Curve sketching (sub math)

The procedures of curve sketching depend on the nature of the curve to be sketched

Graphs of y = f(x) (Non rational functions)

For any graph of the form y = f(x) where f(x) is not linear, some or all the following steps are followed.

(a) Determine if the curve is symmetrical about either or both axes of coordinates.
- Symmetry about the x-axis occurs if the equation contains only even powers of y. here equation will be uncharged when (-y) is substituted for y. this applies to graphs of the type $y^2=f(x)$
- Symmetry about the y-axis occurs if the equation contains only even powers of x. Here the equation will be uncharged when (-x) is substituted for x. Here the graph is said to even i.e. f(x) =f(x). For example the graph of $y = x^2$. **Note** if there are odd powers of x and y then there will be no symmetry.

(b) Determine if there is symmetry about the origin. Here symmetry occurs when a change in the sign of x causes a change in the sign of y without altering its numerical value.

(c) Find the intercepts i.e. the curve cuts the x-axis at a point when y = 0 and cuts the y-axis at the point when x = 0.

(d) The curve passes through the origin if (x, y)= (0, 0)

The behaviour of the neighbour of the curve through the origin is studied by considering the ratio of $\frac{y}{x}$.

- If this ratio is small, the curve keeps close to x-axis near the origin.
- If the ratio is unity, the direction of the curve bisects the angle between the axes.
- If the ratio is large, the curve keeps near the y-axis.

Or:

We consider the behaviour of $\frac{dy}{dx}$ near the origin.

- If $\frac{dy}{dx}$ is very small, then the curve lies near the x-axis.
- If $\frac{dy}{dx}$ is large, then the curve lies near the y-axis.
- If $\frac{dy}{dx}$ is near unity, then the direction of the curve (tangent at origin) bisects the angle between the axes.

(e) Examine the behaviour of the function as x→±∞ and y →±∞ (if any)

(f) Find the turning points and their nature as well as points if inflexion (if any)
Use the second derivative
- For min point, $\frac{d^2y}{dx^2} = +ve$
- For max point, $\frac{d^2y}{dx^2} = -ve$
- Point of inflexion, $\frac{d^2y}{dx^2} = 0$

Example 1

Given the curve $y = 3x^3 – 4x^2 – x$

(a) Find the turning points of the curve

Solution

Turning points when $\frac{dy}{dx} = 0$

=> $9x^2 - 8x -1 = 0$

$(9x + 1)(x – 1) = 0$

Either $9x + 1 = 0$; $x = -\frac{1}{9}$

Or $(x – 1) = 0$; $x = 1$

When $x = -\frac{1}{9}$

$y = 3\left(-\frac{1}{9}\right)^3 - 4\left(-\frac{1}{9}\right)^2 - \left(-\frac{1}{9}\right) = \frac{14}{243}$

Turning point $=\left(-\frac{1}{9}, \ \frac{14}{243}\right)$

When x = 1

$y = 3(1)^3 - 4(1)^2 - (-1) = -2$

Turning point (1, -2)

Hence the turning points (x, y) are $\left(-\frac{1}{9}, \ \frac{14}{243}\right)$ *and* $(1, -2)$

(b) Distinguish between the nature of the turning points.

Solution

$\frac{d^2}{dx^2}(9x^2 - 8x - 1) = 18x – 8$

When x = 1

$\frac{d^2y}{dx^2} = 18 - 8 = 10$

Since $\frac{d^2y}{dx^2} > 0$; the turning point (1, -2) is a minimum

When $x = -\frac{1}{9}$

$\frac{d^2y}{dx^2} = \frac{-18}{9} - 8 = -10$

Since $\frac{d^2y}{dx^2} < 0$; the turning point $\left(-\frac{1}{9}, \ \frac{14}{243}\right)$ is a maximum

Example 2

Determine the coordinates of the stationary point of the curve, $y = \frac{1}{4}x^2 – 2x – 5$

Solution

At the stationary point $\frac{dy}{dx} = 0$

⇨ $\frac{d}{dx}\left(\frac{1}{4}x^2 - 2x - 5\right) = 0$

$\frac{1}{2}x - 2 = 0$

x = 4

Substituting for x = 4 in the equation

$y = \frac{1}{4}(2)^2 – 2(2) – 5 = -9$

Hence stationary point is (4, -9)

Example 3

Find the gradient of the curve $y = 4x^2(3x + 2)$ at the point (1, 20)

Solution

$y = 4x^2(3x + 2) = 12x^3 + 8x^2$

$\frac{dy}{dx} = 36x^2 + 16x$

At x = 1

$\frac{dy}{dx} = 36 + 16 = 52$

Hence gradient = 52

Example 4

The equation of a curve is $y = 3x^2 + 2$

(a) (i) Determine the turning point of the curve

Solution

At turning point $\frac{dy}{dx}(3x^2 - 2) = 0$

$6x = 0$

x = 0 and $y = 3(0)^2 + 2 = 2$

Hence turning point = (0, 2)

(ii) Find the nature of the turning point

S0lution

$\frac{d^2y}{dx^2} = 6$

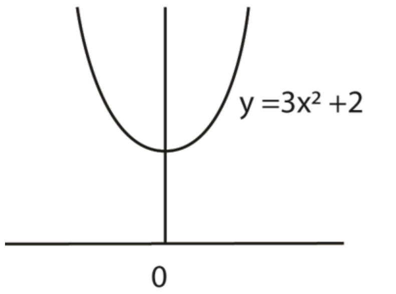
Since $\frac{d^2y}{dx^2} > 0$, the turning point is a minimum

(iii) Sketch the graph of the curve.

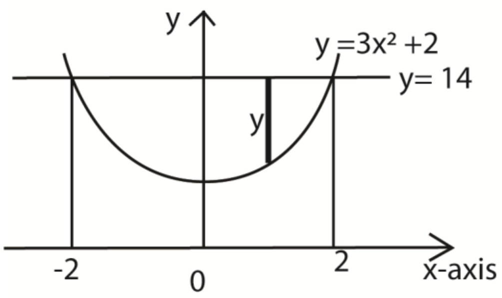
Finding x – intercepts when y = 0

$3x^2 + 2 = 0$, $x^2 = -\frac{-2}{3}$

	0	unidentified
y	2	0



(b) The curve and the line y = 14 intercepts at the point (-2, 14) and (2, 14). Calculate the area of the region enclosed between the line and the curve. (08 marks)



Area of the curve and x-axis $= \int_{-2}^{2} y dx$

$= \int_{-2}^{2} (3x^2 + 2)\, dx$

$= [x^3 + 2x]_{-2}^{2}$ = (8 + 4) – (-8 – 4) =24

Total area of the rectangle = 4 x 14

= 56

Area of the curve and the line (y = 14)

56 – 24 = 36unit^2

Example 5

The equation of a curve is $y = 3 + 2x - x^2$.

(a) Determine the;

(i) coordinates and nature of the turning points of the curve. (06 marks)

Turning points when $\frac{dy}{dx}(3 + 2x - x^2) = 0$

2 – 2x = 0

x = 1

when x = 1, y = 3 + 2 -1 = 4

turning point is (1, 4)

Nature of turning point

$\frac{d^2y}{dx^2} = -2$,

Since $\frac{d^2y}{dx^2} < o$ the turning point is a maxima

(ii) y – and x – intercept of the curve

y intercept when x = 0, i.e. y = 3 or (0, 3)

x intercept when y = 0

$3 + 2x - x^2 = 0$

Or

$x^2 - 2x - 3 = 0$

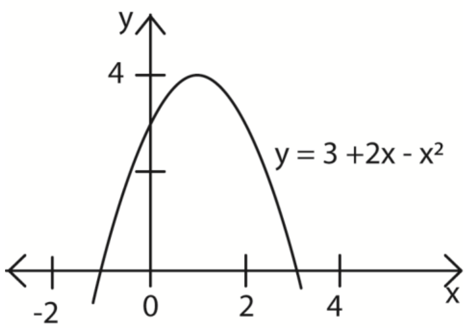
(x – 3)(x + 1) = 0

Either x – 3 = 0 and x = 3

Or x +1 = 0 and x = -1

Hence x intercepts are (-1, 0)and (3 , 0)

(b) (i) sketch the curve (02marks)



(ii) find the area enclosed by the curve and the x – axis.

Area = $\int_{-1}^{3} (3 + 2x - x^3)$

$= \left[3x + x^2 - \frac{x^3}{3}\right]_{-1}^{3}$

= (9 +9-9)-(-3 + 1 + $\frac{1}{3}$)

= $10\frac{2}{3}$ unit^2

Example 6

(a) Sketch the graph of $y= 5 + 4x – x^2$.

Steps taken

- Finding intercepts

  x – intercept; y = 0

  $0 = 5 + 4x – x^2$.

  $5 + 5x –x – x^2 = 0$

  5(1 + x) – x(1 + x) =0

  (5 – x)(1 + x)

  Either 5 – x = 0; x = 5

  Or 1 + x = 0; x = -1

  Hence the curve cuts the x-axis at point (-1, 0) and (0, 5)

  y – intercept, when x = 0, y = 5

  Hence the curve cuts the y-axis at point (0, 5)

- As x→+∞, y →-∞ and x→-∞, y →+∞
- Finding turning point

  $\frac{dy}{dx} = 4 - 2x$

  At turning point $\frac{dy}{dx} = 0$

  2x – 4 = 0; x = 2

  When x = 2; $y= 5 + 4(2) – (2)^2= 9$

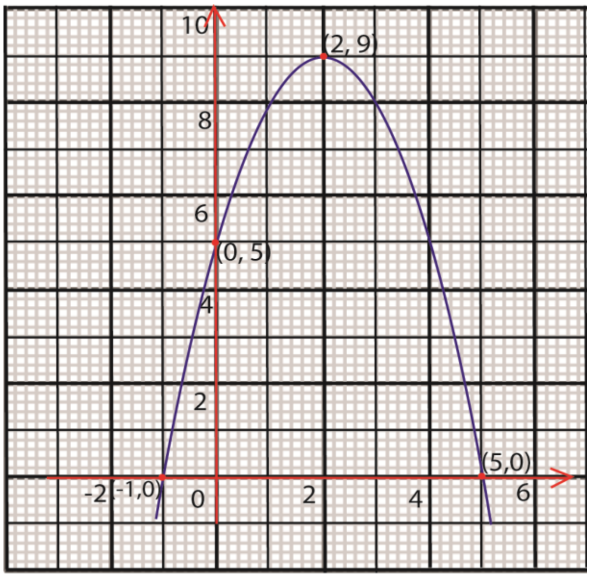
  Hence turning point = (2, 9)

  Finding the nature of turning point

  $\frac{dy}{dx} = 4 - 2x$

  $\frac{d^2y}{dx^2} = -2$

  Since $\frac{d^2y}{dx^2} < 0$, hence the turning point is maximum.



Example 7

Sketch the curve $y = x^3 – x^2 – 5x + 6$

Steps taken

For y – intercept; x = 0, y = 0

Hence the y – intercept is (0, 6)

For x – intercept, y = 0

$x^3 – x^2 – 5x + 6 = 0$

error approach is used to find the first factor i.e. (x-2), then other factor is found by long division

$$\begin{array}{r} x^2 + x - 3 \\ (x-2)\overline{)\,x^3 - x^2 - 5x + 6} \\ -\,x^3 - 2x^2 \qquad\qquad \\ \hline x^2 - 5x + 6 \\ -\;x^2 - 2x \quad\;\; \\ \hline 3x + 6 \\ -\;3x + 6 \\ \hline 0 + 0 \end{array}$$

⇨ $x^3 – x^2 – 5x + 6 = (x – 2)(x^2 + x - 3) = 0$

Solving $x^2 + x - 3 = 0$

$x = \frac{-1\pm\sqrt{1+12}}{2} = \frac{-1\pm3.6}{2}$

x = 1.3 or -2.6

Hence the x- intercepts are (2, 0), (1.3, 0) and (-2.3, 0)

Finding turning points

$y = x^3 - x^2 - 5x + 6$

$\frac{dy}{dx} = 3x^2$-2x-5

At turning point $\frac{dy}{dx} = 0$

$3x^2$-2x-5 = (3x – 5)(x+1) = 0

Either 3x -5 = 0 x = $\frac{5}{3}$

Or x + 1 = 0; x =-1

When x = $\frac{5}{3}$;

$y = \left(\frac{5}{3}\right)^3 - \left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right) + 6 = \frac{-13}{27}$

When x = -1

$y = (-1)^3 - (-1)^2 - 5(-1) + 6$ = 9

Hence turning points are $\left(\frac{5}{3}; \frac{-13}{27}\right)$ and(-1, 9)

Finding the nature of turning points

$\frac{dy}{dx} = 3x^2$-2x-5

$\frac{d^2y}{dx^2} = 6x - 2$

For$\left(\frac{5}{3}; \frac{-13}{27}\right)$

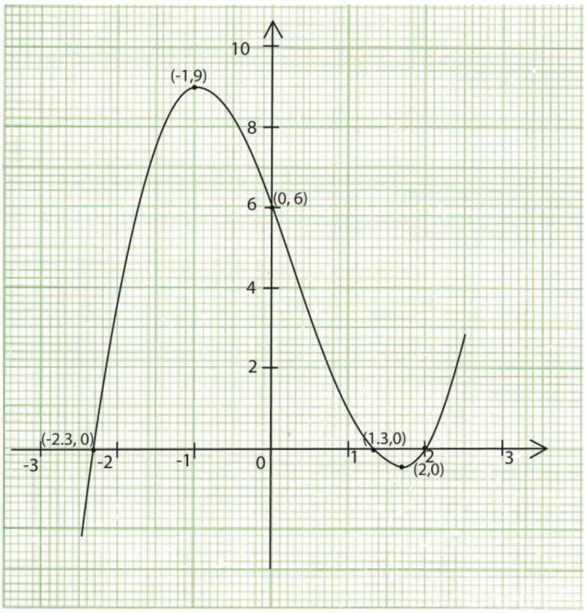
$\frac{d^2y}{dx^2} = 6\left(\frac{5}{3}\right) - 2 = 8\ (> 0)$

$\therefore \left(\frac{5}{3}; \frac{-13}{27}\right)$is minimum

For (-1, 9)

$\frac{d^2y}{dx^2} = 6(-1) - 2 = -8\ (< 0)$

$\therefore (-1; 9\ )$is maximum



(b) sketch the curve y = x^3 – 8

y = x^3 – 8

Intercepts

When x = 0, y = -8

When y = 0, x = 2

(x, y) = (2, 0)

Turning point: $\frac{dy}{dx} = 3x^2$

$3x^2$ = 0

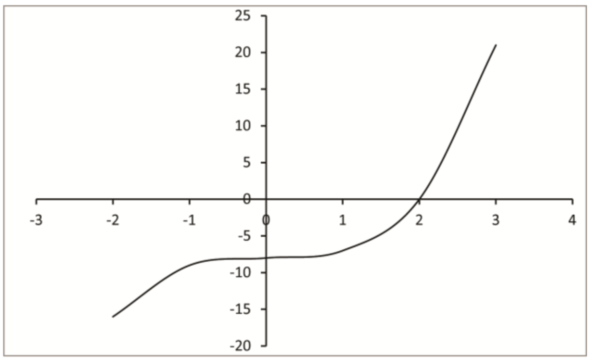
x = 0

$\frac{d^2y}{dx^2} = 6x$

$\frac{d^2y}{dx^2} = 0, x = 0$

Point of reflection= (0, 8)

	x < 2	x > 2
y	-	+



(c) Sketch the curve $y = x^2(x - 4)$

Steps taken

- Finding the intercepts

y – intercept, (0,0)

hence y – intercept is (0, 0)

For x – intercept, y= 0

=>$x^2(x-4)=0$

Either x = 0 or x = 4

Hence x-intercept are (0, 0) and (4, 0)

- As x→+∞, y →+∞ and x→-∞, y →-∞
- Finding turning point(s)

  $y = x^2(x-4) = x^3 - 4x^2$

  $\frac{dy}{dx} = 3x^2 - 8x$

  At turning point, $\frac{dy}{dx} = 0$

  ⇨ $3x^2 - 8x = x(3x-8) = 0$

  Either x = 0

  Or x = $\frac{8}{3}$

  When x = 0; y=0

  When x =$\frac{8}{3}$; =$3\left(\frac{8}{3}\right)^2 - 8\left(\frac{8}{3}\right) = \frac{-256}{27}$

Hence turning points are (0,0) and$\left(\frac{8}{3}, \frac{-256}{27}\right)$

- Finding the nature of turning points

$\frac{dy}{dx} = 3x^2 - 8x$

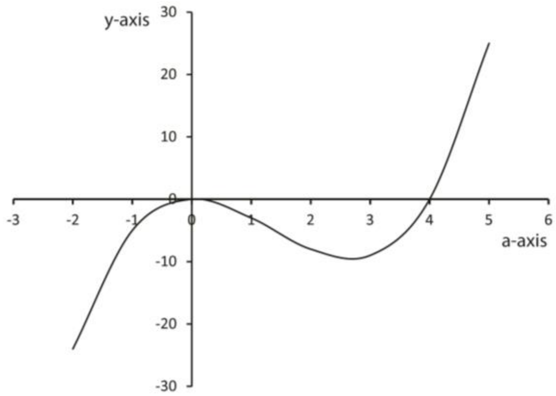
$\frac{d^2y}{dx^2} = 6x - 8$

For (0, 0); $\frac{d^2y}{dx^2} = 6(0) - 8 = -8\ (< 0)$

Hence (0, 0) is maximum

For $\left(\frac{8}{3}, \frac{-256}{27}\right)$; $\frac{d^2y}{dx^2} = 6\left(\frac{8}{3}\right) - 8 = 8\ (> 0)$

Hence $\frac{8}{3}$ is minimum



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Thanks

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