

Proof

Let $\log_a 1 = x$

$a^x = a^0$

x = 0

$\therefore \log_a 1 = 0$

(b) The power rule

$\log_a P^q = \text{q}\log_a P$

Proof

Let $\log_a P = x$

$a^x = P$

Raising each to the power q

$a^{qx} = P^q$

⇨ $\log_a P = \log_a a^{qx} = qx$

$\therefore \log_a P^q = \text{q}\log_a P$

(c) The addition/multiplication rule

$\log_a pq = \log_a p + \log_a q$

Proof

Let $\log_a p = x \; and \; \log_a q = y$

$p = a^x \; and \; q = a^y$

$pq = a^x . a^y = a^{x+y}$

$\log_a pq = \log_a a^{x+y} = x + y$

$\therefore \log_a pq = \log_a p + \log_a q$

(d) The subtraction/division rule

$\log_a \left(\frac{p}{q}\right) = \log_a p - \log_a q$

Proof

Let $\log_a p = x \; and \; \log_a q = y$

$p = a^x \; and \; q = a^y$

$\frac{p}{q} = \frac{a^x}{a^y} = a^{x-y}$

$\log_a \left(\frac{p}{q}\right) = \log_a a^{x-y} = x - y$

$\therefore \log_a \left(\frac{p}{q}\right) = \log_a p - \log_a q$

(e) Change of base

$\log_a p = \frac{\log_q p}{\log_q a}$

Let $\log_a p = x$, then $a^x = p$

⇨ $\log_q a^x = \log_q p$

x$\log_q a = \log_q p$

$x = \frac{\log_q p}{\log_q a}$

$\therefore \log_a p = \frac{\log_q p}{\log_q a}$

Example 6

Evaluate

(a) $\log_2 8\sqrt{2}$

Solution

Either: let $\log_2 8\sqrt{2}$= x

⇨ $2^x = 8\sqrt{2}$

$= 2^3 . 2^{\frac{1}{2}}$

$= 2^{\frac{7}{2}}$

$x = \frac{7}{2}$

$\therefore \log_2 8\sqrt{2} = \frac{7}{2}$

Or $\log_2 8\sqrt{2} = \log_2 \left(2^3 . 2^{\frac{1}{2}}\right)$

$= \log_2 2^{\frac{7}{2}}$

$= \frac{7}{2}\log_2 2$

$= \frac{7}{2}$

(b) $\log_a \frac{1}{a}$

Solution

Let $\log_a \frac{1}{a}$ = x
$a^x = a^{-1}$
$x = -1$
$\therefore \log_a \frac{1}{a}$ = -1

Example 7

Express each of the following as a single logarithm

(a) Log 4 + log3

Solution

Log 4 + log3 = log (4 x 3)

= log 12

(b) Log 5 + log 18 – log3

Solution

Log 5 + log 18 – log3 = log$\left(\frac{5\,x18}{3}\right)$
= log30

Example 8

Show that $\log_a p = \frac{1}{\log_p a}$. Hence solve the equation $\log_5 x + 2\log_x 5 = 3$

Solution

Let $\log_a p = x$

⇨ $a^x = p$

Introducing log to base p on both sides

$\log_p a^x = \log_p p$

x$\log_p a = 1$

$x = \frac{1}{\log_p a}$

$\therefore \log_a p = \frac{1}{\log_p a}$

Then,

$\log_5 x + 2\log_x 5 = 3$

$\log_5 x + \frac{2}{\log_5 x} = 3$

Let y = $\log_5 x$

⇨ y + $\frac{2}{y} = 3$
$y^2 - 3y + 2 = 0$
(y – 1)(y – 2) = 0

Either y = 1 or y = 2

When y = 1: $\log_5 x = 1$; x = $5^1 = 5$

When y = 2: $\log_5 x = 2$; x = $5^2 = 25$

x = 5 and x = 25

Example 9

Solve $\log_x 5 + 4\log_5 x = 4$

Expressing terms on LHS to $\log_5$.

$\frac{\log_5 5}{\log_5 x} + 4\log_5 x = 4$

$\frac{1}{\log_5 x} + 4\log_5 x = 4$

Let $\log_5 x = y$

$\frac{1}{y} + 4y = 4$

$4y^2 - 4y + 1 = 0$

(2y – 1)(2y – 1) = 0

2y = 1
y = $\frac{1}{2}$

⇨ $\log_5 x = \frac{1}{2}$

x = $5^{\frac{1}{2}} = \sqrt{5}$

Example 10

Show that 2log4 + $\frac{1}{2}\log 25 - \log 20 = 2\log 2$.

Solution

Handling the left hand side

2log4 + $\frac{1}{2}\log 25 - \log 20$

= 2log2^2 + $\frac{1}{2}\log 5^2 - (\log 4 + \log 5)$

= 2log2^2 + $\frac{1}{2}\log 5^2 - \log 4 - \log 5$

= 4log2 + $\log 5 - 2\log 2 - \log 5$

= 2log 2

Example 11

(a) (i) Find $\log_9 27\sqrt{3}$ without using tables

Solution

Let $\log_9 27\sqrt{3}$ = x

9^x =$27\sqrt{3}$

$(3^2)^x = 3^3.3^{\frac{1}{2}}$

$3^{2x} = 3^{\frac{7}{2}}$

Equating indices

2x = $\frac{7}{2}$

x =1.75

Or

Changing the base from 9 to 3

$\log_9 27\sqrt{3} = \frac{\log_3 27\sqrt{3}}{\log_3 9}$

$= \frac{\log_3 27+\log_3 \sqrt{3}}{\log_3 9}$

$= \frac{\log_3 3^3+\log_3 3^{\frac{1}{2}}}{\log_3 3^2}$

$= \frac{3+\frac{1}{2}}{2} = \frac{7}{4}$

$= 1.75$

(ii) Simplify $(\log_a b^2)(\log_b a^3)$

$$(\log_a b^2)(\log_b a^3) = (\log_a b^2)\frac{(\log_a a^3)}{\log_a b}$$

$$= (2\log_a b)\frac{(3\log_a a)}{\log_a b}$$

= 2 x 3 = 6

Or

$$(\log_a b^2)(\log_b a^3) = (2\log_a b)(3\log_b a)$$

$$= \left(\frac{2\log_{ba} b}{\log_b a}\right)(3\log_b a)$$

= 2 x 3 = 6

(b) Express $\log_{25} xy$ in terms of $\log_5 x$ *and* $\log_5 y$. Hence solve the simultaneous equations:

$\log_{25} xy = 4\frac{1}{2}$

$\frac{\log_5 x}{\log_5 y}$ = -10

Solution

(ii) By change of base rule

$$\log_{25} xy = \frac{\log_5 xy}{\log_5 25} = \frac{\log_5 x + \log_5 y}{\log_5 5^2}$$

$$= \frac{\log_5 x+\log_5 y}{2}$$

$$\therefore \log_{25} xy = \frac{\log_5 x + \log_5 y}{2}$$

Hence solving

$\log_{25} xy = 4\frac{1}{2}$

$\frac{\log_5 x+\log_5 y}{2} = \frac{9}{2}$

$\log_5 x + \log_5 y = 9$ ………………. (i)

$\frac{\log_5 x}{\log_5 y}$ = -10

$\log_5 x = -10\log_5 y$ ……………(ii)

Substituting eqn. (ii) into eqn. (i)

$-10\log_5 y + \log_5 y = 9$

$\log_5 y = -1$

y = $5^{-1} = \frac{1}{5}$

Substituting $\log_5 y$ into equation (ii)

$\log_5 x = 10$

$x = 5^{10}$

$\therefore\ x = 5^{10}$ and y = $\frac{1}{5}$

Example 12

(a) Given that $\log_b a = x$ show that b =$a^{\frac{1}{x}}$ and deduce $\log_b a = \frac{1}{\log_a b}$

Solution

$\log_b a = x$

$b^x = a$

$(b^x)^{\frac{1}{x}} = a^{\frac{1}{x}}$

$b = a^{\frac{1}{x}}$

Taking log to base a on both sides

$\log_a b = \log_a a^{\frac{1}{x}}$

$\log_a b = \frac{1}{x}\log_a a = \frac{1}{x}$

But x = $\log_b a$

$\therefore \log_a b = \frac{1}{\log_b a}$

(b) Find the value of x and y such that

(i) $\log_{10} x + \log_{10} y = 1.0$

$\log_{10} x - \log_{10} y = \log_{10} 2.5$

Solution

$\log_{10} x + \log_{10} y = 1.0$ ……………(i)

$\log_{10} x - \log_{10} y = \log_{10} 2.5$ …(ii)

Eqn. (i) + eqn. (ii)

2$\log_{10} x$ = $\log_{10} 10 + \log_{10} 2.5$

$\log_{10} x^2 = \log_{10} 25$

⇨ $x^2 = 25$

x = 5

Substituting x into eqn. (i)

$\log_{10} 5 + \log_{10} y = 1.0$

$\log_{10} y = \log_{10} 10 - \log_{10} 5$

$\log_{10} y = \log_{10} 10 \div 5 = \log_{10} 2$

y = 2

Hence x = 5 and y = 2

(ii) Simplify $2^x.3^y = 432$

Solution

$2^x.3^y = 432 = 2^4.3^3$

Comparing exponents

x = 4 and y = 3

(c) Simplify $\frac{1+\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt{3}}$

Solution

By rationalizing

$$\frac{1+\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt{3}} = \frac{(1+\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})}$$

$$= \frac{\sqrt{2}-\sqrt{3}+2-\sqrt{6}+\sqrt{6}-3}{2-3}$$

$$= \frac{\sqrt{2}-\sqrt{3}-1}{-1}$$

$$= 1+\sqrt{3}-\sqrt{2}$$

Example 13

Prove that $\log_6 x = \frac{\log_3 x}{1+\log_3 2}$. Given that $\log_3 2 = 0.631$, find without using tables or calculator $\log_6 4$ correct to 3significant figures

Solution

$$\log_6 x = \frac{\log_3 x}{\log_3 6} = \frac{\log_3 x}{\log_3 (2\,x 3)} = \frac{\log_3 x}{\log_3 3 + \log_3 2}$$

$$= \frac{\log_3 x}{1+\log_3 2}$$

Substituting for $\log_3 2 = 0.631$

$$\log_6 x = \frac{\log_3 2^2}{1+\log_3 2} = \frac{2\log_3 2}{1+\log_3 2} = \frac{2\,x\,0.631}{1+0.631}$$

$$= 0.774$$

Revision exercise

1. Evaluate
   (a) $\log_{\frac{1}{5}} 25\sqrt{5}$ $\left[-\frac{5}{2}\right]$
   (b) $\log_3 27$ [3]
2. Express the following as a single logarithm
   (i) Log15 - $\frac{1}{2}log9$ [log 5]
   (ii) 3log2 + 2log5 – log 20 [log 10]
3. Given that $\log_b a$ and $\log_c b = a$, show that $\log_c a = ac$
4. (a) solve the equation
   (i) $\log_a 4 + \log_4 a^2$
   $[a = 2 \; and \; a = 4]$
   (ii) $\log_{14} x = \log_7 4x$ $\left[\frac{1}{196}\right]$
5. Without using tables or calculator show that $\frac{2log9 + log8 - \log 375}{\frac{1}{3}log6 - log5^{\frac{1}{3}}}$= 9

6. If$\log_2 x + \log_4 x + \log_{16} x = \frac{21}{16}$. Find the value of x [x = 1.6818]
7. Given$\log_a b = \log_d c$, show that $\log_c a = \log_d b$. Hence or otherwise solve the equation$\log_{9x} 64 = \log_x 4$. [x=3]
8. Solve the simultaneous equations
   $\log_{10}(y - x)$= 0
   $2\log_{10}(21 + x)$ [(x,y) =(-5, -4) or (4,5)
9. Given that $\log_2 x + 2\log_4 y = 4$. Show that xy = 16. Solve simultaneous equations
   $10\log_{10}(x + y) = 1$
   $\log_2 x + 2\log_4 y = 4$. [(x, y) =(2,8)or (8,2)
10. (a) If $\log_b a = x$, show that b = $a^{\frac{1}{x}}$ and deduce that $\frac{1}{\log_a b}$.

(b) Solve

(i) $\log_x 4 + \log_4 x^2 = 3$ $[x = 2\ or\ 4]$

(ii) $2^{2x-1} + \frac{3}{2} = 2^{x+1}$[x=0 or 1.585]

11. Prove that $\log_8 x = \frac{2}{3}\log_4 x$. Hence without using tables or calculator, evaluate $\log_8 6$ correct to three significant figure, if $\log_4 3 = 0.7925$ [0.862]
    (a) $\sqrt{x+2} - x = 0$[x=2]
    (b) $\sqrt{1+x} = 1 + \sqrt{1-x}$ $\left[x = \frac{\sqrt{3}}{2}\right]$
    (c) $(3 - x)^{\frac{1}{2}} = (1 + x)^{\frac{1}{2}} + (2 - x)^{\frac{1}{2}}$ [x = -0.92665]
    (d) $\sqrt{x+6}$ =$\sqrt{1-3x} - \sqrt{4-x}$ [-5]

1. Without using mathematical tables or calculators, find the value of
   $\frac{(\sqrt{5}+2)^2 - (\sqrt{5}-2)^2}{8\sqrt{5}}$ [1]
2. Find the square root of the following
   (a) 6 + 2$\sqrt{5}$ $[\pm(1 + \sqrt{5})]$
   (b) 18 - 2$\sqrt{12}$$[\pm(\sqrt{0.695} - \sqrt{17.303})]$
   (c) 18 - 2$\sqrt{2}$$[\pm(\sqrt{0.1118} - \sqrt{17.8882})]$

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Thanks

Dr. Bbosa Science