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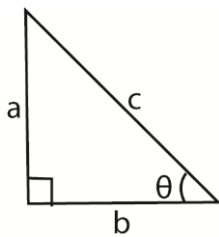
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Trigonometry

The word 'trigonometry' suggests 'tri'-three, 'gono'-angle, 'metry'-measurement. As such, trigonometry is basically about triangles, most especially right-angled triangles.

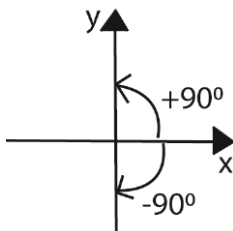
Important to note

(a) For a right angled triangle below

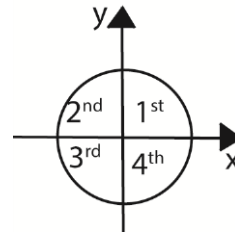


- $\sin\theta = \frac{a}{c}$ $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{c}{a}$
- $\cos\theta = \frac{b}{c}$ $\sec\theta = \frac{1}{\cos\theta} = \frac{c}{b}$
- $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{a}{b}$ $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{b}{a}$

(b) All positive angles are measured anticlockwise from positive x-axis

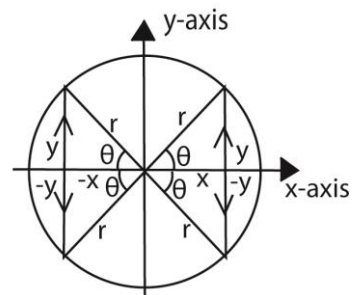


(c) A circle drawn with the centre O, divides the co-ordinate axis into four equal parts called quadrants



The quadrants are also labelled anti-clockwise from the positive x – axis.

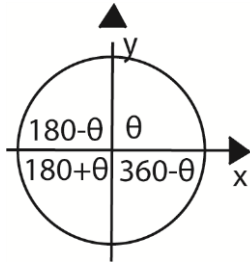
The signs the trigonometric ratios in the quadrants are given below



| Ratio | Quadrant | | | |
|--------|-----------------|-----------------|-----------------|-----------------|
| | 1 st | 2 nd | 3 rd | 4 th |
| cosθ | + | - | - | + |
| sinθ | + | + | - | - |
| tanθ | + | - | + | - |
| secθ | + | - | - | + |
| cosecθ | + | + | - | - |
| cotθ | + | - | + | - |

Note

- If θ is the angle in the 1st quadrat
- In the 2nd quadrat the angle is $(180 - \theta)$
- In the 3rd quadrat the angle is $(180 + \theta)$
- In the 4th quadrat the angle is $(360 - \theta)$



Solving equations

We make use of the quadrants to find the ranges of values within which the angle follows

Example 1

Solve the following equations for $0^\circ \leq \theta \leq 360^\circ$

(i) $3\cos\theta + 2 = 0$

Solution

$$\cos\theta = -\frac{2}{3}$$

But \cos is negative in the 2nd and 3rd quadrants.

Ignoring the negative sign, the angle obtained is referred to as the key or principle angle, i.e.

$$\text{key angle} = \cos^{-1}\frac{2}{3} = 48.2^\circ \text{ (1d.p)}$$

In the 2nd quadrant, $\theta = 180 - 48.2 = 131.8^\circ$

In the 3rd quadrant, $\theta = 180 + 48.2 = 228.2^\circ$

$$\therefore \{\theta: \theta = 131.8^\circ, 228.2^\circ\}$$

Note that: the key angle is not part of the solution but only a guide.

(ii) $4\cos^2\theta - 1 = 0$

Solution

$$\cos\theta = \sqrt{\frac{1}{4}} = \pm\frac{1}{2}$$

$$\text{Key angle, } \theta = \cos^{-1}\frac{1}{2} = 60^\circ$$

When $\cos\theta = \frac{1}{2}$ (positive is 1st and 4th quadrants)

1st quadrant $\theta = 60^\circ$

4th quadrant $\theta = 360 - 60 = 300^\circ$

When $\cos\theta = -\frac{1}{2}$ (positive is 2nd and 3rd quadrants)

3rd quadrant $\theta = 180 - 60 = 120^\circ$

4th quadrant $\theta = 180 + 60 = 240^\circ$

$$\therefore \{\theta: \theta = 60^\circ, 120^\circ, 240^\circ, 300^\circ\}$$

(iii) $\operatorname{cosec}\theta + 2 = 0$

Solution

$$\operatorname{cosec}\theta = -2 \Rightarrow \sin\theta = -\frac{1}{2} \text{ (taking reciprocal)}$$

$$\text{Key angle} = \sin^{-1}\frac{1}{2} = 30^\circ$$

In the 3rd quadrant $\theta = 180 + 30 = 210^\circ$

In the 4th quadrant $\theta = 360 - 30 = 330^\circ$

$$\therefore \{\theta: \theta = 210^\circ, 330^\circ\}$$

(iv) $3\sec^2\theta - 4 = 0$

Solution

$$\sec\theta = \pm\frac{2}{\sqrt{3}} \Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2}$$

$$\text{Key angle} = \cos^{-1}\frac{\sqrt{3}}{2} = 30^\circ$$

For $\cos\theta = \frac{\sqrt{3}}{2}$; $\theta = 30^\circ, 330^\circ$

For $\cos\theta = -\frac{\sqrt{3}}{2}$; $\theta = 120^\circ, 210^\circ$

$$\therefore \{\theta: \theta = 30^\circ, 120^\circ, 210^\circ, 330^\circ\}$$

(d) Definitions of angle

(i) **Acute angle** is an angle between 0° and 90° . It lies in the 1st quadrant

(ii) **Right angle** is an angle = 90°

(iii) **Obtuse angle** is an angle between 90° and 180° . It lies in the 2nd quadrant

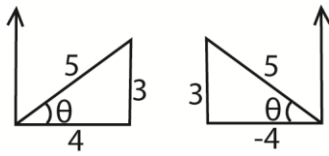
(iv) **Reflex angle** is an angle between 180° and 360° . It lies in the 3rd and 4th quadrant

Example 2

(a) If $\sin\theta = \frac{3}{5}$ and $0^\circ \leq \theta \leq 360^\circ$. Find the possible values of $3\tan\theta - \cot\theta$

Solution

If $\sin\theta = \frac{3}{5}$; θ lies in 1st or 2nd quadrants



In 1st quadrant

$$3\tan\theta - \cot\theta = 3\left(\frac{3}{4}\right) - \left(\frac{4}{3}\right) = \frac{11}{12}$$

In 2nd quadrant

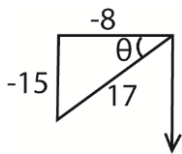
$$3\tan\theta - \cot\theta = 3\left(-\frac{3}{4}\right) - \left(-\frac{4}{3}\right) = -\frac{11}{12}$$

\therefore the possible values are $\pm \frac{11}{12}$

(b) If $\cos\theta = -\frac{8}{17}$ and θ is reflex, find the value of $4\sec^2\theta + \tan\theta$

Solution

If $\cos\theta = -\frac{8}{17}$ and θ is reflex, θ lies in the 3rd quadrant



$$4\sec^2\theta + \tan\theta = 4\left(-\frac{17}{8}\right)^2 + \frac{15}{8} = \frac{319}{16}$$

Example 3

Solve for θ , where $\theta^0 \leq \theta \leq 360^0$

(i) $3\tan^2 3\theta = 1$

Solution

$$\tan 3\theta = \pm \frac{1}{\sqrt{3}}$$

taking $\tan 3\theta = \frac{1}{\sqrt{3}}$

$$\Rightarrow 3\theta = 30^0, 210^0, 390^0, 570^0, 750^0, 930^0$$

$$\theta = 10^0, 70^0, 130^0, 190^0, 250^0, 310^0$$

taking $\tan 3\theta = -\frac{1}{\sqrt{3}}$

$$\Rightarrow 3\theta = 150^0, 330^0, 510^0, 690^0, 870^0, 1050^0$$

$$\theta = 50^0, 110^0, 170^0, 230^0, 290^0, 350^0$$

$$\therefore \{\theta: \theta = 10^0, 50^0, 70^0, 110^0, 130^0, 170^0, 190^0, 230^0, 250^0, 290^0, 310^0, 350^0\}$$

Note

- If $\theta^0 \leq \theta \leq 360^0$ then $\theta^0 \leq 3\theta \leq 1080^0$
[multiply the interval through by 3]

(ii) $2\cos 2\theta + \sqrt{3} = 0$

Solution

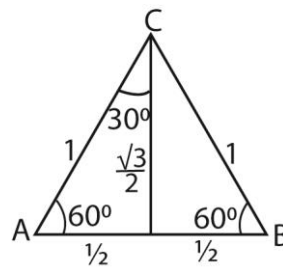
$$\cos 2\theta = -\frac{\sqrt{3}}{2} \text{ and } \theta^0 \leq 2\theta \leq 720^0$$

$$2\theta = 150^0, 210^0, 510^0, 570^0$$

$$\therefore \{\theta: \theta = 75^0, 105^0, 255^0, 285^0\}$$

Set square angles: 30^0 , 45^0 , and 60^0

(i) From equilateral triangle ABC with side equal to 1 unit



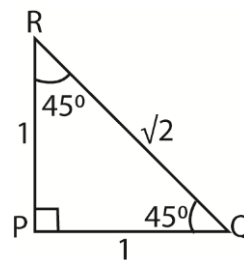
$$\cos 60^0 = \sin 30^0 = \frac{1}{2}$$

$$\cos 30^0 = \sin 60^0 = \frac{\sqrt{3}}{2}$$

$$\tan 30^0 = \cot 60^0 = \frac{1}{\sqrt{3}}$$

$$\tan 60^0 = \cot 30^0 = \sqrt{3}$$

(ii) From the right angled triangle PQR below

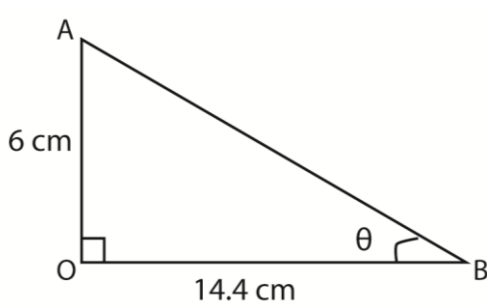


$$\cos 45^0 = \sin 45^0 = \frac{1}{\sqrt{2}}$$

$$\tan 45^0 = 1$$

Example 4

Triangle OAB is such that Angle AOB = 90°, angle ABO = θ, $\overline{OB} = 14.4\text{cm}$ and $\overline{OA} = 6\text{cm}$.



Find $\sin\theta + \cot\theta$

Solution

$$\tan \theta = \frac{6}{14.4} \Rightarrow \theta = \tan^{-1} \frac{6}{14.4} = 22.62^\circ$$

$$\begin{aligned} \sin\theta + \cot\theta &= \sin 22.62 + \cot 22.62 \\ &= 0.3846 + 2.4000 \\ &= 3.7846 \end{aligned}$$

Example 5

Without using tables or calculators find the value of

(i) $\cos 240^\circ$

Solution

$$\cos 240^\circ = -\cos(240 - 180)^\circ = -\cos 60^\circ = -\frac{1}{2}$$

(ii) $\tan 3990^\circ$

Solution

$$\tan 3990^\circ = \tan [(360 \times 11) + 30]^\circ = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

(iii) $\sin 570^\circ$

Solution

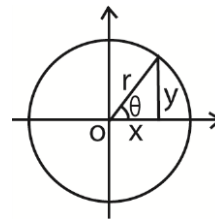
$$\sin 570^\circ = \sin \{(360 \times 1) + 210\}^\circ = -\sin 30 = -\frac{1}{2}$$

(iv) $\sec 225^\circ$

Solution

$$\sec 225^\circ = \sec (225 - 180)^\circ = \sec 45^\circ = -\sqrt{2}$$

The Pythagoras theorem



For any acute angle θ

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

By Pythagoras theorem

$$x^2 + y^2 = r^2$$

Substituting for x and y

$$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$$

$$r^2(\cos^2\theta + \sin^2\theta) = r^2$$

$$\therefore \cos^2\theta + \sin^2\theta = 1$$

$$\text{Now } \tan\theta = \frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta}$$

$$\therefore \frac{\sin\theta}{\cos\theta} = \tan\theta$$

Identities

$$\cos^2\theta + \sin^2\theta = 1 \dots\dots\dots\text{(i)}$$

Identity (i) $\div \cos^2\theta$

$$1 + \tan^2\theta = \sec^2\theta \dots\dots\dots\text{(ii)}$$

Identity (i) $\div \sin^2\theta$

$$1 + \cot^2\theta = \text{cosec}^2\theta \dots\dots\dots\text{(iii)}$$

Example 6

Show that

(a) $\sec^2\theta + \text{cosec}^2\theta = \sec^2\theta\text{cosec}^2\theta$

Solution

$$\begin{aligned} \sec^2\theta + \text{cosec}^2\theta &= \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} \\ &= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta\sin^2\theta} \\ &= \frac{1}{\cos^2\theta\sin^2\theta} \\ &= \sec^2\theta\text{cosec}^2\theta \end{aligned}$$

$$(b) \frac{1-\cos^2\theta}{\sec^2-1} = \cos^2\theta$$

Solution

$$\begin{aligned} \frac{1-\cos^2\theta}{\sec^2-1} &= \frac{\sin^2\theta}{\tan^2\theta} \\ &= \sin^2\theta \times \frac{\cos^2\theta}{\sin^2\theta} \\ &= \cos^2\theta \text{ as required} \end{aligned}$$

$$(i) \sin^2\theta + (1 + \cos\theta)^2 = 2(1 + \cos\theta)$$

Solution

$$\begin{aligned} \sin^2\theta + (1 + \cos\theta)^2 &= \sin^2\theta + 1 + 2\cos\theta + \cos^2\theta \\ &= \sin^2\theta + \cos^2\theta + 1 + 2\cos\theta \\ &= 1 + 1 + 2\cos\theta \text{ (Recall that } \sin^2\theta + \cos^2\theta = 1) \\ &= 2 + 2\cos\theta = 2(1 + \cos\theta) \\ \therefore \sin^2\theta + (1 + \cos\theta)^2 &= 2(1 + \cos\theta) \end{aligned}$$

$$(ii) \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\csc\theta} = \tan\theta$$

Solution

$$\begin{aligned} \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\csc\theta} &= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\frac{1}{\cos\theta}}{1+\frac{1}{\sin\theta}} \\ &= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\frac{\cos\theta+1}{\cos\theta}}{\frac{\sin\theta+1}{\sin\theta}} \\ &= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\cos\theta+1}{\cos\theta} \div \frac{\sin\theta+1}{\sin\theta} \\ &= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\cos\theta+1}{\cos\theta} \times \frac{\sin\theta}{\sin\theta+1} \\ &= \frac{\sin\theta}{\cos\theta} = \tan\theta \end{aligned}$$

$$\therefore \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\csc\theta} = \tan\theta$$

$$(iii) (\tan\theta + \sec\theta)^2 = \frac{1+\sin\theta}{1-\sin\theta}$$

Solution

$$(\tan\theta + \sec\theta)^2 = \left(\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)^2 = \left(\frac{\sin\theta+1}{\cos\theta}\right)^2$$

$$= \frac{(1+\sin\theta)^2}{\cos^2\theta} = \frac{(1+\sin\theta)^2}{1-\sin^2\theta}$$

$$= \frac{(1+\sin\theta)(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} = \frac{1+\sin\theta}{1-\sin\theta}$$

$$\therefore (\tan\theta + \sec\theta)^2 = \frac{1+\sin\theta}{1-\sin\theta}$$

Example 7

Solve the following equations for $-180^\circ \leq x \leq 360^\circ$

$$(a) \sec^2\theta - \tan\theta = 1$$

Solution

$$\sec^2\theta - \tan\theta = 1$$

$$1 + \tan^2\theta - \tan\theta = 1$$

$$\tan^2\theta - \tan\theta = 0$$

$$\tan\theta(\tan\theta-1) = 0$$

$$\text{Either } \tan\theta = 0; \theta = \tan^{-1} 0 = 0^\circ$$

$$\text{Or } \tan\theta = 1, \theta = \tan^{-1} 1 = 45^\circ, 225^\circ$$

$$\text{Hence } \theta = (0^\circ, 45^\circ, 225^\circ)$$

Note that in the 1st and 3rd quadrants $\tan\theta$ is positive.

$$(b) 3\sin^2\theta + \cos\theta + 1 = 0$$

Solution

$$3(1 - \cos^2\theta) + \cos\theta + 1 = 0$$

$$3\cos^2\theta - \cos\theta - 4 = 0$$

$$(\cos\theta + 1)(3\cos\theta - 4) = 0$$

$$\text{Either } \theta = \cos^{-1} -1 = 180^\circ$$

$$\text{Or } \theta = \cos^{-1} \frac{4}{3} = \text{no value}$$

$$\text{Hence } = 180^\circ$$

$$(c) 2\cos^2x = \sin x + 1$$

Solution

$$2(1 - \sin^2x) = \sin x + 1$$

$$2\sin^2x - \sin x - 1 = 0$$

$$(\sin x + 1)(2\sin x - 1) = 0$$

$$\text{Either } \sin x + 1 = 0$$

$$\sin x = -1; x = 270^\circ$$

Or

$$2\sin x - 1 = 0$$

$$\sin x = \frac{1}{2}; x = 30^\circ, 150^\circ$$

$$x = (30^\circ, 150^\circ, 270^\circ)$$

Note that in the 1st and 2nd quadrants $\sin\theta$ is positive while in the 3rd and 4th quadrants $\sin\theta$ is negative

(d) $2\cos^2\theta + \sin\theta - 1 = 0$

Solution

$$2(1 - \sin^2\theta) + \sin\theta - 1 = 0$$

$$2\sin 2\theta - \sin\theta - 1 = 0$$

$$(\sin\theta - 1)(2\sin\theta + 1) = 0$$

$$\text{Either } \sin\theta = 1 \text{ or } \sin\theta = -\frac{1}{2}$$

$$\text{When } \sin\theta = 1; \theta = 90^\circ$$

$$\text{When } \sin\theta = -\frac{1}{2}; \theta = -150^\circ, -30^\circ, 210^\circ, 330^\circ$$

$$[\theta: \theta = -150^\circ, -30^\circ, 90^\circ \text{ for given range}]$$

(e) $\cos\theta + \sqrt{3}\sin\theta = 1$

Solution

1st approach

$$\sqrt{3}\sin\theta = 1 - \cos\theta$$

Squaring both sides

$$3\sin^2\theta = 1 - 2\cos\theta + \cos^2\theta$$

$$3(1 - \cos^2\theta) = 1 - 2\cos\theta + \cos^2\theta$$

$$4\cos^2\theta - 2\cos\theta - 1 = 0$$

$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$

$$\cos\theta = -\frac{1}{2} \quad \left| \quad \cos\theta = 1$$

$$\theta = \pm 120^\circ \quad \left| \quad \theta = 0^\circ$$

$$\therefore [\theta: \theta = 0^\circ, \pm 120^\circ]$$

2nd approach

$$\sqrt{3}\sin\theta = 1 - \cos\theta$$

Dividing through by $\cos\theta$

$$\sqrt{3}\tan\theta = \sec\theta - 1$$

Squaring both sides

$$3\tan^2\theta = \sec^2\theta - 2\sec\theta + 1$$

$$3\tan^2\theta = \sec^2\theta - 2\sec\theta + 1$$

$$3[\sec^2\theta - 1] = \sec^2\theta - 2\sec\theta + 1$$

$$2\sec^2\theta + 2\sec\theta - 4 = 0$$

$$\sec^2\theta + \sec\theta - 2 = 0$$

$$(\sec\theta + 2)(\sec\theta - 1) = 0$$

$$\sec\theta = -2 \text{ or } \sec\theta = 1$$

$$\cos\theta = -\frac{1}{2} \text{ or } \cos\theta = 1$$

$$\therefore [\theta: \theta = 0^\circ, \pm 120^\circ]$$

3rd approach

$$\sqrt{3}\sin\theta = 1 - \cos\theta$$

Dividing through by $\sin\theta$

$$\sqrt{3} = \operatorname{cosec}\theta - \cot\theta$$

Rearranging

$$\sqrt{3} + \cot\theta = \operatorname{cosec}\theta$$

Squaring both sides

$$3 + 2\sqrt{3}\cot\theta + \cot^2\theta = \operatorname{cosec}^2\theta$$

$$3 + 2\sqrt{3}\cot\theta + \cot^2\theta = 1 + \cot^2\theta$$

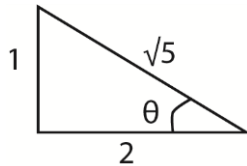
$$\cot\theta = -\frac{1}{\sqrt{3}}; \Rightarrow \tan\theta = -\sqrt{3}$$

$$\therefore [\theta: \theta = -60^\circ, 120^\circ]$$

Example 8

- (a) Given that $\tan\theta = \frac{1}{2}$
Evaluate $\operatorname{cosec}^2\theta - \sec^2\theta$ without using mathematical tables or calculator.

Solution



$$\operatorname{cosec}\theta = \frac{\sqrt{5}}{1} = \sqrt{5}; \operatorname{cosec}^2\theta = 5$$

$$\sec\theta = \frac{\sqrt{5}}{2}; \sec^2\theta = \frac{5}{4}$$

$$\Rightarrow \operatorname{cosec}^2\theta - \sec^2\theta = 5 - \frac{5}{4} = \frac{15}{4} = 3\frac{3}{4}$$

(b) Prove that $\frac{1+\cos 2x}{2\sin 2x} = \frac{1}{2}\cot x$

Solution

$$\begin{aligned} \frac{1+\cos 2x}{2\sin 2x} &= \frac{1+\cos^2 x - \sin^2 x}{4\cos x \sin x} \\ &= \frac{2\cos^2 x}{4\cos x \sin x} \\ &= \frac{\cos x}{2\sin x} \end{aligned}$$

$$= \frac{1}{2}\cot x \text{ as required}$$

Hence solve the equation $\frac{1+\cos 2x}{2\sin 2x} = 1$ for $0^\circ \leq x \leq 180^\circ$.

Solution

$$\text{Hence } \frac{1}{2}\cot x = 1$$

$$\cot x = 2$$

$$x = 26.6^\circ$$

(c) Show that $\frac{1-\cos^2\theta}{\sec^2-1} = \cos^2\theta$

$$\frac{1-\cos^2\theta}{\sec^2-1} = \frac{\sin^2\theta}{\tan^2\theta} = \sin^2\theta \times \frac{\cos^2\theta}{\sin^2\theta} = \cos^2\theta \text{ as required}$$

Hence, solve the equation $\frac{1-\cos^2\theta}{\sec^2-1} = \frac{3}{4}$ for $0^\circ \leq \theta \leq 90^\circ$.

$$\cos^2\theta = \frac{3}{4}$$

$$\cos\theta = \sqrt{\frac{3}{4}} = \pm 0.866$$

$$\text{For } \cos\theta = 0.866; \cos^{-1}(0.866) = 30^\circ, 330^\circ$$

$$\text{For } \cos\theta = -0.866; \cos^{-1}(-0.866) = 150^\circ, 210^\circ$$

$$\text{Hence } \theta = 30^\circ \text{ for } 0^\circ \leq \theta \leq 90^\circ$$

Example 9

(a) Given that $7 \tan\theta + \cot\theta = 5\sec\theta$, derive a quadratic equation for $\sin\theta$. Hence or

otherwise, find all values of θ in the interval $0^\circ \leq \theta \leq 180^\circ$ which satisfy the equation, giving your answer to the nearest 0.10 where necessary

Solution

$$7 \tan\theta + \cot\theta = 5\sec\theta$$

$$7 \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{5}{\cos\theta}$$

$$7 \frac{\sin^2\theta + \cos^2\theta}{\cos\theta \sin\theta} = \frac{5}{\cos\theta}$$

$$7\sin^2\theta + \cos^2\theta = 5\sin\theta$$

$$7\sin^2\theta + (1 - \sin^2\theta) = 5\sin\theta$$

$$6\sin^2\theta - 5\sin\theta + 1 = 0$$

$$(3\sin\theta - 1)(2\sin\theta - 1) = 0$$

$$\sin\theta = \frac{1}{3} \quad \left| \quad \sin\theta = \frac{1}{2}\right.$$

$$\theta = 19.5^\circ, 160.5^\circ \quad \left| \quad \theta = 30^\circ, 150^\circ\right.$$

$$\therefore [\theta: \theta = 19.5^\circ, 30^\circ, 150^\circ, 160.5^\circ]$$

Example 10

Find the solution of $3\cot\theta + \operatorname{cosec}\theta = 2$ for $0^\circ \leq \theta \leq 180^\circ$.

Solution

$$3\cot\theta + \operatorname{cosec}\theta = 2$$

$$3 \frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} = 2$$

$$(3\cos\theta + 1)^2 = (2\sin\theta)^2$$

$$9\cos^2\theta + 6\cos\theta + 1 = 4\sin^2\theta$$

$$9\cos^2\theta + 6\cos\theta + 1 = 4(1 - \cos^2\theta)$$

$$13\cos^2\theta + 6\cos\theta - 3 = 0$$

$$\cos\theta = \frac{-6 \pm \sqrt{6^2 + 4 \times 13 \times 3}}{2 \times 13}$$

$$\cos\theta = 0.3021$$

$$\theta = 72.40$$

$$\cos\theta = 0.7637$$

$$\theta = 40.2$$

$$\therefore [\theta: \theta = 72.4^\circ, 40.2^\circ]$$

Elimination of trigonometric parameter

This involves the use of identities to eliminate the trigonometric values in equation

Example 11

- (a) If $x = \tan\theta + \sec\theta$ and $y = \tan\theta - \sec\theta$; show that $xy + 1 = 0$

Solution

$$x + y = \tan\theta$$

$$x - y = 2\sec\theta$$

$$\sec\theta = \frac{1}{2}(x - y)$$

Using identity: $1 + \tan^2\theta = \sec^2\theta$

$$1 + (x + y)^2 = \left[\frac{1}{2}(x - y)\right]^2$$

$$4 + x^2 + 2xy + y^2 = x^2 - 2xy + y^2$$

$$4xy + 4 = 0$$

$$xy + 1 = 0 \text{ as required}$$

- (b) $x = 2 + 3\sin\theta$ and $y = 3 + 2\cos\theta$ show that $4(x - 2)^2 + (y - 3)^2 = 36$

Solution

$$x = 2 + 3\sin\theta \Rightarrow \sin\theta = \frac{x-2}{3}$$

$$y = 3 + 2\cos\theta \Rightarrow \cos\theta = \frac{y-3}{2}$$

Using identity $\sin^2\theta + \cos^2\theta = 1$

$$\left(\frac{x-2}{3}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$$

$$4(x - 2)^2 + (y - 3)^2 = 36 \text{ as required}$$

- (c) $x = 2\sin\theta$ and $y = \tan\theta$, prove that

$$x = \pm \frac{2y}{\sqrt{1+y^2}}$$

Solution

$$x = 2\sin\theta; \Rightarrow \csc\theta = \frac{2}{x}$$

$$y = \tan\theta; \Rightarrow \cot\theta = \frac{1}{y}$$

Using identity: $1 + \cot^2\theta = \csc^2\theta$

$$1 + \left(\frac{1}{y}\right)^2 = \left(\frac{2}{x}\right)^2$$

$$x = \pm \frac{2y}{\sqrt{1+y^2}}$$

Revision exercise 1

- Solve for θ , where $0^\circ \leq \theta \leq 360^\circ$
 - $\sec\theta \csc\theta + 2\sec\theta - 2\csc\theta - 4 = 0$
[$\theta: \theta = 60^\circ, 210^\circ, 300^\circ, 330^\circ$]
 - $\tan^2\theta - (\sqrt{3} + 1)\tan\theta + \sqrt{3} = 0$

$$[\theta: \theta = 45^\circ, 60^\circ, 225^\circ, 240^\circ]$$

2. Show that

$$(a) \frac{1 - \cos\theta + \sin\theta}{1 - \cos\theta} = \frac{1 + \cos\theta + \sin\theta}{\sin\theta}$$

$$(b) \tan\theta + \cot\theta = \sec\theta \csc\theta$$

$$(c) \cos^4\theta - \sin^4\theta + 1 = 2\cos^2\theta$$

$$(d) \frac{\sin\theta}{1 + \cos\theta} + \frac{1 + \cos\theta}{\sin\theta} = 2\csc\theta$$

$$(e) \sqrt{\frac{1 + \sin\theta}{1 - \sin\theta}} = \sec\theta + \tan\theta$$

3. Solve the following equations for $-180^\circ \leq x \leq 180^\circ$

(i) $2\cos^2\theta + \sin\theta - 1 = 0$

$$[\theta: \theta = -150^\circ, -30^\circ, 90^\circ]$$

(ii) $\sin 2\theta + 5\cos 2\theta = 3$

$$[\theta: \theta = \pm 45^\circ, \pm 135^\circ]$$

(iii) $4\cot^2\theta + 24\csc\theta + 39 = 0$

$$[\theta: \theta = 16.6^\circ, 23.6^\circ, 156.4^\circ, 163.4^\circ]$$

4. Solve each of the following equations in the stated range

(a) $4\cos^2\theta + 2\sin\theta = 4$ $0^\circ \leq \theta \leq 360^\circ$

$$[\theta: \theta = 0^\circ, 48.6^\circ, 131.4^\circ, 180^\circ, 360^\circ]$$

(b) $2\sec^2\theta - 4\tan\theta - 2 = 0$ $-180^\circ \leq \theta \leq 360^\circ$

$$[\theta: \theta = -135^\circ, -161.6^\circ, 18.4^\circ, 45^\circ]$$

(c) $5\cos^2 3\theta = 3(1 + \sin 3\theta)$, $0^\circ \leq \theta \leq 360^\circ$

$$[\theta: \theta = 7.9^\circ, 52.1^\circ, 90^\circ, 127.9^\circ, 172.1^\circ]$$

5. Solve for θ ; $00 \leq \theta \leq 3600$

(a) $\tan\theta + 3\cot\theta = 4$

$$[\theta: \theta = 45^\circ, 71.6^\circ, 225^\circ, 251.6^\circ]$$

(b) $4\cos\theta - 3\sin\theta = 2$

$$[\theta: \theta = 29.50, 256.70]$$

6. Solve

(a) $\cos\theta + \sqrt{3}\sin\theta = 2$ $0 \leq \theta \leq \pi$

$$[\theta = \frac{\pi}{3}]$$

(b) $2\cos\theta - \csc\theta = 0$ $0^\circ \leq \theta \leq 270^\circ$

$$[\theta: \theta = 45^\circ, 225^\circ]$$

(c) $2\sin^2\theta + 3\cos\theta = 0$ $0^\circ \leq \theta \leq 360^\circ$

$$[\theta: \theta = 240^\circ, 120^\circ]$$

(d) $3\sin\theta + 4\cos\theta = 2$ $-180^\circ \leq \theta \leq 180^\circ$

$$[\theta: \theta = -29.55^\circ, 103.29^\circ]$$

(e) $3\tan^2\theta + 2\sec^2\theta = 2(5 - 3\tan\theta)$ for

$$0^\circ < \theta < 180^\circ$$

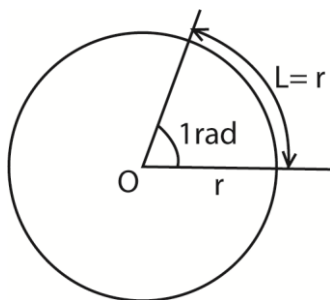
$$[\theta: \theta = 38.66^\circ, 116.57^\circ]$$

7. Without using a tables or calculator, show that $\tan 15^\circ = 2 - \sqrt{3}$

8. Solve equation $8\cos^4\theta - 10\cos^2\theta + 3$ for $0^\circ \leq \theta \leq 180^\circ$
 $[\theta: \theta = 30^\circ, 45^\circ, 135^\circ, 150^\circ]$
9. Eliminate θ from the following equation
- (a) $x = a\sec\theta$ and $y = b + c\cos\theta$
 $[ac = x(y - b)]$
- (b) $x = \sec\theta + \tan\theta$ and $y = \sec\theta - \tan\theta$
 $[xy = 1]$
10. Solve the simultaneous equation
 $\cos x + 4\sin y = 1$
 $4\sec x - 3\operatorname{cosec} y = 5$ for values of x
and y between 0° and 360°
 $[x = 78.8^\circ, 281.5^\circ; y = 11.5^\circ, 168.5^\circ]$
11. Prove each of the following identities
- (a) $\sin x \tan x + \cos x = \sec x$
- (b) $\operatorname{cosec} x + \tan x \sec x = \operatorname{cosec} x \sec^2 x$
- (c) $\operatorname{cosec} x - \sin x = \cot x \cos x$
- (d) $(\sin x + \cos x)^2 - 1 = 2\sin x \cos x$
12. Eliminate θ from each of the following pairs of relationships
- (a) $x = 3\sin\theta, y = \operatorname{cosec}\theta$ $[xy = 3]$
- (b) $5x = \sin\theta, y = 2\cos\theta$ $[100x^2 + y^2 - 4 = 0]$
- (c) $x = 3 + \sin\theta, y = \cos\theta$ $[(x-3)^2 + y^2 = 1]$
- (d) $x = 2 + \sin\theta, \cos\theta = 1 + y$
 $[(x-2)^2 + (y+1)^2 = 1]$

Measuring angles in radians

A radian is defined as an angle subtended at the centre of a circle by an arc that is equal to the radius of the circle. One radian is represented by π , where $\pi = \frac{22}{7}$



How to convert between degrees and radians

1 revolution = circumference of a circle

But circumference of a circle subtends an angle 2π at the centre.

$$\Rightarrow 1 \text{ revolution} = 2\pi = 360^\circ$$

$$\pi = 180^\circ$$

$$1^\circ = \frac{\pi}{180} \text{ radians}$$

$$x^\circ = \frac{\pi}{180} x \text{ radians}$$

Example 12

Convert the following angles to radians

- (a) 330°
(b) 90°
(c) 30°

Solution

- (a) $330^\circ = \frac{\pi}{180} \times 330 = \frac{11\pi}{6}$ radians
- (b) $90^\circ = \frac{\pi}{180} \times 90 = \frac{\pi}{2}$ radians
- (c) $30^\circ = \frac{\pi}{180} \times 30 = \frac{\pi}{6}$ radians

Converting radians to degrees

$$2\pi \text{ radians} = 360^\circ$$

$$1 \text{ radian} = \frac{180^\circ}{\pi}$$

$$x \text{ radians} = \frac{180^\circ}{\pi} x$$

Example 13

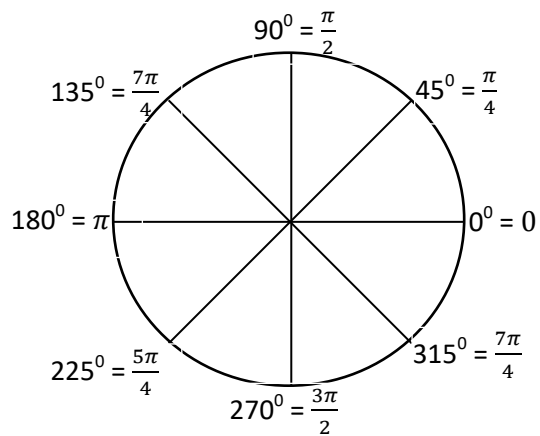
Convert each of the following radians to degrees

- (i) $\frac{\pi}{3}$ radians
(ii) $\frac{2\pi}{5}$ radians
(iii) π radians

Solution

- (i) $\frac{\pi}{3} \text{ radians} = \frac{180^\circ}{\pi} \times \frac{\pi}{3} = 60^\circ$
- (ii) $\frac{2\pi}{5} \text{ radians} = \frac{180^\circ}{\pi} \times \frac{2\pi}{5} = 72^\circ$
- (iii) $\pi \text{ radians} = \frac{180^\circ}{\pi} \times \pi = 180^\circ$

Some equivalent angles in degrees and radians



Example 14

Find each of the following values

- (a) $\sin\left(\frac{2\pi}{3}\right)$
- (b) $\cos\left(\frac{4\pi}{3}\right)$
- (c) $\tan\left(\frac{7\pi}{4}\right)$

Solution

Convert the angles from radian to degrees

- (a) $\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{2 \times 180}{3}\right) = \sin 120^\circ = \frac{\sqrt{3}}{2}$
- (b) $\cos\left(\frac{4\pi}{3}\right) = \cos\left(\frac{4 \times 180}{3}\right) = \cos 240^\circ = -\frac{1}{2}$
- (c) $\tan\left(\frac{7\pi}{4}\right) = \tan\left(\frac{7 \times 180}{4}\right) = \tan 60^\circ = \sqrt{3}$

Compound angles

- $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
 $\Rightarrow \cos 2\theta = \cos^2\theta - \sin^2\theta$
- $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
 $\Rightarrow \cos 2\theta = \cos^2\theta + \sin^2\theta$
- $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
 $\Rightarrow \sin 2\theta = 2\cos\theta\sin\theta$
- $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$
- $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
 $\Rightarrow \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$
- $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Example 15

Solve the equation $\cos\theta = \sin 2\theta$ for values of θ from 0° to 360° . (05 marks)

$$\cos\theta = 2\cos\theta\sin\theta$$

$$\cos(1-2\sin\theta) = 0$$

$$\text{Either } \cos\theta = 0 \text{ or } \theta = 90^\circ, 270^\circ$$

Or

$$1-2\sin\theta = 0$$

$$\sin\theta = \frac{1}{2} \text{ or } \theta = 30^\circ, 150^\circ$$

$$\text{hence } \theta = 30^\circ, 90^\circ, 150^\circ, 270^\circ$$

Example 16

Show that

$$\operatorname{cosec} 2\theta + \cot 2\theta = \cot\theta$$

Solution

$$\begin{aligned} \operatorname{cosec} 2\theta + \cot 2\theta &= \frac{1}{\sin 2\theta} + \frac{\cos 2\theta}{\sin 2\theta} \\ &= \frac{1 + \cos 2\theta}{\sin 2\theta} \\ &= \frac{1 + 2\cos^2\theta - 1}{2\sin\theta\cos\theta} \\ &= \frac{2\cos^2\theta}{2\sin\theta\cos\theta} \\ &= \cot\theta \end{aligned}$$

Example 17

If $\tan\alpha = \frac{3}{4}$ and α is acute, without using tables or calculator work out the value of

(a) $\tan 2\alpha$

$$\begin{aligned} \tan 2\alpha &= \frac{2\tan\alpha}{1 - \tan^2\alpha} \\ &= \frac{2 \times \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2} \\ &= \frac{\frac{3}{2}}{\frac{7}{16}} \\ &= \frac{24}{7} \end{aligned}$$

(b) $\tan \frac{\alpha}{2}$

$$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1 - \tan^2\frac{\alpha}{2}} = \frac{3}{4}$$

$$\begin{aligned} \Rightarrow 3\tan^2\frac{\alpha}{2} + 8\tan\frac{\alpha}{2} - 3 &= 0 \\ (3\tan\frac{\alpha}{2} - 1)(\tan\frac{\alpha}{2} + 3) &= 0 \\ \tan\frac{\alpha}{2} = \frac{1}{3} \text{ or } \tan\frac{\alpha}{2} &= -3 \end{aligned}$$

Since α is acute, $\tan\alpha$ cannot be negative

$$\therefore \tan\frac{\alpha}{2} = \frac{1}{3}$$

Example 18

Prove that $\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$. Hence solve the equation $\cot 2\theta + 2\cot\theta = 2$ for $0^\circ \leq \theta \leq 360^\circ$

Solution

$$\cot 2\theta = \frac{\cos 2\theta}{\sin 2\theta} = \frac{\cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta}$$

dividing through by $\sin^2\theta$

$$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$$

$$\text{Hence, } \cot 2\theta + 2\cot\theta = 0$$

$$\frac{\cot^2\theta - 1}{2\cot\theta} + 2\cot\theta = 0$$

$$5\cot^2\theta - 4\cot\theta - 1 = 0$$

$$(5\cot\theta + 1)(\cot\theta - 1) = 0$$

$$\cot\theta = -\frac{1}{5} \text{ or } \cot\theta = 0$$

$$\Rightarrow \tan\theta = -5 \text{ or } \tan\theta = 1$$

$$\text{When } \tan\theta = -5; \theta = 101.3^\circ, 281.3^\circ$$

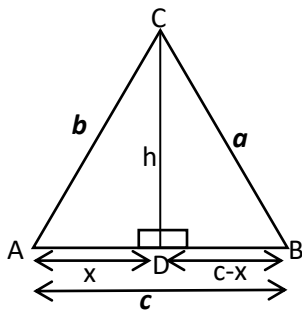
$$\text{When } \tan\theta = 1; \theta = 45^\circ, 225^\circ$$

$$\therefore \{\theta: \theta = 45^\circ, 101.3^\circ, 225^\circ, 281.3^\circ\}$$

Triangles

1. The cosine rule

(a) Given an acute angle A



From triangle

$$ACD: x^2 + h^2 = b^2 \dots\dots\dots(i)$$

$$BCD: (c-x)^2 + h^2 = a^2$$

$$c^2 - 2cx + x^2 + h^2 = a^2 \dots\dots\dots(ii)$$

Substituting eqn. (i) into eqn. (ii)

$$c^2 - 2cx + b^2 = a^2$$

But

$$x = b \cos A$$

$$\Rightarrow b^2 + c^2 - 2bc \cos A = a^2$$

$$a^2 = b^2 + c^2 - 2bc \cos A \quad (1)$$

Similarly;

$$b^2 = a^2 + c^2 - 2accosB \quad (2)$$

$$c^2 = a^2 + b^2 - 2abc \cos C \quad (3)$$

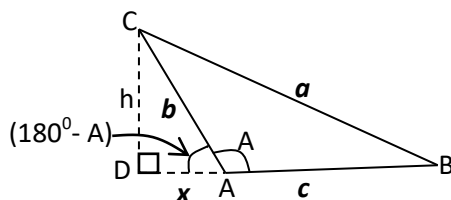
It follows that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

(b) Given an obtuse angle A



In triangle ABC, A is an obtuse angle and CD is the altitude.

From triangle

$$ACD: x^2 + h^2 = b^2 \dots\dots\dots(i)$$

$$BCD: (c-x)^2 + h^2 = a^2$$

$$c^2 - 2cx + x^2 + h^2 = a^2 \dots\dots\dots(ii)$$

Substituting eqn. (i) into eqn. (ii)

$$c^2 - 2cx + b^2 = a^2$$

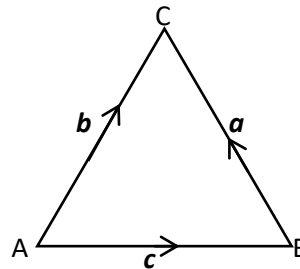
But

$$x = b \cos (180^\circ - A) = -bc \cos A$$

From triangle ACD

$$a^2 = b^2 + c^2 - 2bc \cos A \text{ as before}$$

The cosine rule can be derived using the vector approach.



Given a triangle above with $BC = a$, $AC = c$ and $AB = b$

$$BC = BA + AC = AC - AB$$

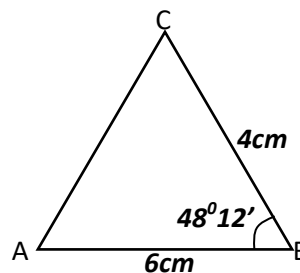
$$a = b - c$$

$$\begin{aligned} \Rightarrow a \cdot a &= (b - c)(b - c) \\ &= b \cdot b - 2b \cdot c + c \cdot c \\ &= b \cdot b + c \cdot c - 2b \cdot c \\ \therefore a^2 &= b^2 + c^2 - 2bc \cos A \\ \text{since } b \cdot c &= |bc| \cos A \end{aligned}$$

Example 19

Solve the triangle in which $AB = 6\text{cm}$, $BC = 4\text{cm}$ and angle $ACB = 48^\circ 12'$

Solution



$$\begin{aligned} \text{Using: } b^2 &= a^2 + c^2 - 2accosB \\ &= 6^2 + 4^2 - 2(6)(4)\cos 48.2^\circ \end{aligned}$$

$$1^\circ (\text{degree}) = 60' (\text{minutes})$$

$$b = 4.47\text{cm}$$

$$\text{Using: } \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{20.0 + 36 - 16}{2(4.47)(6)}$$

$$A = 41.8^\circ$$

$$\text{But } A + B + C = 180^\circ$$

$$41.8^\circ + 48.2^\circ + C = 180^\circ$$

$$C = 90^\circ$$

$\therefore AC = 4.47\text{cm}$, angles $BAC = 41.8^\circ$ and $ACB = 90^\circ$

Example 20

In a triangle ABC, prove that

$$(a) \ a^2 = (b - c)^2 + 4bc \sin^2\left(\frac{A}{2}\right) \text{ hence that}$$

$$a = (b - c) \sec \alpha \text{ where } \tan \alpha = \frac{\sqrt{bc} \sin\left(\frac{A}{2}\right)}{b - c}$$

$$\text{From } \cos A = 1 - 2 \sin^2\left(\frac{A}{2}\right)$$

Substituting for $\cos A$ into the cosine formula $a^2 = b^2 + c^2 - 2bc \cos A$

$$a^2 = b^2 + c^2 - 2bc \left[1 - 2 \sin^2\left(\frac{A}{2}\right)\right]$$

$$a^2 = b^2 + c^2 - 2bc + 4bc \sin^2\left(\frac{A}{2}\right)$$

$$a^2 = (b - c)^2 + 4bc \sin^2\left(\frac{A}{2}\right)$$

Hence, substituting for $\sin^2\left(\frac{A}{2}\right)$ into $\tan \alpha$ expression we get

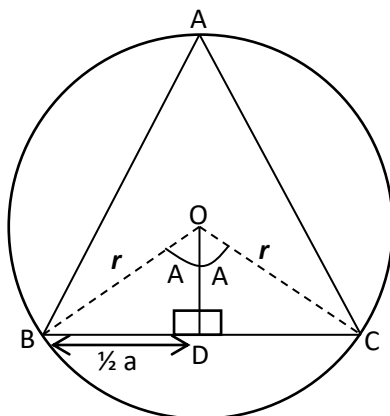
$$a^2 = (b - c)^2 + (b - c)^2 \tan^2 \alpha$$

$$a^2 = (b - c)^2 (1 + \tan^2 \alpha)$$

$$a^2 = (b - c)^2 \sec^2 \alpha$$

$$a = (b - c) \sec \alpha$$

2. The Sine Rule



The figure shows a circle with centre O and radius r circumscribing triangle ABC

Angle $BOC = 2A$ [angle subtended by the same arc at the centre of the circle is twice the angle formed at any point on the circumference]

Triangle BOC is isosceles

OD bisects angle BOC and side BC

$$\therefore BD = \frac{1}{2}a$$

From triangle BOD

$$\sin A = \frac{a}{2r} \text{ i.e. } \frac{a}{\sin A} = 2r$$

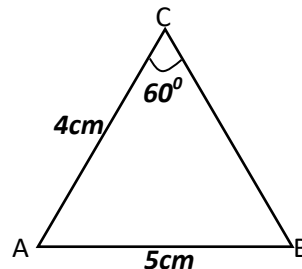
if instead we consider triangles AOC and AOB, we obtain $\frac{b}{\sin B} = 2r$ and $\frac{c}{\sin C} = 2r$

$$\text{In general: } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Example 21

Solve the triangle in which $AB = 5\text{cm}$, $AC = 4\text{cm}$ and angle $ACB = 60^\circ$

Solution



Using sine rule

$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow B = \sin^{-1}\left(\frac{b \sin C}{c}\right)$$

$$B = \sin^{-1}\left(\frac{4}{5} \sin 60^\circ\right) = 43.9^\circ$$

From $A + B + C = 180^\circ$

$$A = (180 - 60 - 43.9)^\circ = 76.1^\circ$$

$$\text{Similarly } a = \frac{b \sin A}{\sin B} = \frac{4 \sin 76.1^\circ}{\sin 43.9^\circ} = 5.6\text{cm}$$

$$\therefore \overline{AB} = 5.6\text{cm}, \hat{BAC} = 76.1^\circ, \hat{ABC} = 43.9^\circ$$

Example 23

Prove that in any triangle

$$\frac{a^2 - b^2}{c^2} = \frac{\sin(A - B)}{\sin(A + B)}$$

Solution

From sine rule formula;

$$a = 2r \sin A, b = 2r \sin B, c = 2r \sin C$$

By substitution

$$\frac{a^2 - b^2}{c^2} = \frac{(2r \sin A)^2 - (2r \sin B)^2}{(2r \sin C)^2} = \frac{\sin^2 A - \sin^2 B}{\sin^2 C}$$

$$\text{But } A + B + C = 180^\circ$$

$$C = 180^\circ - (A + B)$$

$$\sin C = \sin[180^\circ - (A + B)] = \sin(A + B)$$

By substitution

$$\frac{a^2 - b^2}{c^2} = \frac{\sin^2 A - \sin^2 B}{\sin(A + B)} = \frac{(\sin A + \sin B)(\sin A - \sin B)}{\sin(A + B)}$$

$$= \frac{2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B) \cdot 2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)}{\sin(A + B)}$$

$$= \frac{2 \sin \frac{1}{2}(A - B) \cos \frac{1}{2}(A - B)}{\sin(A + B)} = \frac{\sin(A - B)}{\sin(A + B)}$$

$$\text{Hence } \frac{a^2 - b^2}{c^2} = \frac{\sin(A - B)}{\sin(A + B)}$$

Example 24

Prove that in any triangle ABC,

$$\sin \frac{1}{2}(B - C) = \frac{b - c}{a} \cos \frac{1}{2}A$$

Solution

From sine rule formula;

$$a = 2r \sin A, b = 2r \sin B, c = 2r \sin C$$

By substitution

$$\frac{b - c}{a} = \frac{2r \sin B - 2r \sin C}{2r \sin A} = \frac{\sin B - \sin C}{\sin A}$$

$$\text{But } A + B + C = 180^\circ$$

$$A = 180^\circ - (B + C)$$

$$\sin A = \sin[180^\circ - (B + C)] = \sin(B + C)$$

By substitution

$$\frac{b - c}{a} = \frac{\sin B - \sin C}{\sin A} = \frac{\sin B - \sin C}{\sin(B + C)}$$

$$= \frac{2 \cos \frac{1}{2}(B + C) \sin \frac{1}{2}(B - C)}{2 \cos \frac{1}{2}(B + C) \cos \frac{1}{2}(B + C)}$$

$$= \frac{\sin \frac{1}{2}(B - C)}{\cos \frac{1}{2}(B + C)}$$

$$\text{From } A + B + C = 180^\circ$$

$$B + C = 180^\circ - A$$

$$\frac{1}{2}(B + C) = \left(90^\circ - \frac{1}{2}A\right)$$

$$\sin \frac{1}{2}(B - C) = \sin \left(90^\circ - \frac{1}{2}A\right) = \cos \frac{1}{2}A$$

By substitution

$$\frac{b - c}{a} = \frac{\sin \frac{1}{2}(B - C)}{\cos \frac{1}{2}A}$$

$$\therefore \sin \frac{1}{2}(B - C) = \frac{b - c}{a} \cos \frac{1}{2}A$$

Thank you

Dr. Bbosa Science