SENIOR FIVE TERM 2

TOPIC 5/6: Integration 1

Competency: The learner applies integration techniques to solve problems involving areas under curves, accumulation of quantities, and other real-world applications.

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Integration (A-level)

It is the reverse of differentiation.

Integration of polynomial functions

$\int x^n dx = \frac{1}{n+1} x^{n+1} + 1 \; where \; n \neq -1$

i.e. increase the power by 1 and divide the term by the new power;

Example 1

Integrate the following

(i) $\int 1dx = \int (x^0)dx = x + c$

During integration the following concepts should be considered.

(ii) $\int xdx = \int x^1 dx$

$= \frac{1}{1+1} x^{1+1}$

$= \frac{1}{2} x^2$

(iii) $\int x^4 dx = \frac{1}{4+1} x^{4+1} = \frac{1}{4} x^5$

(iv) $\int 4x^3 dx = 4 \int x^3 dx = \frac{4}{(3+1)} x^4 = x^4$

(v) $\int x^{-3} = \frac{1}{-3+1} x^{-3+1} = -\frac{1}{2} x^{-2} = \frac{-1}{2x^2}$

Methods of integration

The choice of the method depends on judgment. Below are some of the methods:

Integration by change of variable where a derivative exist/integration by recognition or inspection

Example 2

(i) $\int x\sqrt{(x^2 - 2)}$dx

Solution

Let u = x^2-2

- $du = 2u \; i.e. xdx = \frac{1}{2} du$

$\int x\sqrt{(x^2 - 3)}\, dx = \int \sqrt{(x^2 - 3)}\, xdx$

$= \int \sqrt{u}. \frac{1}{2} du$

$= \frac{1}{2} \int u^{\frac{1}{2}} du$

$= \frac{1}{2}. \frac{2}{3} u^{\frac{3}{2}} + c$

$= \frac{1}{3} (x^2 - 3)^{\frac{3}{2}} + c$

$= \frac{1}{3} (x^2 - 3)\sqrt{(x^2 - 3)} + c$

Or let u =$\sqrt{(x^2 - 32)}$ => $u^2 = x^2 - 3$

2du = 2xdx i.e. xdx = udu

$\int x\sqrt{(x^2 - 3)}\, dx = \int \sqrt{(x^2 - 3)}\, xdx$

$\int u.udu = \int u^2 du$

$= \frac{1}{3} u^3 + c$

$= \frac{1}{3} (x^2 - 3)\sqrt{(x^2 - 3)} + c$

(ii) $\int_0^1 \frac{x^2 - 1}{\sqrt{(x^3 - 3x + 5)}} dx$

Solution

Let u = $x^3 - 3x + 5$ =>$du = (3x^2 - 3)dx$

i.e. $(3x2 - 3)dx = \frac{1}{3} du$

$\therefore \int_0^1 \frac{x^2 - 1}{\sqrt{(x^3 - 3x + 5)}} dx = \frac{1}{3} \int_0^1 \frac{1}{\sqrt{u}} dx$

$= \frac{1}{3} u^{\frac{1}{2}}(2) + c$

$= \frac{2}{3} \sqrt{(x^3 - 3x + 5)}$

- $\int_0^1 \frac{x^2 - 1}{\sqrt{(x^3 - 3x + 5)}} dx = \frac{2}{3} \sqrt{(x^3 - 3x + 5)}\Big|_0^1$

$= \frac{2}{3}\left(\sqrt{1-3+5} - \sqrt{5}\right)$
$= \frac{2}{3}\left(\sqrt{3} - \sqrt{5}\right) = 0.336$

Revision exercise 1

Integrate the following using the suggested substitution in each case.

1. $\int x(x+4)^3 dx$, u = x + 4
   $\left[\frac{1}{5}(x-1)(x+4)^4 + c\right]$
2. $\int (x-4)(x-1)^3 dx$, u = x – 1
   $\left[\frac{1}{4}(4x-19)(x-1)^4 + c\right]$
3. $\int x(2x-3)^2 dx$, u = 2x – 3
   $\left[\frac{1}{16}(2x+1)(2x-3)^3 + c\right]$
4. $\int (3x+1)\,(2x-5)^2 dx$, u = 2x – 5
   $\left[\frac{1}{48}(18x+23)(2x-5)^3 + c\right]$
5. $\int \frac{x}{x+3} dx$, u= x+3 $\left[x - 3In(x+3) + c\right]$
6. $\int \frac{x}{(x+1)^2} dx$, u= x+1 $\left[\frac{1}{x+1} + In(x+1) + c\right]$
7. $\int \frac{x+1}{(2x-3)^3} dx$, u= 2x – 1 $\left[-\frac{4x+1}{8(2x-3)^2}\right]$
8. $\int \sqrt{(x+1)}\, dx$, u = x + 1
   $\left[\frac{2}{15}(3x-2)\sqrt{(x+1)^2} + c\right]$
9. $\int x\sqrt{(x-1)}\, dx, u = \sqrt{(x-1)}$
   $\left[\frac{2}{15}(3x+2)\sqrt{(x-1)^2} + c\right]$
10. $\int (x-4)\sqrt{(x+5)}$, u = x + 5
    $\left[\frac{2}{5}(x-10)\sqrt{(x+1)^3} + c\right]$
11. $\int (3x-2)\sqrt{(1-2x)}dx, u = \sqrt{(1-2x)}$
    $\left[\frac{1}{15}(7-9x)\sqrt{(1-2x)^2} + c\right]$
12. $\int \frac{x}{\sqrt{x+1}} dx$, u= x+1
    $\left[\frac{2}{3}(x-2)\sqrt{x+1} + c\right]$
13. $\int \frac{x}{\sqrt{x-3}} dx$, u= $\sqrt{x-3}$
    $\left[\frac{2}{3}(x+6)\sqrt{x-3} + c\right]$
14. $\int \frac{x-2}{\sqrt{x-4}} dx$, u= $x-4$
    $\left[\frac{2}{3}(x+2)\sqrt{x-4} + c\right]$
15. $\int \frac{x+3}{\sqrt{5-x}} dx$, u= $\sqrt{5-x}$
    $\left[-\frac{2}{3}(x+19)\sqrt{5-x} + c\right]$
16. Use the substitution $x = \frac{1}{u}$ to evaluate $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ $\left[\frac{\pi}{3}\right]$
17. Use the substitution u = $\sqrt{x-2}$ to evaluate $\int_3^4 \frac{3x}{\sqrt{x-2}} dx$
    $\left[16\sqrt{2} - 4\right]$
18. Evaluate
    (a) $\int_3^5 x(x-3)^2 dx$ [12]
    (b) $\int_4^7 \frac{5-x}{\sqrt{x-3}}$dx $\left[-\frac{2}{3}\right]$
    (c) $\int_1^3 (3x+1)(2-x)^4\, dx$ $\left[2\frac{4}{5}\right]$
19. By using the substitution u = $\sqrt{1+x^2}$, show that $\int_0^{\sqrt{3}} x^2\sqrt{(1+x^2)}\, dx = 3\frac{13}{15}$

Integration by change of variable where a derivative not exist

Here a term is solved by changing it to another variable

Example 3

Find

(a) $\int_5^6 x\sqrt{(x-5)}\, dx$

Solution

Let u= $\sqrt{(x-5)}$ hence $u^2 = x - 5$ => x =u^2 + 5

dx = 2udu

Changing limits

x	θ
6	1
5	0

$\therefore \int_5^6 x\sqrt{(x-5)}\,dx = \int_0^1 (u^2+5)u.\,2udu$

$= 2\int_0^1 (u^4+5u^2)\,du$

$= 2\left[\frac{1}{5}u^5 + \frac{5}{3}u^3\right]_0^1$

$= \frac{2}{15}(3+25) = \frac{56}{15}$

(b) $\int \frac{x-3}{\sqrt{x+1}}dx$

Solution

Let u= $\sqrt{x+1}$ i.e.$u^2 = x+1$

- x = $u^2 - 1$ and dx = 2udu

$\therefore \int \frac{x-3}{\sqrt{x+1}}dx = \int \frac{(u^2-1)}{u}.\,2udu$

$= 2\int (u^2-4)du$

$= 2\left(\frac{1}{3}u^3 - 4u\right) + c$

$= \frac{2}{3}u(u^2-12) + c$

$= \frac{2}{3}\sqrt{(x+1)}(x+1-12) + c$

$= \frac{2}{3}(x-11)\sqrt{(x+1)} + c$

(c) $\int (2\text{x}-1)(\text{x}+2)^3\,dx$

Solution

Let u = x+2 => x = u-2 and dx = du

$\therefore \int (2\text{x}-1)(\text{x}+2)^3\,dx$

$= \int [2(u-2) - 1u^3du]$

$= \int (2u^4 - 5u^3)\,du$

$= \frac{2}{5}u^5 - \frac{5}{4}u^4 + c$

$= \frac{1}{20}u^4(8u-25) + c$

$= \frac{1}{20}(x+2)^4(8(x+2+-25) +$

c

$= \frac{1}{20}(8x-9)(x+2)^4 + c$

Revision exercise 2

1. Integrate each of the following with respect to x using suitable substitution
   (a) x(x+3)^3
   $\left[\frac{1}{20}(4x-3x+3^4)+c\right]$
   (b) $x\sqrt{5-x}$
   $\left[-\frac{2}{15}\left(3x+10\sqrt{(5-3)^3}+c\right)\right]$
   (c) $\frac{x-3}{(x+2)^2}$ $\left[\frac{5}{x+2} + In(x+2) + c\right]$
   (d) $\frac{x}{\sqrt{2x+1}}$ $\left[\frac{1}{3}(x-1)\sqrt{2x+1} + c\right]$
   (e) $(x-3)(5-2x)^4$
   $\left[\frac{1}{120}(31-10x)(5-2x)^5 + c\right]$
   (f) $\frac{x}{\sqrt{(x+1)^3}}$ $\left[\frac{2(x+2)}{\sqrt{x+1}} + c\right]$
   (g) $\frac{x+3}{(3-x)^2}$ $\left[\frac{6}{3-x} + In(3-x) + c\right]$
   (h) $x^2(x-1)^4$
   $\left[\frac{1}{105}(15x^2+5x+1)(x-1)^5 + c\right]$
   (i) $x\sqrt{(1-x)^3}$
   $\left[-\frac{2}{35}(5x+2)\sqrt{(1-x)^5}\right]$

Integration of exponential and logarithmic functions.

A. From $\frac{d}{dx}e^x = e^x$

- $\int e^x dx = e^x + c$

Example 4

Find

(a) $\int xe^{x^2}dx$

Solution

Let u = $3x^2$ => du = 6xdx i.e. xdx =$\frac{1}{6}du$

$\int xe^{x^2}dx = \frac{1}{6}\int e^u du = \frac{1}{6}e^u + c$

$\therefore \int xe^{x^2}dx = \frac{1}{6}e^{x^2} + c$

(b) $\int \frac{e^{-\frac{1}{x}}}{x^2}dx$

Solution

Let u = $-\frac{1}{x}$ => du = $\frac{1}{x^2}dx$

$\int \frac{e^{-\frac{1}{x}}}{x^2}dx = \int e^u du = -e^u + c$

$\therefore \int \frac{e^{-\frac{1}{x}}}{x^2}dx = e^{-\frac{1}{x}} + c$

B. From $\frac{d}{dx}(Inx) = \frac{1}{x}$

- $\int \frac{1}{x}dx = Inx + c \equiv InAx$

This result shows that

$\int \frac{f\prime(x)}{f(x)}dx = In[f(x)] + c$ **i.e.**

- $\int cot2xdx = \int \frac{cos2x}{sin2x}dx$
  $= \frac{1}{2}In(sin2x) + c$
- $\int \frac{a}{b+cx}dx = \frac{a}{c}In(b+cx) + k$

Example 5

Find

(a) $\int \frac{1}{3x+4}dx$

Solution

Let u = 3x+4 => du 3dx i.e. dx=$\frac{1}{3}du$

$\therefore \int \frac{1}{3x+4}dx = \frac{1}{3}\int du = \frac{1}{3}Inu + c$

$= \frac{1}{3}In(3x+4) + c$

(b) $\int \frac{x}{1-5x^2}dx$

Solution

Let u = $1-5x^2$

=> du -10xdx i.e. dx=$-\frac{1}{10x}du$

$\therefore \int \frac{x}{1-5x^2}dx = \frac{1}{10}\int \frac{1}{u}du = \frac{1}{10}Inu + c$

$= \frac{1}{3}In(1-5x^2) + c$

(c) $\int_0^1 \frac{x+1}{3+4x^2}dx$

Solution

$\int_0^1 \frac{x+1}{3+4x^2}dx = \int_0^1 \frac{1}{3+4x^2}dx + \int_0^1 \frac{1}{3+4x^2}dx$

$= \left[\frac{1}{8}In(3+4x^2 + \frac{1}{2\sqrt{3}}tan^{-1}\left(\frac{2}{\sqrt{3}}\right)\right]_0^1$

$= \frac{1}{8}In\left(\frac{7}{3}\right) + \frac{1}{2\sqrt{3}}tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$

(d) $\int_a^{2a} \frac{x^3}{x^4+a^4}dx$

Solution

$\int_a^{2a} \frac{x^3}{x^4+a^4}dx = \frac{1}{4}[In(x^4+a^4]_a^{2a}$

$= \frac{1}{4}(In17a^4 - In2a^4)$

$= \frac{1}{4}In\left(\frac{17}{2}\right)$ =0.535

C. From $\frac{d}{dx}a^x = a^xIna$

⇨ $\int a^xdx = \frac{1}{Ina}a^x + c$

It follows that $\int 2^xdx = \frac{2^x}{In2} + c$

Example 6

Integrate

(a) $\int x^2 2^{3x^2}dx$

Solution

Let u = 3x[3], => du = $9x^2$ i.e. $x^2dx = \frac{1}{9}du$

$\int x^2 2^{3x^2}dx = \frac{1}{9}\int 2^u du = \frac{1}{9}\frac{2^u}{In2} + c$

$= \frac{1}{9}\frac{2^{3x^2}}{In2} + c$

Revision exercise 6

1. Find the following integrals

(a) $\int e^x(3+e^x)^2dx$ $\left[\frac{1}{3}(3+e^x)^3 + c\right]$

(b) $\int 2e^x(e^x-4)^3dx$ $\left[\frac{1}{2}(e^x-4)^4 + c\right]$

(c) $\int \frac{4e^{-2x}}{(1+e^{-2x})^2}dx$ $\left[\frac{2}{1+e^{-2x}} + c\right]$

(d) $\int \frac{(e^{-x}+7)^2}{e^x}dx$ $\left[-\frac{1}{3}(e^{-x}+7)^3 + c\right]$

(e) $\int e^x\sqrt{4+e^x}\,dx$ $\left[\frac{2}{3}\sqrt{(4+e^x)^3} + c\right]$

(f) $\int e^{5x}\sqrt{e^{5x}+2}\,dx$ $\left[\frac{2}{15}\sqrt{(e^{5x}+2)^3} + c\right]$

(g) $\int \frac{e^{3x}}{\sqrt{e^{3x}-1}}$ $\left[\frac{2}{3}\sqrt{e^{3x}-1}+c\right]$

(h) $\int \frac{1}{2e^x\sqrt{1-e^{-x}}}dx$ $\left[\sqrt{1-e^{-x}}+c\right]$

(i) $\int 5^x dx$ $\left[\frac{5^x}{In5}+c\right]$

(j) $\int 3^{2x}$ $\left[\frac{3^{2x}}{In9}+c\right]$

(k) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}$ $\left[2e^{\sqrt{x}}+c\right]$

(l) $\int x^2e^{x^3}$ $\left[\frac{1}{3}e^{x^3}\right]$

(m) $\int 4^x dx$ $\left[\frac{4x}{In4}\right]$

(n) $\int x10^x dx$ $\left[\frac{x10^x}{In10}-\frac{10^x}{(In10)^2}+c\right]$

2. Evaluate

(a) $\int_1^3 e^x dx$ $[e(e^2-1)]$

(b) $\int_0^3 e^{-x}dx$ $\left[1-\frac{1}{e^3}\right]$

(c) $\int_1^2 2e^{(2x+1)}$ $[e^3(e^2-1)]$

(d) $\int_{-1}^1 2e^{(1-2x)}$ $\left[e^3-\frac{1}{e}\right]$

(e) $\int_0^1 (4xe^{x^2}+1)dx$ $[2e-1]$

Integration involving partial fractions

There are three established types of partial fractions depending on the nature of the denominator.

Denominators with linear factors e.g. 3x – 1, x + 2 and 3x – 4.

Each linear factor (ax + b) in the denominator has a corresponding partial fraction of the form $\frac{c}{(ax+b)}$, where a, b and c are constants.

Example 7

(a) Express each of the following in partial fraction. Hence find the integral of each with respect to x.

(i) $\frac{x-1}{(x+1)(x-2)}$

Solution

Let $\frac{x-1}{(x+1)(x-2)}=\frac{A}{(x+1)}+\frac{B}{(x-2)}$

Multiplying by (x+1)(x-2)

⇨ x – 1 = A(x – 2) + B(x+1)

then we find the values of A and B

Putting x = 2: 1 = 3B, => B = $\frac{1}{3}$

Putting x = -1: -2 = -3A, => A = $\frac{2}{3}$

$\therefore \frac{x-1}{(x+1)(x-2)}=\frac{\frac{2}{3}}{(x+1)}+\frac{\frac{2}{3}}{(x-2)}$

$=\frac{2}{3(x+1)}+\frac{1}{3(x-2)}$

Hence,

$\int \frac{x-1}{(x+1)(x-2)}dx=\frac{2}{3}\int\frac{1}{(x+1)}dx+\frac{1}{3}\int\frac{1}{(x-2)}dx$

$=\frac{2}{3}In(x+1)+\frac{1}{3}In(x-2)+c$

$=\frac{2}{3}In(x+1)^2(x-2)+c$

(ii) $\frac{1}{x^3-9x}$

Solution

$\frac{1}{x^3-9x}=\frac{1}{x(x^2-9)}=\frac{1}{x(x-3)(x+3)}$

⇨ $\frac{1}{x^3-9x}=\frac{A}{x}+\frac{B}{(x-3)}+\frac{C}{(x+3)}$

Multiplying through with $x(x-3)(x+3)$

1= $A(x^2-9)+B(x^2+3x)+C(x^2-3x)$

Putting x = 0; 1 = -9A => $A=-\frac{1}{9}$

Putting x = 3; 1 = 18B => $B=\frac{1}{18}$

Putting x = -3; 1 = 18C => $C=\frac{1}{18}$

⇨ $\frac{1}{x^3-9x}=-\frac{1}{9x}+\frac{1}{18(x-3)}+\frac{1}{18(x+3)}$

Hence,

$\int\frac{1}{x^3-9x}dx$

$=-\frac{1}{9}\int\frac{1}{x}dx+\frac{1}{18}\int\frac{1}{(x-3)}dx+\frac{1}{18}\int\frac{1}{(x+3)}dx$

$=-\frac{1}{9}Inx+\frac{1}{18}In(x+3)+\frac{1}{18}In(x-3)$+c

$=\frac{1}{18}(In(x+3)+In(x-3)-2Inx)+c$

$=\frac{1}{18}\left[In\frac{(x+3)(x-3)}{x^2}\right]+c$

(iii) $\frac{2x+1}{(x-1)(3x^2+7x+2)}$

Solution

$\frac{2x+1}{(x-1)(3x^2+7x+2)}=\frac{2x+1}{(x-1)(x+2)(3x+1)}$

$\frac{2x+1}{(x-1)(3x^2+7x+2)}=\frac{A}{(x-1)}+\frac{B}{(x+2)}+\frac{1}{(3x+1)}$

Multiplying by $(x-1)(x+2)(3x+1)$

2x + 1= A(x+2)(3x+1)+B(x-1)(3x+1)+C(x-1)(x+2)

Putting x = 1; 3 = 12A => $A=\frac{1}{4}$

Putting x = -2; -3 = 15B => $B=-\frac{1}{5}$

Putting x = $-\frac{1}{3}$; $\frac{1}{3}=-\frac{20}{9}$C => $C=-\frac{3}{20}$

$\therefore \frac{2x+1}{(x-1)(3x^2+7x+2)}=\frac{1}{4(x-1)}-\frac{1}{5(x+2)}-\frac{3}{20(3x+1)}$

Hence,

$\int\frac{2x+1}{(x-1)(3x^2+7x+2)}dx$

$=\frac{1}{4}\int\frac{1}{(x-1)}dx-\frac{1}{5}\int\frac{1}{(x+2)}dx-\frac{3}{20}\int\frac{1}{(3x+1)}dx$

$=\frac{1}{4}In(x-1)-\frac{1}{5}In(x+2)-\frac{3}{20}In(3x+1)$

$=\frac{1}{20}In\frac{(x-1)^5}{(x+2)^4(3x+1)^3}$

(iv) $\frac{2x^2-x+1}{(x^2-1)(x+2)}$

Solution

$\frac{2x^2-x+1}{(x^2-1)(x+2)}=\frac{2x^2-x+1}{(x+1)(x-1)(x+2)}$

⇨ $\frac{2x^2-x+1}{(x^2-1)(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x-1)}+\frac{C}{(x+2)}$

Multiplying through by (x+1)(x-1)(x+2)

$2x^2-x+1$= A(x-1)(x+2)+B(x+1)(x+2)+C(x+1)(x-1)

Putting x = -1; 4 = -2A => $A=-2$

Putting x = 1; 2 = 6B => $B=\frac{1}{3}$

Putting x = -2; 11 = 3C => $C=\frac{11}{3}$

$\therefore \frac{2x^2-x+1}{(x^2-1)(x+2)}=\frac{1}{3(x-1)}-\frac{2}{(x+1)}+\frac{11}{3(x+2)}$

Hence,

$\int\frac{2x^2-x+1}{(x^2-1)(x+2)}dx$

$=\frac{1}{3}\int\frac{1}{(x-1)}dx-2\int\frac{1}{(x+1)}dx+\frac{11}{3}\int\frac{1}{(x+2)}dx$

$=\frac{1}{3}In(x-1)-2In(x+1)+\frac{11}{3}In(x+2)+c$

(b) Evaluate $\int_1^3\frac{x^2+1}{x^3+4x^2+3x}dx$

Solution

$\frac{x^2+1}{x^3+4x^2+3x}=\frac{x^2+1}{x(x+1)(x+3)}$

Let $\frac{x^2+1}{x^3+4x^2+3x}=\frac{A}{x}+\frac{B}{(x-3)}+\frac{C}{(x+3)}$

Multiplying with x(x – 3)(x + 3)

x^2+ 1 = A(x – 3)(x+3)+ B(x)(x+3)+C(x)(x-3)

Putting x = 0; 1 = 3A => $A=\frac{1}{3}$

Putting x = -1; 2 = -2B => $B=-1$

Putting x = -3; 10 = 6C => $C=\frac{5}{3}$

$\therefore \frac{x^2+1}{x^3+4x^2+3x}=\frac{1}{3x}-\frac{1}{(x+1)}+\frac{5}{3(x+3)}$

Hence

$\int_1^3\frac{x^2+1}{x^3+4x^2+3x}dx$

$=\frac{1}{3}\int_1^3\frac{1}{x}dx-\int_1^3\frac{1}{(x+1)}dx+\frac{5}{3}\int_1^3\frac{1}{(x+3)}dx$

$=\left[\frac{1}{3}Inx-In(x+1)+\frac{5}{3}In(x+3)\right]_1^3$

$=\left\{\frac{1}{3}In3-In4+\frac{5}{3}In6\right\}-\left\{\frac{1}{3}In1-In2+\frac{5}{3}In4\right\}$

$=0.3488$

Denominators with Quadratic factors

Each quadratic factors (ax^2+bx+c) has a corresponding partial fraction of the form $\frac{Ax+B}{(ax^2+bx+c)}$ where a, b, c and A and B are constants.

Example 8

(a) Express $\frac{7x^2+2x-28}{(x-6)(x^2+3x+5)}$ in partial fraction.

Solution

Let $\frac{7x^2+2x-28}{(x-6)(x^2+3x+5)} = \frac{A}{x-6} + \frac{Bx+c}{x^2+3x+5}$

Multiplying through by $(x-6)(x^2+3x+5)$

$7x^2+2x-28 = A(x^2+3x+5) + (Bx+C)(x-6)$

Putting x = 6; 236= 59A, => A = 4

Equating coefficients of x^2

7 = A + B

7 = 4 + B; => B = 3

Equating constants

-28 = 5A – 6C

-28 = 20 – 6C

C = 8

$\therefore \frac{7x^2+2x-28}{(x-6)(x^2+3x+5)} = \frac{4}{x-6} + \frac{3x+8}{x^2+3x+5}$

(b) Find the integral of $f(x) = \frac{2x-1}{(x-1)(x^2+1)}$

Solution

Let $\frac{2x-1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)}$

Multiplying through by $(x-1)(x^2+1)$

$2x-1 = A(x^2+1) + (Bx+C)(x-1)$

Putting x = 1; 1 = 2A => A $=\frac{1}{2}$

Putting x= 0; -1 = A –C => C = $\frac{3}{2}$

Putting x= -1; 2A + 2B – 2C => B = $-\frac{1}{2}$

$\therefore \frac{2x-1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} + \frac{-\frac{1}{2}x+\frac{3}{2}}{(x^2+1)}$

$\frac{2x-1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} + \frac{3-x}{2(x^2+1)}$

Note the values of x = 0 and x = -1 are conveniently chosen, but the constants B and C by expansion of the expression and equating constants, i.e.

-1 = A –C => C = $\frac{3}{2}$

2 = C – B

B = $\frac{3}{2} - 2 = -\frac{1}{2}$

Thus,

$\int \frac{2x-1}{(x-1)(x^2+1)}dx$

$= \frac{1}{2}\int \frac{1}{(x-1)}dx + \frac{3}{2}\int \frac{1}{(x^2+1)}dx - \frac{1}{2}\int \frac{x}{(x^2+1)}dx$

$= \frac{1}{2}In(x-1) + \frac{3}{2}tan^{-1}x - \frac{1}{4}In(x^2+1) + c$

(c) Evaluate

(i) $\int_2^3 \frac{3+3x}{x^3-1}dx$

Solution

Note memorize the identities

$x^3 - 1 = (x-1)(x^2+x+1)$

$x^3 + 1 = (x-1)(x^2-x+1)$

Then

$\frac{3+3x}{x^3-1} = \frac{3+3x}{(x-1)(x^2+x+1)}$

$Let \frac{3+3x}{x^3-1} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)}$

Multiplying through by $(x-1)(x^2+x+1)$

$3+3x = A(x^2+x+1) + (Bx+C)(x-1)$

Putting x = 1, 6 = 3A, => A = 2

By expanding and equating coefficients

x^2: A + B = 0, => B = 0 – 2 = -2

x^0: A – C = 3, => C = 2 – 3 = -1

$\therefore \frac{3+3x}{x^3-1} = \frac{2}{(x-1)} - \frac{2x+1}{(x^2+x+1)}$

$\int_2^3 \frac{3+3x}{x^3-1}dx = 2\int_2^3 \frac{1}{(x-1)}dx - \int_2^3 \frac{2x+1}{(x^2+x+1)}$

$= [2In(x-1) - In(x^2+x+1)]_2^3$

$= 2In(2) + In\left(\frac{7}{13}\right)$

$= 0.7673$

(ii) $\int_2^3 \frac{x^2}{x^4-1}dx$

Solution

$\frac{x^2}{x^4-1} = \frac{x^2}{(x-1)(x+1)(x^2+1)}$

Let $\frac{x^2}{(x-1)(x+1)(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{Cx+D}{(x^2+1)}$

By multiplying through by $(x-1)(x+1)(x^2+1)$

$x^2 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)$

By equating coefficients

x^3: A + B + C = 0 …………………… (i)

x^2: A – B + D= 1 ………………………(ii)

x^1: A + B – C = 0 ………………………(iii)

x^0: A – B – D = 0 ………………………(iv)

Eqn. (ii) – Eqn. (iv)

2D = 2 => D = $\frac{1}{2}$

Eqn.(i)+ (iii)

2A+ 2B = 0 ………………………… (v)

Eqn. (ii) + Eqn. (iv)

2A – 2B = 1…………………………(vi)

Eqn. (v)+ Eqn. (vi)

4A = 1 => A = $\frac{1}{4}$

Eqn. (v)

B = $-\frac{1}{4}$

Eqn. (i)

C = 0

$\therefore \frac{x^2}{x^4-1} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} + \frac{1}{2(x^2+1)}$

$\int \frac{x^2}{x^4-1}dx$

$= \frac{1}{4}\int \frac{1}{(x-1)}dx - \frac{1}{4}\int \frac{1}{(x+1)}dx + \frac{1}{2}\int \frac{1}{(x^2+1)}dx$

$= \frac{1}{4}In(x-1) - \frac{1}{4}In(x+1) + \frac{1}{2}tan^{-1}x + c$

$\int_2^3 \frac{x^2}{x^4-1}dx$

$= \left[\frac{1}{4}In(x-1) - \frac{1}{4}In(x+1) + \frac{1}{2}tan^{-1}x\right]_2^3$

$= \frac{1}{4}\left[In\frac{x-1}{x+1} + 2tan^{-1}x\right]_2^3$

$= \frac{1}{4}\left\{\left[In\frac{2}{4} + 2tan^{-1}3\right] - \left[In\frac{1}{3} + 2tan^{-1}2\right]\right\}$

$= \frac{1}{4}\left[In\frac{1}{2} - In\frac{1}{3} + 2(tan^{-1}3 - tan^{-1}2)\right]$

$= \frac{1}{4}(0.405 + 0.1\pi)$

$= 0.18$

(iii) $\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)}dx$

Solution

Let $\frac{x^2+6}{(x^2+4)(x^2+9)} = \frac{Ax+B}{(x^2+4)} + \frac{Cx+D}{(x^2+9)}$

Multiplying by $(x^2+4)(x^2+9)$

$x^2+6 = (Ax+B)(x^2+9) + (Cx+D)(x^2+4)$

$x^2+6 = (A+C)x^3+(B+D)x^2+(9A+4C)x+9B+4D$

Equating coefficients

x^3: A + C = 0 ………………(i)

x^2: B+D = 1…………………(ii)

x^1: 9A + 4C = 0 …………..(iii)

x^0: 9B + 4D = 6 ………….(iv)

Solving simultaneously

A= C= 0; B=$\frac{2}{5}$ and D =$\frac{3}{5}$

$\therefore \frac{x^2+6}{(x^2+4)(x^2+9)} = \frac{2}{5(x^2+4)} + \frac{3}{5(x^2+9)}$

$\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)}dx$

$= \frac{2}{5}\int_0^1 \frac{1}{(x^2+4)}dx + \frac{3}{5}\int_0^1 \frac{1}{(x^2+9)}dx$

$= \frac{1}{5}\left[tan^{-1}\frac{1}{2}x + tan^{-1}\frac{1}{3}x\right]_0^1$

$= \frac{1}{5}tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right) = 0.1571$

Repeated factors

Each repeated factor $(ax^2+ b)^n$ in the denominator has corresponding partial fraction of the form: $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$, where a, b, A_i are constants (i = 1, 2, ….n)

Example 9

Express each of the follow in partial fraction and hence find their integrals.

(a) $\frac{4x-9}{(x-3)^2}$

Solution

Let $\frac{4x-9}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$

Multiplying through by $(x-3)^2$

4x – 9 = A(x-3) + B = Ax – 3A + B

Equating coefficients

x^1: x = 4

x^0: -3A + B = 4; B = 3

$\therefore \frac{4x-9}{(x-3)^2} = \frac{4}{x-3} + \frac{3}{(x-3)^2}$

Hence

$\int \frac{4x-9}{(x-3)^2}dx = 4\int \frac{1}{x-3}dx + 3\int (x-3)^{-2}dx$

$= 4In(x-3) - \frac{3}{x-3} + c$

(b) $\frac{3x-14}{x^2-8x+16}$

Solution

$\frac{3x-14}{x^2-8x+16} = \frac{3x-14}{(x-4)^2}$

Let $\frac{3x-14}{(x-4)^2} = \frac{A}{x-4} + \frac{B}{(x-4)^2}$

Multiplying through by $(x-4)^2$

3x – 14 = A(x-4) + B = Ax – 4A + B

Equating coefficients

x^1: x = 3

x^0: -4A + B = -14; B = -2

$\therefore \frac{3x-14}{(x-3)^2} = \frac{3}{x-3} - \frac{2}{(x-4)^2}$

Hence

$\int \frac{3x-14}{(x-4)^2}dx = 3\int \frac{1}{x-4}dx - 2\int (x-4)^{-2}dx$

$= 3In(x-4) + \frac{2}{x-4} + c$

(c) $\frac{2x^2-5x+7}{(x-2)(x-1)^2}$

Solution

Let $\frac{2x^2-5x+7}{(x-2)(x-1)^2} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

Multiplying through by $(x-2)(x-1)^2$

$2x^2-5x+7 = A(x-1)^2+B(x-2)(x-1)+C(x-2)$

Putting x = 1: 4 = -C, => C = -4

Putting x = 2: A = 5

Putting x = 0, 7 =A – 2B – 2C; B = -2

$\therefore \frac{2x^2-5x+7}{(x-2)(x-1)^2} = \frac{5}{x-2} - \frac{2}{x-1} - \frac{4}{(x-1)^2}$

Hence

$\int \frac{2x^2-5x+7}{(x-2)(x-1)^2}dx$

$= 5\int \frac{1}{x-2}dx - 2\int \frac{1}{x-1}dx - 4\int (x-1)^{-2}dx$

$= 5In(x-2) - 2In(x-1) - \frac{4}{x-1} + c$

(d) $\frac{7x+2}{3x^3+x^2}$

Solution

$\frac{7x+2}{3x^3+x^2} = \frac{7x+2}{x^2(3x+1)}$

Let $\frac{7x+2}{x^2(3x+1)} = \frac{A}{(3x+1)} + \frac{B}{x} + \frac{C}{x^2}$

Multiplying through by $x^2(3x+1)$

$7x+2 = Ax^2 + Bx(3x+1) + C(3x+1)$

Putting x = 0; c = 2

Putting x = $-\frac{1}{3}$; $\frac{A}{9} = 2 - \frac{7}{3}$ => A = -3

Putting x= -1; -5 = A + 2B- 2C, => B = 1

$\therefore \frac{7x+2}{x^2(3x+1)} = \frac{-3}{(3x+1)} + \frac{1}{x} + \frac{2}{x^2}$

Hence

$\int \frac{7x+2}{x^2(3x+1)} dx$

$= -\int \frac{3}{(3x+1)} dx + \int \frac{1}{x} dx + 2\int x^{-2} dx$

$= -In(3x+1) + Inx - \frac{2}{x} + c$

$= In\frac{x}{3x+3} - \frac{2}{x} + c$ln

Integration of improper fractions

Improper fractions are those whose index of the numerator is equal to or greater than that of the denominators.

They are first changed to proper fraction by long division or otherwise, before being integrated.

Example 10

(a) Express $\frac{5x^2-71}{(x+5)(x-4)}$ in partial fractions.

Hence find $\int \frac{5x^2-71}{(x+5)(x-5)} dx$

Solution

$\frac{5x^2-71}{(x+5)(x-4)} = \frac{5x^2-71}{x^2+x-20}$

Using long division

```
                5
x^2+x-20 ) 5x^2+0x-71
         - 5x^2+5x-100
           -----------
               -5 +29
```

=> $\frac{5x^2-71}{(x+5)(x-4)} = 5 + \frac{-5x+29}{x^2+x-20}$

Let $\frac{-5x+29}{(x+5)(x-4)} = \frac{A}{x+5} + \frac{B}{x-4}$

Multiplying through by (x+5)(x-4)

-5x + 29 = A(x – 4)+ B(x + 5)

Putting x = 4, B = 1

Putting x = -5; A = -6

$\therefore \frac{-5x+29}{(x+5)(x-4)} = \frac{-6}{x+5} + \frac{1}{x-4}$

Hence

$\int \frac{5x^2-71}{(x+5)(x-4)} dx$

$= 5\int dx - 6\int \frac{1}{x+5} dx + \int \frac{1}{x-4} dx$

$= 5x - 6In(x+5) + In(x-4) + c$

(b) Evaluate $\int_0^1 \frac{3-2x}{1+x} dx$

Solution

$\frac{3-2x}{1+x} = \frac{-2x+3}{x+1}$

Using long division

```
      -2
x+1 ) -2x+3
      -2x -2
      ------
          5
```

$\therefore \frac{3-2x}{1+x} = -2 + \frac{5}{x+1}$

Hence

$\int_0^1 \frac{3-2x}{1+x} dx = -2\int_0^1 dx + 5\int_0^1 \frac{1}{x+1} dx$

$= [-2x + 5In(x+1)]_0^1$

= -2+ 5In2

= 1.4657

Revision exercise 7

1. Express the following into partial fraction

(a) $\frac{8x}{x^2-4x-12}$ $\left[\frac{6}{x-6} + \frac{2}{x+2}\right]$

(b) $\frac{x^4-x^3+x^2+1}{x^3+x}$ $\left[x - 1 + \frac{1}{x} + \frac{x-1}{x^2+1}\right]$

(c) $\frac{5x-1}{2x^2+x} - 10$ $\left[\frac{3}{2x+5} + \frac{1}{x-2}\right]$

(d) $\frac{2x^2-7x+1}{(2x+1)(2x-1)(x-2)}$ $\left[\frac{1}{2x+1} + \frac{2}{3(2x-1)} - \frac{1}{3(x-2)}\right]$

(e) $\frac{6x+7}{(x^2+2)(x+3)}$ $\left[\frac{x+3}{x^2+2} - \frac{1}{x+3}\right]$

(f) $\frac{5x+7}{(x+1)^2(x+2)}$ $\left[\frac{3}{x+1} + \frac{2}{(x+1)^2} - \frac{3}{x+2}\right]$

(g) $\frac{2x^3+3x^2-x-4}{x^2(x+1)}$ $\left[2 + \frac{3}{x} + \frac{4}{x^2} - \frac{2}{x+1}\right]$

2. Find

(a) $\int \frac{x^2}{x^4-1}dx$ $\left[\frac{1}{4}In\left(\frac{x-1}{x+1}\right)+tan^{-1}x+c\right]$

(b) $\int \frac{x^2-4}{(x+1)^2(x-5)}dx$

$\left[\frac{5}{12}In(x+1)-\frac{1}{2(x+1)}+\frac{7}{12}In(x-5)\right]$

(c) $\int \frac{3x^2+x+1}{(x-2)(x+1)^3}dx$

$\left[\frac{5}{9}In(x-2)-\frac{5}{9}In(x+1)-\frac{4}{3(x+1)}+\frac{1}{2(x+1)^2}\right]$

(d) $\int \frac{x^4-x^3+x^2+1}{x^3+x}dx$

$\left[\frac{x^2}{2}-x+Inx+tan^{-1}x-\frac{1}{2}In(1+x^2)+c\right]$

(e) $\int \frac{5x-1}{2x^2+x-10}dx$

$\left[\frac{3}{2}In(2x+5)+In(x-2)+c\right]$

(f) $\int \frac{x^2-9x+2}{(x11)(x-1)(x-2)}dx$

$[2In(x+1)+3In(x-1)-4In(x-2)]+c$

(g) $\int \frac{9x+7}{(2x^2+3)(x+2)}dx$

$\left[\frac{1}{2}In(2x^2+3)+\frac{5}{\sqrt{10}}tan^{-1}\left(\sqrt{\frac{2}{3}}x\right)-In(x+2)+c\right]$

(h) $\int \frac{7+5x-6x^2}{(2x+1)^2(x+2)}dx$

$\left[\frac{3}{2}In(2x+1)-\frac{1}{2x+1}-3In(x+2)+c\right]$

(i) $\int \frac{x^2+7x-14}{(x+5)(x-3)}dx$

$[x+3In(x+5)+2In(x-3)+c]$

3. Evaluate

(a) $\int_0^2 \frac{3x^4+7x^3+8x^2+53-186}{(x+4)(x^2+9)}dx$ [-4.5489]

(b) $\int_2^3 \frac{x^2}{x^4-1}dx$ [0.18]

(c) $\int_1^3 \frac{x^2+1}{x^3+4x^2+3x}$dx [0.3489]

(d) $\int_0^1 \frac{x^3}{x^2+1}dx$ [0.1535]

(e) $\int_6^7 \frac{x^2-4}{(x+1)^2(x-5)}dx$ [0.4689]

(f) $\int_3^4 \frac{3x^2+x+1}{(x-2)(x+1)^3}dx$ [0.3165]

(g) $\int_0^2 \frac{8x}{x^2-4x-12}dx$ [1.05]

Integration by parts

This stems from differentiating the product of a function, y = uv,

$\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}$

$u\frac{dv}{dx}=\frac{d}{dx}(uv)-v\frac{du}{dx}$

$\int u\frac{dv}{dx}dx=\int \frac{d}{dx}(uv)dx-\int v\frac{du}{dx}dx$

$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$

Or simply $\int udv=uv-\int vdu$

The function chosen as u should be easily differentiated whereas the other function chosen as v should be easily integrated.

The above expression of the integration by parts can be summarized by using a technique of integration by parts

This is summarized in the table below

Sign	Differentiate	Integrates
+	u_1	$\frac{dv}{dx}$
-	u_2	v_1
+	u_3	v_2
-	u_4	v_3

NB: the signs change as +, -, + etc.

The u function is differentiated until a zero value is obtained otherwise we continue with differentiation.

The integral of the function is equal to the sum of result shown in the table above.

Integration by parts is applied in the following areas:

Integration products of polynomials by parts

Example 11

(a) Find

(i) $\int x(x+2)^3dx$

Solution

Let u = x and $\frac{dv}{dx} = (x+2)^3$

$\frac{du}{dx} = 1$; v = $\frac{1}{4}(x+2)^4$

From $\int udv = uv - \int vdu$

$\int x(x+2)^3dx$

$=\frac{1}{4}x(x+2)^4 - \int 1.\frac{1}{4}(x+2)^4dx$

$=\frac{1}{4}x(x+2)^4 - \frac{1}{4}\int(x+2)^4dx$

$=\frac{1}{4}x(x+2)^4 - \frac{1}{20}(x+2)^5 + c$

$=\frac{1}{2o}(x+2)^4(5x - x - 2) + c$

$=\frac{1}{2o}(x+2)^4(4x-2) + c$

$=\frac{1}{1o}(x+2)^4(x-1) + c$

Or by using basic techniques

Sign	Differentiate	Integrates
+	x	$(x+2)^3$
-	1	$\frac{1}{4}(x+2)^4$
+	0	$\frac{1}{20}(x+2)^5$

$\int x(x+2)^3dx = \frac{1}{4}x(x+2)^4 - \frac{1}{20}(x+2)^5 + c$

$=\frac{1}{1o}(x+2)^4(x-1) + c$

$\therefore \int x(x+2)^3dx = \frac{1}{1o}(x+2)^4(x-1) + c$

(ii) $\int(x+3)(x-4)^5dx$

Solution

Let u = (x+3) and $\frac{dv}{dx} = (x-4)^5$

$\frac{du}{dx} = 1$; v = $\frac{1}{6}(x-4)^6$

$\int(x+3)(x-4)^5dx$

$=\frac{1}{6}(x+3)(x-4)^6 - \frac{1}{6}\int 1.(x-4)^6 dx$

$=\frac{1}{6}(x+3)(x-4)^6 - \frac{1}{6}\int(x-4)^6dx$

$=\frac{1}{6}(x+3)(x-4)^6 - \frac{1}{42}(x-4)^7 + c$

$=\frac{1}{42}(x-4)^6((7(x+3) - x + 4) + c$

$=\frac{1}{42}(x-4)^6(6x+25) + c$

Sign	Differentiate	Integrates
+	x+3	$(x-4)^5$
-	1	$\frac{1}{6}(x+2)^6$
+	0	$\frac{1}{42}(x+2)^7$

$\int(x+3)(x-4)^5dx$

$=(x+3)(x-4)^5 - \frac{1}{42}x(x+2)^7 + c$

$\therefore \int(x+3)(x-4)^5dx$

$=\frac{1}{42}(x-4)^6(6x+25) + c$

(iii) $\int\frac{3x-4}{(x+2)^4}dx$

Solution

$\int\frac{3x-4}{(x+2)^4}dx = \int(3x-4)(x+2)^{-4}dx$

Let u =(3x – 4) and $\frac{dv}{dx} = (x+2)^{-4}$

$\frac{du}{dx} = 3$; v = $-\frac{1}{3}(x+2)^{-3}$

$\int\frac{3x-4}{(x+2)^4}dx$

$= -\frac{1}{3}(3x-4)(x+2)^{-3} - \int 3.-\frac{1}{3}(x+2)^{-3}dx$

$= -\frac{1}{3}(3x-4)(x+2)^{-3} + \int(x+2)^{-3}dx$

$= -\frac{1}{3}(3x-4)(x+2)^{-3} - \frac{1}{2}(x+2)^{-2} + c$

$= \frac{4-3x}{3(x+2)^3} - \frac{1}{2(x+2)^2} + c$

$= \frac{2(4-3x)-3(x+2)}{6(x+2)^3} + c$

$= \frac{2-9x}{6(x+2)^3} + c$

$\therefore \int \frac{3x-4}{(x+2)^4} dx = \frac{2-9x}{6(x+2)^3} + c$

Sign	Differentiate	Integrates
+	3x-4	$(x-4)^{-4}$
-	3	$-\frac{1}{3}(x-4)^{-3}$
+	0	$\frac{1}{6}(x-4)^{-2}$

$\int \frac{3x-4}{(x+2)^4} dx$

$= -\frac{1}{3}(3x-4)(x-4)^{-3} - \frac{1}{2}(x-4)^{-2} + c$

$= \frac{2-9x}{6(x+2)^3} + c$

(b) Evaluate

(i) $\int_0^2 x(x-3)^2 dx$

Solution

Sign	Differentiate	Integrates
+	x	$(x-3)^2$
-	1	$\frac{1}{3}(x-3)^3$
+	0	$\frac{1}{12}(x-3)^4$

$\int x(x-3)^2 dx = \frac{1}{3}x(x-3)^3 - \frac{1}{12}(x-3)^4 + c$

$= \frac{1}{12}(x-3)^3(4x - x + 3) + c$

$= \frac{1}{12}(x-3)^3(3x+3) + c$

$= \frac{1}{4}(x-3)^3(x+1) + c$

$\Rightarrow \int_0^2 x(x-3)^2 dx = \left[\frac{1}{4}(x-3)^3(x+1)\right]_0^2$

$= \frac{-3}{4} - \frac{-27}{4} = \frac{24}{4} = 6$

(i) $\int_3^6 \frac{x}{\sqrt{x-2}} dx$

Solution

Sign	Differentiate	Integrates
+	x	$(x-2)^{-\frac{1}{2}}$
-	1	$2(x-2)^{\frac{1}{2}}$
+	0	$\frac{4}{3}(x-2)^{\frac{3}{2}}$

$\int \frac{x}{\sqrt{x-2}} dx = 2x(x-2)^{\frac{1}{2}} - \frac{4}{3}(x-2)^{\frac{3}{2}} + c$

$= \frac{2}{3}(x-2)^{\frac{1}{2}}[3x - 2(x-2)] + c$

$= \frac{2}{3}(x-2)^{\frac{1}{2}}(x+4) + c$

$\Rightarrow \int_3^6 \frac{x}{\sqrt{x-2}} dx = \left[\frac{2}{3}(x-2)^{\frac{1}{2}}(x+4)\right]_3^6$

$= \left[\frac{2}{3}(6-2)^{\frac{1}{2}}(6+4)\right] - \left[\frac{2}{3}(3-2)^{\frac{1}{2}}(3+4)\right]$

$= \frac{2}{3}(20-7) = \frac{26}{3} = 8\frac{2}{3}$

Revision exercise 8

1. Integrate

(a) $\int (x-1)(x+2)^2$dx

$\left[\frac{1}{4}(x-2)(x+2)^3 + c\right]$

(b) $\int (3x-1)(2x+3)^2$dx

$\left[\frac{1}{48}(18x-17)(2x+3)^3 + c\right]$

(c) $\int (2-5x)(4-x)^4$dx

$\left[\frac{1}{30}(25x+8)(4-x)^5\right]$

(d) $\int \frac{x-2}{(2x-3)^2}$dx

$\left[\frac{1}{4}In(2x-3) + \frac{1}{4(2x-3)} + c\right]$

(e) $\int \frac{x+4}{\sqrt{3x-2}}$dx

$\left[\frac{2}{27}(3x+40)\sqrt{3x-2} + c\right]$

(f) $\int \frac{3x+1}{\sqrt{1-2x}}$dx

$In\left(\frac{2-x}{5-x}\right) + \frac{1}{5-x} + c$

2. Evaluate

(a) $\int_{-1}^1 x^2(x+3)^3 dx$ $\left[\frac{108}{5}\right]$

(b) $\int_3^6 \frac{x^2}{\sqrt{1-2x}}dx$ $\left[\frac{586}{15}\right]$

Integration products of polynomials and exponential functions by parts

Examples 12

(a) Find

(i) $\int xe^x dx$

Solution

Let u = x and $\frac{dv}{dx} = e^x$

$\frac{du}{dx} = 1$; v =e^x

$\int xe^x dx = xe^x - \int 1.e^x dx$

$= xe^x - e^x + c$

Or by using basic technique

Sign	Differentiate	Integrates
+	x	e^x
-	1	e^x
+	0	e^x

$\int xe^x dx = xe^x - e^x + c$

(ii) $\int xe^{-x} dx$

Solution

Let u = x and $\frac{dv}{dx} = e^{-x}$

$\frac{du}{dx} = 1$; v =$-e^{-x}$

$\int xe^x dx = -xe^{-x} - \int 1.-e^{-x} dx$

$= -xe^{-x} + \int e^{-x} dx$

$= -xe^x - e^x + c$

Or by using basic technique

Sign	Differentiate	Integrates
+	x	e^{-x}
-	1	$-e^{-x}$
+	0	e^x

$\int xe^x dx = -xe^x - e^x + c$

(iii) $\int xe^{3x} dx$

Solution

Let u = x and $\frac{dv}{dx} = e^{3x}$

$\frac{du}{dx} = 1$; v =$\frac{1}{3}e^{3x}$

$\int xe^{3x} dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$

Or by using basic technique

Sign	Differentiate	Integrates
+	x	e^{3x}
-	1	$\frac{1}{3}e^{3x}$
+	0	$\frac{1}{9}e^{3x}$

$\int xe^x dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$

(b) Find

(i) $\int x.2^x$dx

Solution

Let u = x and $\frac{dv}{dx} = 2^x$

$\frac{du}{dx} = 1$; v = $\frac{2^x}{In2}$

$\int x.2^x$dx=$\frac{x.2^x}{In2} - \frac{1}{In2}\int 2^x dx$

=$\frac{x.2^x}{In2} - \frac{1}{In2}\left(\frac{2^x}{In2}\right) + c$

=$\frac{2x}{In2}(x-1) + c$

(ii) $\int 3^{\sqrt{(2x-1)}}$dx

Solution

Let p =$\sqrt{(2x-1)}$, p^2 = 2x – 1

2pdp = 2dx

pdp = dx

⇨ $\int 3^{\sqrt{(2x-1)}}$dx=$\int 3^p . pdp$

Let u =p and $\frac{dv}{dp} = 3^p$

$\frac{du}{dp} = 1$, v = $\frac{3^p}{In3}$

$\int 3^p . pdp = \frac{3^p.p}{In3} - \frac{1}{In3}\int 3^p dp$

$= \frac{3^p.p}{In3} - \frac{1}{In3}\left(\frac{3^p}{In3}\right) + c$

∴ $\int 3^{\sqrt{(2x-1)}}$dx

$= \frac{\sqrt{(2x-1)}3^{\sqrt{(2x-1)}}}{In3} - \frac{1}{In3}\left(\frac{3^{\sqrt{(2x-1)}}}{In3}\right) + c$

$= \frac{3^{\sqrt{(2x-1)}}}{In3}\left(\sqrt{(2x-1)} - \frac{1}{In3}\right) + c$

(c) Evaluate

(i) $\int_0^1 xe^{-x}dx$

Solution

$\int_0^1 xe^{-x}dx = [-xe^{-x} - e^{-x}]_0^1$

$= (-e^{-1} - e^{-1}) - (0 - e^0)$

$= -2e^{-1} + 1$

$= 1 - \frac{2}{e}$

$= 0.2642$

(ii) $\int_0^1 xe^{3x}dx$

Solution

$\int_0^1 xe^{3x}dx = \left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right]_0^1$

$= \left[\frac{1}{3}e^3 - \frac{1}{9}e^3\right] - \left[0 - \frac{1}{9}e^0\right]$

$= \frac{2}{9}e^3 + \frac{1}{9} = 4.5746$

Revision exercise 10

1. Integrate each of the following with respect to x
   (a) xe^{3x} $\left[\frac{e^{3x}}{9}(3x-1) + c\right]$
   (b) x^2e^x $[e^x(x^2 - 2x + 2) + c]$
   (c) $x^3e^{x^2}$ $\left[\frac{e^{x^2}}{2}(x^2 - 1) + c\right]$
   (d) x^2e^{-2x} $\left[-\frac{e^{-2}}{4}(2x^2 + 2x + 1) + c\right]$
   (e) $\frac{x^2}{e^{-x^3}}$ $\left[-\frac{1}{3}e^{-x^3} + c\right]$
   (f) $e^x(3 + e^x)^2$ $\left[\frac{1}{3}(3 + e^x)^3 + c\right]$
2. Evaluate each of the following
   (a) $\int_0^1 x^2 e^{2x}dx$ [1.5973]
   (b) $\int_0^1 (x-1)e^x dx$ [2]

Integration products of polynomials and logarithmic functions by parts

Example 13

(a) Integrate

(i) $\int Inx^2 dx$

Solution

$\int Inx^2 dx = \int 1.Inx^2 dx$

Let u = Inx^2, $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{2x}{x^2} = \frac{2}{x}$; v = x

$\int Inx^2 dx = xInx^2 - 2\int x.\frac{1}{x}dx$

$= xInx^2 - 2x + c$

$= 2xInx - 2x + c$
$\therefore \int Inx^2 dx = 2xInx - 2x + c$

(ii) $\int xIn(x^2-1)dx$

Solution

Let u = $In(x^2-1)$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{2x}{x^2-1}$; $v = \frac{1}{2}x^2$

$\int xIn(x^2-1)dx$
$= \frac{1}{2}x^2 In(x^2-1) - \int \frac{1}{2}x^2.\frac{2x}{x^2-1}dx$
$= \frac{1}{2}x^2 In(x^2-1) - \int \frac{x^3}{x^2-1}dx$
For $\int \frac{x^3}{x^2-1}dx$
By using long division
$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$
⇨ $\int \frac{x^3}{x^2-1}dx = \int xdx + \int \frac{x}{x^2-1}dx$
$= \frac{1}{2}x^2 + \frac{1}{2}In(x^2-1) + c$
$\therefore \int xIn(x^2-1)dx$
$= \frac{1}{2}x^2 In(x^2-1) - \frac{1}{2}x^2 - \frac{1}{2}In(x^2-1) + c$

(iii) $\int x^{-3}Inxdx$

Solution

Let u = Inx and $\frac{dv}{dx} = x^{-3}$

$\frac{du}{dx} = \frac{1}{x}$ and v = $-\frac{1}{2}x^{-2}$

$\int x^{-3}Inxdx = -\frac{1}{2}x^{-2}Inx + \frac{1}{2}\int \frac{1}{x}.x^{-2}$ dx
$= -\frac{1}{2}x^{-2}Inx + \frac{1}{2}\int x^{-3}dx$
$= -\frac{1}{2}x^{-2}Inx - \frac{1}{4}x^{-2} + c$
$= -\frac{1}{4}x^{-2}(Inx + 1) + c$

(b) Evaluate $\int_1^{10} x\log_{10} xdx$

Solution

Changing from base 10 to base e

$\log_{10} x = \frac{Ine}{In10}$

$\int_1^{10} x\log_{10} xdx = \frac{1}{In10}\int_1^{10} xInxdx$

Let u= Inx; $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x}$; $v = \frac{1}{2}x^2$

$$\frac{1}{In10}\int_1^{10} xInxdx = \frac{1}{In10}\left[\frac{1}{2}x^2Inx - \frac{1}{4}x^2\right]_1^{10}$$

$= \frac{1}{In10}\left[(50In10 - 25) - \frac{1}{4}\right]$

$= \frac{1}{In10}\left[50In10 - \frac{99}{4}\right] \quad = 50 - \frac{99}{4In10}$

Revision exercise 12

1. Integrate each of the following
   (a) $xInx$ $\left[\frac{x^2}{4}(2Inx - 1) + c\right]$
   (b) x^2Inx $\left[\frac{x^2}{9}(3Inx - 1) + c\right]$
   (c) $\sqrt{x}Inx$ $\left[\frac{2}{9}\sqrt{x^3}(3Inx - 2) + c\right]$
   (d) $(Inx)^2$ $[x(2 - 2Inx + (Inx)^2) + c]$
   (e) $\frac{Inx}{x^2}$ $\left[-\frac{1}{x}(Inx + 1) + c\right]$
   (f) $3^x x$ $\left[\frac{3^x}{(In3)^2}(xIn3 - 1) + c\right]$
   (g) $x(Inx)^2$ $\left[\frac{1}{4}x^2(1 - 2Inx + 2(Inx)^2) + c\right]$
2. Evaluate the following
   (a) $\int_2^4 x^3Inxdx$ [70.9503]
   (b) $\int_2^4 (x-1)In(2x)dx$ [1.0794]
   (c) $\int_1^4 \frac{Inx}{x^2}dx$ [0.4034]

Integration of special cases involving splitting the numerator

Case 1

When a fractional integrand with quadratic denominator expressed in the form of $\frac{f(x)}{g(x)}$ is such that g(x) cannot be factorized or written in simple partial fractions, it is normally very useful to express it as a fraction by splitting the numerator.

i.e. Numerator = A(derivative of denominator + B

Example 14

Find the integral of each of the following

(a) $\int \frac{2x-1}{4x^2+3} dx$

Solution

Numerator=$A\left[\frac{d}{dx}(4x^2+3)\right]+B$

2x – 1 =A(8x)+B

Putting x = 0, B= -1

Putting x = 1, A = $\frac{1}{4}$

$\int \frac{2x-1}{4x^2+3} dx = \frac{1}{4}\int \frac{8x}{4x^2+3} dx - \int \frac{1}{4x^2+3} dx$

$= \frac{1}{4} In(4x^2+3) - \frac{\sqrt{3}}{6} tan^{-1}\left(\frac{2\sqrt{3}}{3}x\right) + c$

(b) $\int \frac{2x+3}{x^2+2x+10} dx$

Solution

Numerator=$A\left[\frac{d}{dx}(x^2+2x+10)\right]+B$

2x +3 =A(2x+2)+B

Putting x = -1, B= 1

Putting x = 0, A = 1

$\int \frac{2x+3}{x^2+2x+10} dx$

$= \frac{1}{4}\int \frac{2x+2}{x^2+2x+10} dx + \int \frac{1}{x^2+2x+10} dx$

$= In(x^2+2x+10) + \int \frac{1}{9+(x+1)^2}$dx

$= In(x^2+2x+10) + \frac{1}{3} tan^{-1}\left(\frac{x+1}{3}\right) + c$

(c) $\int \frac{x}{x^2+3x+5} dx$

Solution

Numerator=$A\left[\frac{d}{dx}(x^2+3x+5)\right]+B$

x =A(2x+3)+B

Putting x = $-\frac{3}{2}$, B= $-\frac{3}{2}$

Putting x = 0, A = $-\frac{1}{2}$

$\int \frac{x}{x^2+3x+5} dx$

$= \frac{1}{2}\int \frac{2x+3}{x^2+3x+5} dx - \frac{3}{2}\int \frac{1}{x^2+3x+5} dx$

$= \frac{1}{2} In(x^2+3x+5) - \frac{3}{2}\int \frac{1}{\frac{11}{4}+\left(x+\frac{3}{2}\right)^2}$dx

$= \frac{1}{2} In(x^2+3x+5) - \frac{3}{\sqrt{11}} tan^{-1}\left(\frac{2x+3}{\sqrt{11}}\right) + c$

(d) $\int \frac{1-2x}{9-(x+2)^2} dx$

Solution

$\int \frac{1-2x}{\sqrt{9-(x+2)^2}} dx$

$= \int \frac{1}{\sqrt{9-(x+2)^2}} dx - \int \frac{2x}{\sqrt{9-(x+2)^2}} dx$

$= sin^{-1}\left(\frac{x+2}{3}\right) - \int \frac{2x}{\sqrt{9-(x+2)^2}} dx$

For $\int \frac{2x}{\sqrt{9-(x+2)^2}} dx$

Let sinu = $\frac{x+2}{3}$

3sinu= x +2

3cosudu=dx

$\int \frac{2x}{\sqrt{9-(x+2)^2}} dx = \int \frac{2(3sinu-2)}{\sqrt{9-9sin^2u}}.3cosudu$

$= \int \frac{6sinu-4}{3\sqrt{(1-sin^2u)}}.3cosudu$

$= \int (6sinu-4)du$

$= -6cosu - 4u + c$

$= -6\sqrt{1-\left(\frac{x+2}{3}\right)^2} - 2sin^{-1}\left(\frac{x+2}{3}\right) + c$

Substituting for $\int \frac{2x}{\sqrt{9-(x+2)^2}} dx$

$\int \frac{1-2x}{\sqrt{9-(x+2)^2}} dx$

$= sin^{-1}\left(\frac{x+2}{3}\right) + -6\sqrt{1-\left(\frac{x+2}{3}\right)^2} - 4sin^{-1}\left(\frac{x+2}{3}\right) +$
c

$= 5sin^{-1}\left(\frac{x+2}{3}\right) + 6\sqrt{1-\left(\frac{x+2}{3}\right)^2} + c$

Case II

When finding the integral of fractional trigonometric function expressed in the form $\int \frac{acosx+bsinx}{c\,cosx+dsinx}$, a, b, c and d are constants, we split the numerator as:

Numerator = A(derivative of denominator)+ (denominator)

Example 15

1. Find

(a) $\int \frac{2cosx+9sinx}{3cosx+sinx}dx$

Solution

Let 2cosx+ 9sinx $=A\frac{d}{dx}$(3cosx+sinx)+B(3cosx+sinx)

2cosx+ 9sinx =A(-3sinx+cosx)+B(3cosx+sinx)

2cosx+ 9sinx =(A+3B)cosx +(-3A+B)sinx

Equating coefficients:

For cosx: A+3B = 2 .... (i)

For sinx: -3A+B = 9.....................(ii)

Solving Eqn. (i) and Eqn. (ii) simultaneously

A= $-\frac{5}{2}$ and B = $\frac{3}{2}$

⇨ $\int \frac{2cosx+9sinx}{3cosx+sinx}dx$

$=-\frac{5}{2}\int \frac{-3sinx+cosx}{3cosx+sinx}dx + \frac{3}{2}\int \frac{3cosx+sinx}{3cosx+sinx}$

$=-\frac{5}{2}$ln(3cosx+sinx)+ $\frac{3}{2}x + c$

(b) $\int \frac{3sinx}{4cosx-sinx}dx$

Solution

Let 3sinx = $A\frac{d}{dx}$(4cosx–sinx) + B(4cosx–sinx)

3sinx = A(-4sinx–cosx) + B (4cosx – sinx)

3sinx = (-A+B)cosx +(-4A-B)sinx

Equating coefficients

For cosx: -A + 4B = 0 .... (i)

For sin x: -4A- B = 3 .... (ii)

Solving Eqn. (i) and Eqn. (ii) simultaneously

A= $-\frac{12}{17}$ and B= $-\frac{3}{17}$

$\int \frac{3sinx}{4cosx-sinx}dx$

$= -\frac{12}{17}\int \frac{-4sinx-cosx}{4cosx-sinx}dx - \frac{3}{17}\int \frac{4cosx-sinx}{4cosx-sinx}dx$

$= -\frac{12}{17}In(4cosx - sinx) - \frac{3}{17}x$+c

2. Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cosx-sinx}{3cosx+2sinx}dx$

Solution

Let 3sinx = $A\frac{d}{dx}$(3cosx+2sinx) + B(3cosx+2sinx)

cosx-sinx = A(-3sinx+2cosx) + B (3cosx +2sinx)

cosx- sinx = (2A+3B)cosx +(-3A+2B)sinx

Equating coefficients

For cosx: 2A + 3B = 1 .... (i)

For sin x: -3A+2B = -1 .... (ii)

Solving Eqn. (i) and Eqn. (ii) simultaneously

A= $\frac{5}{13}$ and B= $-\frac{1}{13}$

$\int \frac{cosx-sinx}{3cosx+2sinx}dx$

$=\frac{5}{13}\int \frac{2cosx-3sinx}{3cosx+2sinx}dx + \frac{1}{13}\int \frac{3cosx+2sinx}{3cosx+2sinx}dx$

$=\frac{5}{13}In(3cosx + 2sinx) + \frac{1}{13}x + c$

$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cosx-sinx}{3cosx+2sinx}dx$

$=\left[\frac{5}{13}In(3cosx + 2sinx) + \frac{1}{13}x + c\right]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}$

$=\left[\frac{5}{13}In\left(3cos\frac{\pi}{2}+2sin\frac{\pi}{2}\right)+\frac{1}{13}.\frac{\pi}{2}\right]-\left[\frac{5}{13}In\left(3cos\frac{\pi}{6}+2sin\frac{\pi}{6}\right)+\frac{1}{13}.\frac{\pi}{6}\right]$

$=\left[\frac{5}{13}In2+\frac{\pi}{26}\right]-\left[\frac{5}{13}In\frac{2+\sqrt{3}}{2}+\frac{\pi}{78}\right]$

$=\frac{5}{13}In\left(\frac{4}{2+3\sqrt{3}}\right)+\frac{\pi}{39}$

Revision exercise 13

1. Integrate each of the following

(a) $\int\frac{x+2}{x^2+2x+4}dx$

$\left[\frac{1}{2}In(x^2+2x+4)+\frac{\sqrt{3}}{3}tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+c\right]$

(b) $\int\frac{x}{x^2-x+3}dx$

$\left[\frac{1}{2}In(x^2-x+3)+\frac{2}{\sqrt{11}}tan^{-1}\left(\frac{2x-1}{\sqrt{11}}\right)+c\right]$

(c) $\int\frac{2(x+1)}{x^2+4x+8}dx$

$\left[In(x^2+4x+8)-tan^{-1}\left(\frac{x+2}{2}\right)+c\right]$

(d) $\int\frac{5x+7}{x^2+4x+8}dx$

$\left[\frac{5}{2}In(x^2+4x+8)-\frac{3}{2}tan^{-1}\left(\frac{x+2}{2}\right)+c\right]$

2. Integrate the following

(a) $\int\frac{cosx-2sinx}{3cosx+4sinx}dx$

$\left[\frac{2}{5}In(4sinx+3cox)-\frac{1}{5}x+c\right]$

(b) $\int\frac{cosx}{2cosx-sinx}dx$

$\left[-\frac{1}{5}In(2cosx-sinx)+\frac{2}{5}x+c\right]$

(c) $\int\frac{cosx}{cosx-2sinx}dx$

$\left[-\frac{2}{5}In(cosx-2sinx)-\frac{1}{5}x+c\right]$

(d) $\int\frac{2cosx+sinx}{4cosx+3sinx}dx$

$\left[-\frac{14}{15}In(4cosx+3sinx)-\frac{11}{5}x+c\right]$

Topical revision questions

1. Find

(a) $\int sinxdx$ $\left[xsin^{-1}x+\sqrt{1-x^2}+c\right]$

(b) $\int xsec^2xdx$ $[xtanx+Incosx+c]$

(c) $\int\frac{x^2}{\sqrt{1-x^2}}dx$ $\left[\sqrt{1-x^2}\left(\frac{-2-x^2}{3}\right)+c\right]$

(d) $\int In(x^2-4)dx$

$\left[xIn(x^2-4)-2x+2\left(In\frac{x+2}{x-2}\right)+c\right]$

(e) $\int\frac{dx}{3-2cosx}dx$ $\left[\frac{2}{\sqrt{5}}tan^{-1}\left(\sqrt{5}tan\frac{x}{2}\right)+c\right]$

(f) $\int 3^{\sqrt{2x-1}}dx$

$\left[\frac{3^{\sqrt{2x-1}}}{In3}\left(\sqrt{2x-1}-\frac{1}{In3}\right)+c\right]$

(g) $\int sin^2xdx$ $\left[\frac{1}{2}\left(x-\frac{1}{2}sin2x\right)+c\right]$

(h) $\int tan^3xdx$ $\left[\frac{1}{2}tan^2x-Incosx+c\right]$

(i) $\int\frac{4x^2}{\sqrt{1-x^6}}dx$ $\left[\frac{4}{3}sin^{-1}(x^3)+c\right]$

(j) $\int\frac{x^2}{x^4-1}dx$ $\left[\frac{1}{4}In\left(\frac{x-1}{x+1}\right)+\frac{1}{2}tan^{-1}x+c\right]$

(k) $\int\frac{2x}{\sqrt{x^2+4}}dx$ $\left[2\sqrt{x^2+4}+c\right]$

(l) $\int xInxdx$ $\left[\frac{x^2}{2}Inx-\frac{x^2}{4}+c\right]$

(m) $\int x^3e^{x^4}dx$ $\left[\frac{1}{4}e^{x^4}+c\right]$

(n) $\int\frac{1}{1+sin^2x}$ $\left[\frac{\sqrt{2}}{2}tan^{-1}(\sqrt{2}tanx)+c\right]$

(o) $\int Inxdx$ $[x(Inx-1)+c]$

(p) $\int x^2sin2xdx$

$\left[-\frac{1}{2}x^2cos2x+\frac{1}{2}xsin2x+\frac{1}{4}cos2x+c\right]$

(q) $\int Inx^2dx$ $[2x(Inx-1)+c]$

(r) $\int\frac{dx}{e^x-1}$ $[In(1-e^{-x})+c]$

(s) $\int\frac{x^2}{(1+x^2)^{\frac{1}{2}}}dx$ $\left[\frac{1}{3}(1+x^2)^{\frac{1}{2}}(x^2-2)+c\right]$

(t) $\int\frac{dx}{1-cosx}$ $\left[-cot\left(\frac{x}{2}\right)+c\right]$

(u) $\int\frac{x^4-x^3+x^2+1}{x^3+x}dx$

$\left[\frac{x^2}{2} - x + Inx + tan^{-1}x - \frac{1}{2}In(1+x^2) + c\right]$

(v) $\int \frac{sin^{-1}2x}{\sqrt{1-4x^2}}dx$ $\left[\left(\frac{sin^{-1}2x}{2}\right)^2 + c\right]$

(w) $\int x(1- x^2)^{\frac{1}{2}}dx$ $\left[\frac{1}{3}(1- x^2)^{\frac{3}{2}} + c\right]$

(x) $\int \frac{1+\sqrt{x}}{2\sqrt{x}}$dx $\left[\sqrt{x} + \frac{x}{2} + c\right]$

(y) $\int x^2e^x dx$ $[x^2e^x - 2xe^x + 2e^x + c]$

(z) $\int \frac{dx}{x^2\sqrt{(25-x^2)}}dx$ $\left[-\frac{1}{25}\left(\frac{5\sqrt{25-x^2}}{x^2}\right) + C\right]$

2. Evaluate
   (a) $\int_0^{\frac{\pi}{2}} xcos^2xdx$ [0.3669]
   (b) $\int_1^{\sqrt{3}}(x + tanx)dx$ [1.0003]
   (c) $\int_0^{\frac{\pi}{2}} sin2xcosxdx$ $\left[\frac{2}{3}\right]$
   (d) $\int_1^3 \frac{x^2+1}{x^3+4x^2+3x}dx$ [0.3489]
   (e) $\int_0^1 \frac{x}{\sqrt{1+x}}dx$ [0.3905]
   (f) $\int_0^{\frac{\pi}{2}} \frac{cosx}{1+sin^2x}dx$ [0.7854]
   (g) $\int_0^{\frac{\pi}{6}} sinxsin3xdx$ [0.1083]
   (h) $\int_0^1 \frac{x^3}{x^2+1}dx$ [0.15345]
   (i) $\int_0^{\sqrt{\frac{\pi}{2}}} 2xcosx^2dx$ [1]
   (j) $\int_0^2 \frac{8x}{x^2-4x-12}dx$ [1.05]
   (k) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+sinx+cosx}$ $[In2]$
   (l) $\int_0^{\frac{\pi}{2}} sin2xcosxdx$ $\left[\frac{2}{3}\right]$
   (m) $\int_4^6 \frac{dx}{x^2-2x-3}$ [0.1905]
   (n) $\int_0^{\frac{\pi}{2}} xsin^2 2xdx$ $\left[\frac{\pi^2}{16}\right]$
   (o) $\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{x^4-x^2}}dx$ [2]
   (p) $\int_1^3 \frac{3x^2+4x+1}{x^3+2x^2+x}dx$ $[In12]$
   (q) $\int_0^{\frac{\pi}{2}} x^2sinxdx$ $[\pi - 2]$
   (r) $\int_0^1 xe^{2x}dx$ [2.0973]
   (s) $\int_{\frac{\pi}{3}}^{\pi} xsinxdx$ [2.7992]
   (t) $\int_{\frac{1}{2}}^1 10x\sqrt{(1- x^2)}\,dx$ [2.165]
   (u) $\int_0^{\frac{\pi}{2}} sin5xcos3xdx$ $\left[\frac{1}{2}\right]$
   (v) $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{9+4x^2}dx$ $\left[\frac{\pi}{36}\right]$
3. Show that
   (a) $\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)} = \frac{\pi}{20}$
   (b) $\int_0^{\frac{\pi}{2}} xtan^2xdx = \frac{1}{32}(8\pi - \pi^2 - 16\log_e 2)$
   (c) $\int_2^4 xInxdx = 14In2 - 3$
   (d)
4. Given that
   $\frac{3x^3+2x^2-6x-2}{(x^2+x-2)(x^2-2)} = \frac{1}{x+2} + \frac{B}{x-1} + \frac{Cx+D}{x^2-2}$
   Determine the values of A, B, C, D
   Hence evaluate $\int_3^4 \frac{3x^3+2x^2-6x-2}{(x^2+x-2)(x^2-2)}$dx
   $[A = B = C = 1, D = 0; 2.4770]$
5. Use the substitution of x = $\frac{1}{u}$ to evaluate
   $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ $\left[\frac{\pi}{3}\right]$
6. Express $\frac{x^3-3}{(x-2)(x^2+1)}$ as partial fractions
   $\left[\frac{x^3-3}{(x-2)(x^2+1)} = 1 + \frac{1}{x-2} + \frac{x+1}{x^2+1}\right]$
   Hence find $\int \frac{x^3-3}{(x-2)(x^2+1)}dx$
   $\left[x + In(x-2) + \frac{1}{2}In(x^2+1) + tan^{-1}x + c\right]$
7. Express f(x) = $\frac{2x^2-x+14}{(4x^2-1)(x+3)}$ in partial fraction
   $\left[\frac{2x^2-x+14}{(4x^2-1)(x+3)} = \frac{-3}{2x+1} + \frac{2}{2x-1} + \frac{1}{x+3}\right]$
   Hence evaluate $\int_1^3 f(x)dx$ [0.7440]
8. Using the substitution 2x+1 = u, find
   $\int_0^1 \frac{xdx}{(2x+1)^2}$ $\left[\frac{1}{18}\right]$
9. Express
   (a) f(x) = $\frac{6x}{(x-2)(x+4)^2}$ in partial fraction
   $\left[\frac{6x}{(x-2)(x+4)^2} = \frac{1}{3(x-2)} - \frac{1}{3(x+4)} + \frac{4}{(x+4)^2}\right]$
   Hence evaluate $\int f(x)dx$
   $\left[\frac{1}{3}In\left(\frac{x-2}{x+4}\right) - \frac{4}{(x+4)} + c\right]$
   (b) f(x) = $\frac{3x^2+x+1}{(x-2)(x+1)^2}$ in partial fraction
   $\frac{3x^2+x+1}{(x-2)(x+1)^2}$
   $= \frac{5}{9(x-2)} - \frac{5}{9(x+2)} + \frac{4}{3(x+1)^2} - \frac{1}{(x+1)^3}$

Hence evaluate

$\int_3^4 \frac{3x^2+x+1}{(x-2)(x+1)^2}dx$ [0.317]

(c)

10. Using the substitution x = 3sinθ, evaluate

(a) $\int_0^3 \sqrt{\left(\frac{3+x}{3-x}\right)}\,dx$ [7.7125]

(b) $\int_0^{\pi} \frac{dx}{3+5cosx}$ [0.2747]

(e)

11. Use t= tan$\frac{1}{2}x$ to evaluate

(a) $\int_0^{\frac{\pi}{2}} \frac{dx}{3-cosx}$ [0.6755]

(b)

12. Given that $\int_0^a (x^2+2x-6)\,dx = 0$, find the value of a [a=-6]

13. Use the substitution $x^2 = \theta$ to find

$\int \frac{x}{1+\cos x^2}dx \left[\frac{1-3x}{3(x+1)^{\frac{5}{3}}(x-1)^{\frac{4}{5}}}\right]$

14. Resolve $y = \frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)}$ into partial fraction

$\left[\frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)} \equiv \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{4}{(x^2+2)}\right]$

Hence find $\int ydx$

$\left[In(x-1) + \frac{-2}{(x-1)} + \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c\right]$

15. Express f(x) = $\frac{1}{x^2(x-1)}$ in partial fraction

$\left[\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}\right]$

Hence evaluate $\int_2^3 f(x)dx$ [0.12102]

Application of integration

Determining areas enclosed between two functions and the volume of a solid of revolution

Example 16

The equation of a curve is y = 3x^2 + 2

(a) (i) Determine the turning point of the curve

Solution

At turning point $\frac{dy}{dx}(3x^2-2)$ = 0

$6x = 0$

x = 0 and y = 3(0)^2+ 2 = 2

Hence turning point = (0, 2)

(ii) Find the nature of the turning point

S0lution

$\frac{d^2y}{dx^2} = 6$

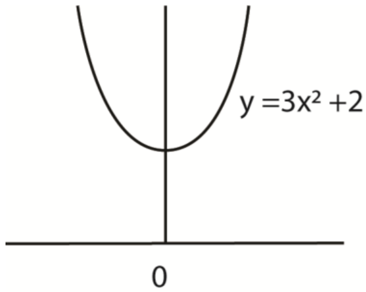
Since $\frac{d^2y}{dx^2} > 0$, the turning point is a minimum

(iii) Sketch the graph of the curve.

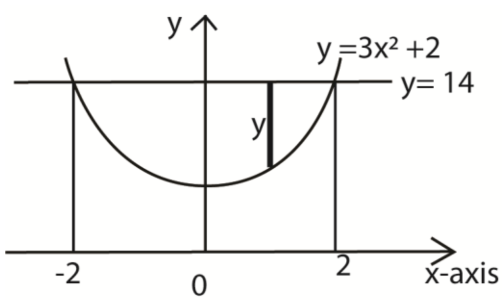
Finding x – intercepts when y = 0

3x^2 +2 = 0, x^2 = -$\frac{-2}{3}$

	0	unidentified
y	2	0



(b) The curve and the line y = 14 intercepts at the point (-2, 14) and (2, 14). Calculate the area of the region enclosed between the line and the curve.



Area of the curve and x-axis $=\int_{-2}^{2} ydx$

$= \int_{-2}^{2}(3x^2 + 2)\, dx$

$=[x^3 + 2x]_{-2}^{2}$ = (8 + 4) – (-8 – 4) =24

Total area of the rectangle = 4 x 14

= 56

Area of the curve and the line (y = 14)

56 – 24 = 36unit^2

Example 17

1. (a) On the same axis, sketch the curves y = x(x+ 2) and y = x(4x –x)

Solution

Considering y = x(x+2)

If x = 0, y = 0

If y = 0

x(x + 2) = 0

x = 0 or x = -2

(x, y) = (0,0) and (-2, 0)

Finding the turning point

y = x^2 + 2x

$\frac{dy}{dx} = 2x + 2$

⇨ 2x + 2 = 0

x = -1; y = 1 – 2 = -1

The turning point = (-1, -1)

Finding the nature of the turning point;

x	-2	-1	0
$\frac{dy}{dx}$	-2	0	2

min

OR

By finding the second derivative

$\frac{dy}{dx} = 2x + 2$

$\frac{d^2y}{dx^2} = 2$ (min)

Hence the turning point is minimum

Considering y = x(x – 4)

Finding intercept

If x = 0, y = 0

(x, y) = (0, 0)

If y = 0

x(x – 4) = 0

Either x =0 or x = 4

(x,y) =(0, 0) and (0, 4)

Finding the turning point

y = 4x – x^2

$\frac{dy}{dx} = 4 - 2x$

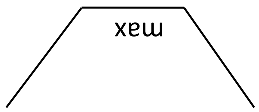
⇨ $0 = 4 - 2x$

x = 2, y = 4

The turning point = (2, 4)

Finding the nature of the turning point

x	1	2	3
$\frac{dy}{dx}$	3	0	-2



Or

By finding the second derivative;

$\frac{dy}{dx} = 4 - 2x$

$\frac{d^2y}{dx^2} = -2$ (max)

Hence the turning point is maximum

Finding the point of intersection of the curves

x^2 + 2x = 4x – x^2

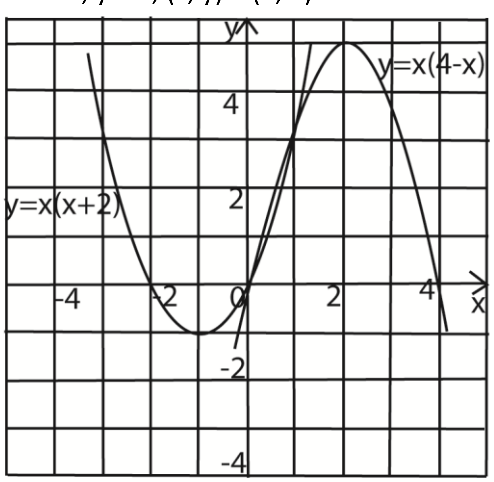
2x^2 – 2x = 0

2x(x – 1) = 0

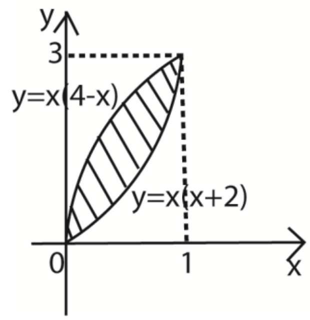
Either x = 0 or x = 1;

If x = 0, y = 0; (x, y) = (0, 0)

If x = 1, y = 3; (x, y) = (1, 3)



(b) Find the area enclosed by the two curve in (a)



Area of shaded region

$=\int_0^1 (4x - x^2)dx - \int_0^1 (x^2 + 2x)dx$

$=\left[2x^2 - \frac{x^3}{3}\right]_0^1 - \left[\frac{x^3}{3} + x^2\right]_0^1$

$=\left[2 - \frac{1}{3}\right]_0^1 - \left[\frac{1}{3} + 1\right]_0^1 = \frac{1}{3}$sq. units

(c) Determine the volume of the solid generated when the area in (a) is rotated about the x-axis.

Solution

V $=\pi \int_0^1 [(4x - x^2)^2 - (x^2 + 2x)^2]$

$=\pi \int_0^1 (12x^2 - 12x^3)$

$=\pi[4x^3 + 3x^4]_0^1$

$=\pi(4 - 3)$

$=\pi$ *cubic units*

Example 18

A curve has the equation $y = \frac{2}{1+x^2}$.

(a) Determine the nature of the turning point on the curve.

$y = \frac{2}{1+x^2} = 2(1 + x^2)^{-1}$

$\frac{dy}{dx} = -2(1 + x^2)^{-2}.2x = \frac{-4x}{(1+x^2)^2}$

Or

$y = \frac{2}{1+x^2}$

$Iny = In2 - In\ (1 + x^2)$

$\frac{1}{y}dy = \left(0 - \frac{2x}{1+x^2}\right)dx$

$\frac{dy}{dx} = \frac{-2x}{1+x^2}.\frac{2}{1+x^2} = \frac{-4x}{(1+x^2)^2}$

At turning points, $\frac{dy}{dx} = 0$

⇨ $\frac{-4x}{(1+x^2)^2} = 0$

$x = 0$

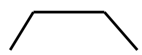
$y = \frac{2}{1+0} = 2$

Hence the turning points is (x, y) = (0, 2)

Finding the nature of the turning point

X	-1	0	1
$\frac{dy}{dx}$	+1	0	-1

Max



Hence the nature of the turning point is maximum

(b) Find the equation of the asymptote. Hence sketch the curve.

$y = \frac{2}{1+x^2}$

For vertical asymptote

$1 + x^2 = 0$

$x^2 = -1$

$x = \sqrt{-1}$ which does not exist

Or

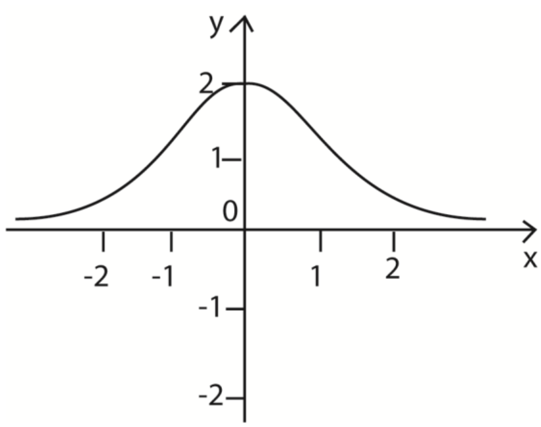
Since $1 + x^2$ cannot be factorized, there is no vertical asymptote

For horizontal asymptote

$y = \frac{2}{1+x^2}$

As $x \to \infty$, $y \to 0$

Hence the horizontal asymptote is y = 0



Example 19

(a) Sketch the curve $y = x^3 - 8$

Solution

$y = x^3 - 8$

Intercepts

When x = 0, y = -8

When y = 0, x = 2

(x, y) = (2, 0)

Turning point: $\frac{dy}{dx} = 3x^2$

$3x^2 = 0$

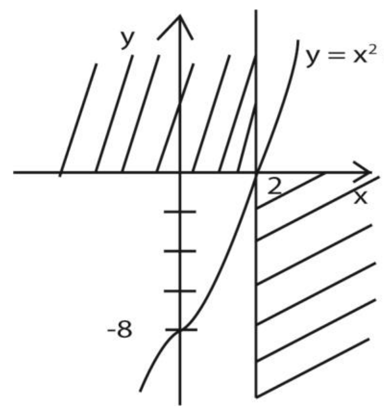
x = 0

$\frac{d^2y}{dx^2} = 6x$

$\frac{d^2y}{dx^2} = 0, x = 0$

Point of reflection= (0, 8)

	x < 2	x > 2
y	-	+



(b) The area enclosed by the curve in (a), the y-axis and x-axis is rotate about the line y = 0 through 360^0. Determine the volume of the solid generated. (04 marks)

V = $\pi \int_0^2 y^2 dx$

$= \pi \int_0^2 (x^3 - 8)^2 dx$

$= \pi \int_0^2 (x^6 - 16x^3 + 64) dx$

$= \pi \left[\frac{x^7}{7} - 4x^4 + 64x\right]_0^2$

$= \pi \left(\frac{128}{7} - 64 + 128\right)$

$= \frac{576\pi}{7}$ = 250.5082 units^3

Revision exercise

1. (i) Find the Cartesian equation of the curve
   $x = \frac{1+t}{1-t}$ and $y = \frac{2t^2}{1-t}$= $\left[y = \frac{(x-1)^2}{1+x}\right]$
   (ii) Sketch the curve
   (iii) Find the area enclosed between the curve and the line y = 1 [1.95 sq units]
2. (i) Sketch the curve $y = \frac{x}{x^2+1}$ from -1≤x≤5
   (ii) Find the area enclosed by the curve and x-axis [64/3 sq units]
3. Find the volume of the solid of revolution formed by the area enclosed by the curve y = x(1+x), the x-axis, the line x = 2 and x = 3 through four right angles about the x-axis [81.033π cubic units]
4. Sketch the curve y = x - $\frac{8}{x^2}$ for x>0, showing any asymptote. Find the area enclosed by x-axis, the line x =4 and the curve y = x - $\frac{8}{x^2}$ [10 sq. units]
   If this area is now rotated about the x-axis through 3600, determine the volume of the solid generated, correct to 3 significant figures. [42.1 cubic units]
5. Show that the tangents at (-1, 3) and (1, 3) on the curve y = 2x2 + x + 2 pass through the origin. Find the area enclosed by the curve and the two tangents [4/3 sq. units)
6. In a culture of bacteria the rate of population growth is proportional to the population at time t. the population double every day. Given that the initial population is P0 is one million, determine the day when the population will be 100 million. [7th day]
7. Show that the tangents to the curve 4-2x-2x^2 at (-1, 4) and $\left(\frac{1}{2}, 2\frac{1}{2}\right)$ respectively pass through the point $\left(-\frac{1}{4}, 5\frac{1}{2}\right)$.
   Calculate the area of the curve enclosed between the curve and x-axis (9 sq. units)

Determine displacement, velocity and acceleration

Example 20

A force F = (2t i + j – 3t k)N acts on a particle of mass 2kg. The particle is initially at a point (0,0,0) and moving with a velocity (i + 2j-k)ms^{-1}. Determine the:

(a) Magnitude of the acceleration of the particle after 2 seconds (04marks)

F = (2t i + j – 3t k) = $\begin{pmatrix} 2t \\ 1 \\ -3t \end{pmatrix}$N

a = $\frac{F}{m} = \frac{1}{2}\begin{pmatrix} 2t \\ 1 \\ -3t \end{pmatrix} = \begin{pmatrix} t \\ 0.5 \\ -1.5t \end{pmatrix} ms^{-1}$

At t = 2s

$\underline{a}$=2i + 0.5j – 3k

$|\underline{a}| = \sqrt{2^2 + 0.5^2 + (-3)^2}$ = 3.64ms^{-2}

(b) Velocity of the particle after 2seconds (04marks)

$\underline{v} = \int \underline{a}dt = \int \begin{pmatrix} t \\ 0.5 \\ -1.5t \end{pmatrix} dt$

= $\begin{pmatrix} 0.5t^2 \\ 0.5t \\ -7.5t^2 \end{pmatrix}$ + C

At t = 0 initial velocity = (i + 2j –k)

$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + C$ => C = $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

∴ $\underline{v}$= $\begin{pmatrix} 0.5t^2+1 \\ 0.5t+2 \\ -7.5t^2-1 \end{pmatrix}$

At t = 2s

$$\underline{v} = \begin{pmatrix} 0.5(2)^2 + 1 \\ 0.5(2) + 2 \\ -7.5(2)^2 - 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix} ms^{-1}$$

(c) Displacement of the particle after 2 seconds (04marks)

$$\underline{r} = \int \underline{v} dt \int \begin{pmatrix} 0.5t^2 + 1 \\ 0.5t + 2 \\ -7.5t^2 - 1 \end{pmatrix} dt$$

$$= \begin{pmatrix} \frac{t^3}{6} + t \\ \frac{t^2}{4} + 2t \\ -t - \frac{t^3}{4} \end{pmatrix} + C$$

At t = 0; $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + C$

$$C = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} t^3 + t \\ \frac{t^2}{4} + 2t \\ -t - \frac{t^3}{4} \end{pmatrix}$$

At t = 2s

$$\underline{r} = \begin{pmatrix} \frac{2^3}{6} + 2 \\ \frac{2^2}{4} + 2x2 \\ -2 - \frac{2^3}{4} \end{pmatrix} = \begin{pmatrix} \frac{10}{3} \\ 5 \\ -4 \end{pmatrix} m$$

Revision exercise

1. A particle of mass 3g is acted on by a force F = 6i – 36t^{2}j + 54k Newton at time t. At time t = 0 the particle is at the point with position vector i – 5j –k and its velocity is 3i + 3j ms^{-1}.
   Determine the
   (a) position vector of the particle at time t= 1second (09marks) [5i -3j + 2k]
   (b) distance of the particle from the origin at time t = 1 second [6.164]

Thank you

Dr. Bbosa Science