SENIOR FIVE TERM 3

TOPIC 1/4: Series

Competency: The learner applies principles and techniques of series and binomial expansions to understand and solve mathematical and scientific problems in the real world.

Contents

Mathematical Series

Introductions

Numbers arranged in a definite order a sequence. Each number in the sequence is derived from a particular rule.

The terms below are examples of sequences

(a) 1, 3, 5, 7, 9 ….. is a sequence of odd numbers

Arithmetic progression (A.P)

This is a series in which each term is obtained from the preceding one by addition or subtraction of a constant quantity.

The series 1 + 3 + 5 + 7 + 9 ….. is an A.P

Note the following in an A.P

(i) The first term of an A.P is denoted a. the first letter of the English alphabet
(ii) There is a common difference d. in the progression, a = 1 and d = 2.
(iii) Given the first term, a and the common difference, d
  1^{st} term = a
  2^{nd} term = a + d
  3^{rd} term = a + 2d
  n^{th} term (U_n) = a + (n – 1)d

Example 1

Find the 30^{th} term of a series that has an n^{th} term given by $\frac{1}{2}(32-n)$

Solution

$U_n = \frac{1}{2}(32-n)$

$U_{30} = \frac{1}{2}(32-30) = 1$

(b) 2, 3, 5, 7, 11 … is a sequence of prime number
(c) 4, 16, 64 …. is a sequence formed by multiplying the proceding number by 4 to give the next number

Series are categorized into two:

- Arithmetic progression (A.P)
- Geometric progression

Example 2

The first term of an arithmetic progression (A.P) is 73 and the 9^{th} term is 25. Determine the common difference

Solution

U_n = a + (n – 1)d

25 = 73 + (9 – 1)d

25 = 73 + 8d

d = -6

Example 3

The 3^{rd}, 5^{th} and 8^{th} terms of A.P are 3n + 8, n + 34, and n^3 + 15 respectively. Find the value of n and hence the common difference of the A.P

Solution

a + 2d = 3n + 8 ……………… (i)

a + 4d = n + 24 ………………(ii)

a + 7d = n^3 + 15 …………….(iii)

Eqn. (ii) – eqn. (i)

2d = -2n + 16

d = -n + 8………………….. (iv)

eqn. (iii) – eqn. (i)

5d = n^3 – 3n + 7 ………………… (v)

Substituting eqn. (iv) into eqn. (v)

$5(-n + 8) = n^3 – 3n + 7$

$n^3 + 2n – 33 = 0$

By factorizing the equation

$(n- 3)(n^2 + 3n + 11) = 0$

Either $n – 3 = 0$ or $n^2 + 3n + 11 = 0$

$n = 3$ since $n^2 + 3n + 11 = 0$ has no real roots

Substituting for n in eqn. (iv)

$d = -n + 8$

$d = -3 + 8 = 5$

Hence n = 3 and the common difference is 5

Example 4

An A.P has the first term 3, common difference -2 and nth term – 15. Find n and the $(n -3)^{th}$ term

Solution

Given, a = 3, d =-2 , $U_n = -15$

But $U_n = a + (n – 1)d$

$-15 = 3 + (n – 1) \times (– 2)$

n = 10

substitute n – 3 for n

$U_{(n – 3)} = 3 + [(10 – 3) – 1](-2)$

$= -9$

Example 5

The n^{th} term of a series is $Un = a3^n + bn + c$ given that $U_1 = 4$, $U_2 = 13$ and $U_3 = 46$, find the values of a, b and c.

Solution

By substituting for n

n = 1:

3a + b + c = 4 ....(i)

n =2

9a + 2b + c = 13 ....(ii)

n = 3

27a + 3b + 2 = 46 ....(iii)

Eqn. (ii) – eqn. (i)

6a + b = 9 .... (iv)

Eqn.(iii) – eqn (ii)

18a + b = 33 .... (v)

Eqn. (v) – eqn. (iv)

12a = 24

a = 2

Substitute for a in (v)

36 + b = 33

b = -3

Substituting for a and b in eqn. (i)

3 x 2 – 3 + c = 4

c = 1

Hence a = 2, b = -3 and c = 1

The sum of the first n terms of an A.P

There are two formulas used for finding the sum of the first n terms of the A.P depending on the terms given

Formula A

If the first term, a and common difference are given, then sum (S_n) of the first n terms is given by

$S_n = \frac{n}{2}(2a + (n - 1)d)$

Formula B

If the first term is a and the last term is L, then sum of the first n terms (S_n)is given by

$S_n = \frac{n}{2}(a + L)$

Example 6

The first term of A.P is 73 and the common difference is -6, find the number of terms that must be added to give a sum of 96

Solution

$S_n = \frac{n}{2}(2a + (n-1)d$

$\frac{n}{2}(2x\ 73 - 6(n-1) = 93$

n(73 – 3(n – 1)) = 96

$73n - 3n^2 + 3n = 96$

$3n^2 - 76n + 96 = 0$

$n = \frac{76 \pm \sqrt{76^2 - 4x\ 3\ x\ 96}}{2\ x\ 3}$

$n = \frac{76 \pm 68}{6}$

n = 24

Hence the number of terms that must be added to give a sum of 96 are 24

Example 7

The tenth term of an arithmetic progression (A.P) is 29 and the fifteenth term is 44. Find the value of the first term and the common difference. Hence find the sum of the first 30 terms

Solution

U_n = a + (n – 1)d

U_{10} = a + (10 – 1)d

a + 9d = 29 …………….(i)

U_{15} = a + (15 – 1)d

a + 14d = 44 ……….. (ii)

Eqn (ii) – eqn. (i)

5d = 15

d = 3

Substituting d in eqn. (i0

a + 9 x 2 = 29

a = 2

Sum of the first 30 terms

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_n = \frac{30}{2}(2\ x\ 2 + 3(30 - 1))$ = 1365

Example 8

The 5th term of an arithmetic progression (A.P) is 12 and the sum of the first 5 terms is 80. Determine the first term and common difference.

Solution

U_n = a + (n – 1)d [Un= nth term, a = first term and d = common difference}

U_5 = a + (5 – 1)d

a + 4d = 12 ……………………………(i)

The sum of the first n terms,

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_5 = \frac{5}{2}(2a + (5-1)d)$

10a + 20d = 80 x 2 = 160 ………(ii)

Eqn. (ii) – 5eqn. (i)

5a = 100

a = 20

Substituting for a in eqn. (i)

20 + 4d = 12

d = -2

Hence the first term = 20 and the common difference = -2

Example 9

(a) Prove that $\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$

Solution

S_n = 1 + 2 + 3 + ….. + n

+ S_n = n + (n – 1) + (n – 2)+ …. + 1

$2S_n$ = (n+1) + (n+1) + (n+1) + …. + (n+ 1)

$2S_n$ = n(n + 1)

Sn = $\frac{n}{2}(n+1)$

(b) Use your answer in (a) to deduce

(i) $\sum_{r=1}^{n}(3r-1) = \frac{n}{2}(3n+1)$

Note to deduce is to use the already existing result to work out other problems

$S_n = \frac{n}{2}[last\ term + first\ term]$

$\sum_{r=1}^{n}(3r-1) = \frac{n}{2}\left(3n-1\ + (3-1)\right)$

$= \frac{n}{2}(3n-1\ + 2)$

$= \frac{n}{2}(3n+1)$

(ii) $\sum_{r=0}^{n}(r+5) = \frac{1}{2}(n+1)(n+10$

$= \sum_{r=1}^{n}(r+5) + \sum_{r=0}(r+5)$

$= \frac{n}{2}\left((n+1)\ + (1+5)\right) + 5$

$= \frac{1}{2}(n^2+11n) + 5$

$= \frac{1}{2}(n^2+11n+10)$

$= \frac{1}{2}(n+1)(n+10)$

Inserting geometric means

Like for A.Ps, the terms inserted between given two values of a G.P are known as geometric means.

If n terms are inserted, then the total number of terms will be n+ 2 with the two extreme values representing the first and last terms respectively

Example 10

(a) Insert two geometric means between 2 and 16

Solution

1st term a = 2

4th term, ar^3 = 16

$2(r^3)$ = 18

r = 2

the second term, ar = 2 x 2 = 4

the third term, ar^2 = 2 x 2^2 = 8

(b) Insert three geometric means between 1 and 81

a = 1

the 5th term ar^4 = 81

$1(r^4)$ = 81

r = 3

the second term, ar = 1 x 3 = 3

the third term, ar^2 = 1 x 3^2 = 9

the third term, ar^3 = 1 x 3^3 = 27

Revision exercise 1

1. Find the 5th and 8th terms of a series that has an nth term given by $(-1)^n(2n + 1)$ [-11, 17]

2. The first term of an arithmetic progression (A.P) is $\frac{1}{2}$. The sixth term of th A.P is four

times the fourth term. Find the common difference of the A.P $\left[\frac{-3}{14}\right]$

3. The sum of p terms of an arithmetic progression is q and the sum of q terms is p; find the sum of p + q terms
4. (a) the first four terms of an A.P are 5, 11, 17 and 23. Find the 30th term and the sum of the first 30 terms [179, 2760]
(b) the second term of an A.P is 7 and the 7th term is -8. Find the first term, common difference and the sum of the first 14 terms [10, -3, -133]
5. (a) An A.P has the first term of 2 and common difference 5. Given that the sum of the first n terms of the progression is 119, calculate n [7]
(b) the sum of the first five terms of an A.P is $\frac{65}{2}$. Also , five times the 7th term is the same as six times the second term. Find the first term and the common difference $\left[a = 6, d = \frac{1}{4}\right]$
6. The sum of the first n terms of the a series is n(n + 2). Find the first three terms [3, 5, 7]
7. In an A.P, the 1st term is 13 and 15th term is 11. Find the common difference and sum of the first 20 terms [7, 1590]
8. (a) Show that In2^r, r = 1, 2,3, is an arithmetic progression
(b) find the sum of the first 10 terms of the progression [38.1231]
(c) Determine the least value of m for which the sum of the first 2m terms exceeds 883.7 [25]
9. In an arithmetic progression $u_1 + u_2 + u_3 + u_4 = 15$ and $u_{16} = -3$. Find the greatest integer N such that $U_N \geq 0$. Determine the sum of the first N terms of the progression. [N = 14, S_{14} = 136.5]

Geometric progression (G.P)

It is a series in which each term is obtained from the preceding one by multiplication or division by a constant quantity.

Observations

- The first term of G.P is also denoted, a
- The common ratio is r
- Given a and r
  1st term = a
  2nd term = ar
  3rd term = ar^2
  4th term = ar^3
  nth term = ar^{n-1}

The sum (S_n) of the first n terms of G.P

The sum (S_n) of the first n terms of G.P is given by

$S_n = \frac{a(r^n-1)}{r-1}$ for r > 1 or

$S_n = \frac{a(1-r^n)}{1-r}$ for r< 1

Application of Arithmetic progression

1. Finance and Savings

(i) **Loan repayments:** Many installment schemes follow AP, where payments increase or decrease by a fixed amount.
(ii) **Saving plans:** If you save money with a fixed increment each month (e.g., sh. 100, then sh. 120, then sh.140…), the total savings can be calculated using AP formulas.

Example 1: A person saves sh. 200 in January, sh. 250 in February, sh.300 in March…

This is an AP with a=200a, d=50.

Total savings in 6 months:

$S_6 = \frac{n}{2}[2a+(n-1)d] = \frac{6}{2}[400+250] = 3\times650 = 1950$

2. Population Growth (Linear)

In cases where a population grows by a fixed number each year (not percentage), AP helps predict future population size.

3. Sports and Competitions

Tournament scheduling: If matches increase by a fixed number each round, AP helps calculate total matches.

Athletics training: Increasing running distance by a fixed amount daily forms an AP.

4. Architecture and Construction

Staircases: Steps are often designed with equal height difference, forming an AP

Example A staircase has 10 steps, each 15 cm higher than the last.

Height of the 10th step:

$a+(n-1)d = 15+(10-1)(15) = 150$ cm

Seating arrangements: Rows in stadiums or theaters often increase by a fixed number of seats.

5. Science and Nature

Patterns in plants: Some leaf arrangements follow arithmetic sequences.

Temperature changes: If temperature rises or falls steadily, AP models the change.

6. Education

Marks distribution: If marks increase by a fixed difference across assignments, AP helps in analysis.

Roll numbers: Often assigned in sequential order, forming an AP.

The geometric mean (G.M)

Suppose three numbers a, b and c are consecutive terms of GP, then, the middle term is the geometric mean.

The common ratio, $r = \frac{b}{a} = \frac{c}{b}$

⇨ $b^2 = ac$

$b = \sqrt{ac}$, the geometric mean

Example 10

A GP has 3rd term 7 and 5th term 847. Find the possible values of the common ratio and the corresponding 4th terms

Solution

$U_3 = ar^2 = 7$

⇨ $ar^2 = 7$ ……………….(i)

$U_5 = ar^4 = 847$

⇨ $ar^4 = 847$ ……………(ii)

Eqn. (ii) ÷ eqn. (i)

$r^2 = 121$

$r = \pm11$

From eqn (i)

$a = \frac{7}{121}$

The 4th term, $U_4 = ar^3$

If r = 11, $U_4 = \frac{7}{121}(11)^3$ = 77

If r = -11, $U_4 = \frac{7}{121}(-11)^3$ = -77

Example 11

In a G.P the 2nd term is 15 and the 5th term is -405. Find the sum of the first 8 terms

Solution

U_2 = ar = 15

⇨ ar = 15 ……………………..(i)

$U_5 = ar^4$ = -405

⇨ ar^4 = -405

Eqn. (ii) ÷ eqn. (i)

r^3 =-27

r = -3

From eqn. (i), a = -5

Since r < 1

$S_n = \frac{a(1-r^n)}{r-1} = \frac{-5(1-(-3)^8)}{1-(-3)} = \frac{32800}{4} = 8200 =$

Example 12

In the geometric series $u_1 + u_2 + u_3 + …$

$u_1 + u_3 = 26$ and $u_3 + u_5 = 650$.

Find the possible values of u_4

Solution

$u_1 + u_3 = 26$

$a + ar^2 = 26$

$a(1 + r^2) = 26$ ………………… (i)

$u_3 + u_5 = 650$.

$ar^2 + ar^4 = 650$

$ar^2(1 + r^2) = 650$ ……………(ii)

Eqn. (ii) ÷ eqn. (i)

$r^2 = 25$

r = ±5

From eqn. (i)

a(1 + 25) = 26

a = 1

$u_4 = ar^3$

If r = 5; $u_4 = a(5)^3 = 125$

If r = 5; $u_4 = a(-5)^3 = -125$

Example 13

In a Geometric Progression (G.P), the difference between the fifth and the second term is 156. The difference between the seventh and the fourth is 1404 Find the possible values of the common ratio.

$U_5 – U_2 = 156$
$ar^4 – ar = 156$
$ar(r^3 -1) = 156$ ……………. (i)
$U_7 – U_4 = 156$
$Ar^6 – ar^3 = 1404$
$ar^3(r^3 -1) = 156$ ……………. (ii)
Eqn. (ii) ÷ eqn. (i)
$\frac{ar^3(r^3-1)}{ar(r^3-1)} = \frac{1404}{156}$
$r^2 = 9$
r =±3
∴ r = 3 and r = -3

Example 14

(a) The first three terms of a Geometric progression (G.P) are 4, 8 and 16. Determine the sum of the first ten terms of the G.P. (04marks)

solution

a = 4, ar = 6
4r = 8
r = 2
$S_n = \frac{a(1-r^n)}{r-1}$

$S_{10} = 4\left(\frac{2^{10}-1}{2-1}\right) = 4092$

(b) An Arithmetic Progression (A.P) has a common difference of 3. A Geometric Progression (G.P) has a common ration of 2. A sequence is formed by subtracting the term of the A.P from the corresponding terms of the G.P. The third term of the sequence is 4. The sixth term of the sequence is 79. Find the first term of the

(i) A.P (08 marks)

(ii) G.P (06 mars)

A.P
x, x+3, x +6, x + 9, x + 12, x+ 15, ……
G.P
y, 2y, 4y, 8y, 16y, 32y, ….
4y – (x + 6) = 4
4y – x = 10 ……………..(i)
32y – (x + 15) = 79
32y – x = 94…………. (ii)
Eqn. (ii) – Eqn. (i)
28y = 84, => y = 3
Substituting for y into eqn. (i)
12 – x = 10
x =2
(i) A.P, U_1 = 2
(ii) G.P, U_1 =3

Example 15

The sum of the first n terms of a geometric progression (G.P) is $\frac{4}{3}(4^n - 1)$. Find the n^{th} term as an integral power of 2

Solution

$S_n = \frac{a(1-r^n)}{r-1}$

Comparing with $S_n = \frac{4}{3}(4^n - 1)$

a = 4

r – 1 = 3, r = 4

The n^{th} term, $U_n = ar^{n-1}$

$= 4 \times 4^{n-1} = 2^2 \times 2^{2(n-1)} = 2^{2+2n-2} = 2^{2n}$

Example 16

Find three numbers in geometrical progression such that their sum is 26 and their product is 216

Solution

Let the numbers be $\frac{a}{r}$, a and ar

Product = $\left(\frac{a}{r}\right)(a)(ar)$ = 216

$a^3 = 216 = 6^3$

a = 6

∴ the terms are $\frac{6}{r}, 6\ and\ 6r$

Sum of terms $\frac{6}{r} + 6 + 6r$ = 26

⇨ $6r^2 - 26r + 6 = 0$
$3r^2 - 13r + 3 = 0$
(r – 3)(3r – 1) = 0
Either r – 3 = 0; r = 3
Or 3r - 1 = 0; r = $\frac{1}{3}$

When r = 3

the terms are $\frac{6}{3} = 2, 6\ and\ 6x3 = 18$

when r = $\frac{1}{3}$

the terms are $6 \div \frac{1}{3} = 18, 6\ and\ 6\ x\ \frac{1}{3} = 2$

Hence the terms in their order are 2, 6, 18 or 18, 6, 2

Inserting geometric means

Like for A.Ps, the terms inserted between given two values of a G.P are known as geometric means.

If n terms are inserted, then the total number of terms will be n+ 2 with the two extreme values representing the first and last terms respectively

Example 16

(c) Insert two geometric means between 2 and 16

Solution

1^{st} term a = 2

4^{th} term, $ar^3 = 16$

$2(r^3) = 18$

r = 2

the second term, ar = 2 x 2 = 4

the third term, $ar^2 = 2 \times 2^2 = 8$

(d) Insert three geometric means between 1 and 81

a = 1

the 5^{th} term $ar^4 = 81$

$1(r^4) = 81$

r = 3

the second term, ar = 1 x 3 = 3

the third term, $ar^2 = 1 \times 3^2 = 9$

the third term, $ar^3 = 1 \times 3^3 = 27$

Mixed terms of A.P and G.P

These are problems involving both A.Ps and G.Ps. when handling we make use of their respective properties.

Example 17

A geometric progression (G.P) and an arithmetic progression (A.P) have the same first term. The sum of their first, second and third terms are 6, 10.5 and 18 respectively. Calculate the sum of their 5^{th} terms.

Solution

Terms	G.P	A.P	Sum
1st	a	a	2a = 6 …………………(i)
2nd	a + d	ar	a + d + ar = 10.5 ….(ii)
3^{rd}	a + 2d	ar^2	$a + 2d + ar^2 = 18$ ..(iii)

From eqn. (i): 2a = 6; a = 3

From eqn. (ii): 3 + d + 3r = 10.5

d + 3r = 7.5 ………………. (iv)

From eqn. (iii) $3 + 2d + 3r^2 = 18$

$2d + 3r^2 = 15$ …………….(v)

Eqn. (v) – 2eqn. (iv)

$3r^2 - 6r = 0$

3r(r -2) = 0

r -2 = 0

r = 2

Substitute for r into eqn. (iv)

d + 6 = 7.5

d = 1.5

Sum of their fifth terms

$= (a + 4d) + ar^4$

$= (3 + 4 \times 1.5) + 3 \times 2^4 = 57$

Example 18

The 1^{st}, 4thand 8^{th} terms of A.P form a G.P. if the first term is 9, find the

(i) Common difference of the A.P
(ii) Common ratio of the G.P
(iii) Difference in sums of the first 6 terms of the progressions.

Solution

Given that a, a + 3d, a + 7d form a G.P

Substituting for a = 9, the terms are

9, 9 + 3d, 9 + 7d

For a G.P, r = $\frac{9+3d}{9} = \frac{9+7d}{9+3d}$

⇨ (9 + 3d)2 = 9(9 + 7d)
81 + 54d + 9d^2 = 81 + 63d
9d^2 – 9d = 0
9d(d – 1) = 0
Either d – 1 = 0; d = 1
Or d = 0

When d = 0 all terms of A.P are equal

Hence the common difference d = 1

Example 19

(a) The sum of the first m terms of a progression is m(2m + 11)
   (i) Show that the progression is an A.P
   (ii) Determine the nth term of the progression

Solution

Given S_m = m(2m + 11)

First term = S_1 = 1(2 x 1 + 11) = 13

S_2 = 2(2 x 2 + 11) = 30

Second term = 30 – 13 = 17

S_3 = 3(2 x 3 + 11) = 51

Third term = 51 – 20 = 21

The progression is 13, 17, 21, ….. Hence A.P with the first term 13 and common difference, d= 4

(ii) $U_n = S_n - S_{n-1}$

= n(2n + 11) – (n – 1)(2(n -1) + 11)

= 9 + 4n

Example 20

(a) The first, second and last term of an A.P are a, b, c respectively. Prove that the sum of all terms is $\frac{(a+b)(b+c-2a)}{2(b-a)}$

Solution
If the 1st term is a and the second term is b; the common difference, d = (b – a)

Last term (nth terms c = a + (n – 1)d

c = a + (n – 1)(b – 1)

i.e. n – 1 = $\frac{c-a}{b-a}$ => n = $\frac{b+c-2a}{b-a}$

but $S_n = \frac{1}{2}n(a+L)$

$=\frac{1}{2}\left(\frac{b+c-2a}{b-a}\right)(a+c)$

$= \frac{(a+b)(b+c-2a)}{2(b-a)}$

(b) The first, second and last terms of a GP are a and b. show that the sum of the first n terms is $\frac{a^n-b^n}{a^{n-2}(a-b)}$

Solution

Common ratio =$\frac{b}{a}$

$S_n = \frac{a(1-r^n)}{1-r} = \frac{a\left(1-\left(\frac{b}{a}\right)^n\right)}{1-\left(\frac{b}{a}\right)}$

$= \frac{a(a^n-b^n)a}{a^n(a-b)} = \frac{a^n-b^n}{a^{n-2}(a-b)}$

Sum to infinity of a G.P

We have seen that the sum of n terms of a G.P for r < 1 is $S_n = \frac{a(1-r^n)}{1-r}$

Now for -1 < n < 1 i.e. $|r| < 1$, as n →∞, r^n →0

Therefore $S_n = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$

Hence the sum of a GP to infinity for $|r| < 1$ converges to $S_\infty = \frac{a}{1-r}$ and diverges for r > 1 and r < 1

Example 21

(a) Calculate the sum to infinity of the following terms

(i) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

Solution

a = 1 and r = $\frac{1}{2}$

$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

(ii) $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$

Solution

a = $\frac{1}{5}$ and r = $\frac{1}{5}$

$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{1}{4}$

(b) Work out the following

(i) $\sum_{r=0}^{\infty}\left(\frac{1}{3}\right)^r$

Solution

$\sum_{r=0}^{\infty}\left(\frac{1}{3}\right)^r = 1 + \frac{1}{3} + \frac{1}{9} + \cdots$

a = 1 and r = $\frac{1}{3}$

$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$

(ii) $\sum_{r=2}^{\infty}\left(-\frac{1}{8}\right)^r$

Solution

$\sum_{r=2}^{\infty}\left(-\frac{1}{8}\right)^r = \frac{1}{64} - \frac{1}{512} + \frac{1}{4096} - \cdots$

A = $\frac{1}{64}$ and r = $-\frac{1}{8}$

$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{64}}{1+\frac{1}{8}} = \frac{1}{72}$

(iii) $\sum_{r=0}^{\infty} a^r$

Solution

a = 1 and r = a

$S_\infty = \frac{a}{1-r} = \frac{1}{1-a}$

(iv) $\sum_{r=1}^{\infty}(3x)^{r+1}$

Solution

$\sum_{r=1}^{\infty}(3x)^{r+1}$ = $9x^2 + 27x^3 + 81x^4 + \ldots$

a = $9x^2$ and r = 3x

$S_\infty = \frac{a}{1-r} = \frac{9x^2}{1-3x}$

Example 22

(a) Express the following as fractions using approach of sum of a G.P to infinity

(i) $0.\dot{4}$

Solution

$0.\dot{4} = \frac{4}{10} + \frac{4}{100} + \frac{1}{1000} + \cdots$

$= \frac{4}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right)$

$= \frac{4}{10}\left(\frac{1}{1-\frac{1}{10}}\right) = \frac{4}{10} x \frac{10}{9} = \frac{4}{9}$

(ii) $3.1\dot{2}\dot{7}$

Solution

$3.1\dot{2}\dot{7} = 3 + \frac{1}{10} + \frac{27}{1000} + \frac{27}{10000} + \cdots$

$= 3 + \frac{1}{10} + \frac{27}{1000}\left(1 + \frac{1}{100} + \frac{1}{10000} + \cdots\right)$

$= \frac{31}{10} + \frac{27}{1000}\left(\frac{1}{1-\frac{1}{100}}\right) = \frac{31}{10} + \frac{27}{1000}\left(\frac{100}{99}\right)$

$= \frac{31}{10} + \frac{3}{110} = \frac{344}{110} = \frac{172}{55}$

Hence $3.1\dot{2}\dot{7} = \frac{172}{55}$

(b) The sum to infinity of a GP is 7 and the sum of the first two terms is $\frac{48}{7}$. Find the common ratio and the first term of the GP with positive common ratio

Solution

$S_\infty = 7$

$=> \frac{a}{1-r} = 7$

$a = 7(1-r)$ ……………………(i)

But $S_2 = \frac{48}{7}$

a + ar = $\frac{48}{7}$

a(1 + r) = $\frac{48}{7}$ ………………………(ii)

substituting eqn. (i) into eqn. (ii)

$7(1-r)(1+r) = \frac{48}{7}$

$49\,(1 - r^2) = 48$

$49r^2 = 1$

$r^2 = \frac{1}{49}$

$r = \pm\frac{1}{7}$

Considering appositive ratio

From eqn. (i)

$a = 7(1 - \frac{1}{7}) = 6$

Revision exercise 2

1. The common ration of a GP is -5 and the sum of the first seven terms of the progression is 449. Find the first three terms. $\left[\frac{1}{29}, \frac{-5}{29}, \frac{25}{29}\right]$
2. In the geometrical series $\sum_{r=1}^{n} u_r$, $u_5 - u_2 = 156$ and $u_7 - u_4 = 1404$. Find the possible values of the common ratio and corresponding values of u_1 [r = 3, a = 156; r = -3, a = $\frac{13}{7}$]
3. The sum of the second and third terms of a G.P is 9. It the seventh term is eight times the fourth term, find the
   (a) The first term and the common ration [a = $^3/_2$ and r = 2]
   (b) The sum of the fourth and first term [36]
4. Find the sum of ten terms of geometrical series 2, -4, 8 ….. [-682]
5. The second and the third terms of a G.P progression are 24 and 12(b + 1) respectively. Find b if the sum of the first three terms of the progression is 76 $\left[\frac{1}{2} \text{ or } 2\right]$
6. The sum of the 2nd and 3rd terms of a G.P is 12. The sum of the 3rd and 4th terms is -36. Find the 1st term and common ratio [a = 2, r = -3]
7. What is the smallest number of terms of GP 5, 10, 20 …. that can give a sum greater than 500, 000 [n= 17]
8. The first, fourth and eighth terms of Arithmetic progression (A.P) form a geometric progression. If the first term is 9, find
   (i) The common difference of A.P [1]
   (ii) The common ratio of the G.P $\left[\frac{4}{3}\right]$
   (iii) The difference in the sums of the first 6 terms of the progressions [55.7049]
9. The second, third and ninth terms of an A.P form a G.P. find the common ratio of the G.P [6]
10. (a) The sum of the first 10 terms of an AP is 120. The sum of the next 8 terms is 240. Find the sum of the next 6 terms [264]
    (b) the arithmetic mean of the a and b is three times their geometric mean. Show that $\frac{a}{b} = 7 \pm 12\sqrt{2}$
11. The first three terms of a geometric series are 1, p, and q. Given also that 10, q and p are the first three terms of an arithmetic series. Show that $2p^2 - p - 10 = 0$
    Hence find the possible values of p and q [p = -2 and q = 4 or p = $\frac{5}{2}$ and q = $\frac{25}{4}$]

Application of A.Ps and G.Ps to interest rates

If a sum of money P is invested at a simple interest rate of r% per annum, the amount received after n years is given by A = P + I where $I = \frac{P \times r \times n}{100}$

By substitution we have

$A = P + \frac{P \times r \times n}{100} = P\left(1 + \frac{nr}{100}\right)$

The interest for one year is $\frac{Pr}{100}$, for 2 years is $\frac{2Pr}{100}$, for n year = $\frac{nPr}{100}$. Therefore the various amounts of interest after one, two, three, etc. years form an AP

On the other hand, if the principal P is invested at compound interest rate of r% per annum, the interest being added annually, the amount after one year is $\left(1 + \frac{r}{100}\right)$, after two years is $P\left(1 + \frac{r}{100}\right)^2$, after 3 years is

$P\left(1 + \frac{r}{100}\right)^3$ and after n years $P\left(1 + \frac{r}{100}\right)^n$

Hence the amounts after one, two, three, etc. years for a GP.

Note: if with compound interest is added half annually as much as when added yearly, but it is added twice as much. Hence amount A = $P\left(1 + \frac{r}{100}\right)^{2n}$

Now suppose that instead of adding the interest annually, it is the principal, P which is added annually,

Amount after 1st year = $P\left(1 + \frac{r}{100}\right)^1$

Amount after 2nd year = $P\left(1 + \frac{r}{100}\right)^2$

Amount after 3rd year = $P\left(1 + \frac{r}{100}\right)^3$

Amount after nth year = $P\left(1 + \frac{r}{100}\right)^n$

Total amount after n years

$= P\left(1 + \frac{r}{100}\right)^1 + P\left(1 + \frac{r}{100}\right)^2 + \ldots + P\left(1 + \frac{r}{100}\right)^n$

$= P\left[\left(1 + \frac{r}{100}\right)^1 + \left(1 + \frac{r}{100}\right)^2 + \cdots + \left(1 + \frac{r}{100}\right)^n\right]$

This is a G.P with:

first term = $\frac{100+r}{100}$ = (100 + r)%

And common ratio $\frac{100+r}{100}$ = (100 + r)%

$s_n = P(100 + r)\%\left[\frac{(100 + r)\%-1}{(100 + r)\%-1}\right]$

Example 23

(a) Find the amount at the end of ten years when 500000 shillings is invested at 5% compound interest

(i) the interest being added annually

Solution

A = $P\left(1 + \frac{r}{100}\right)^n$

$= 500000\left(1 + \frac{5}{100}\right)^{10}$

A = 814,447.3134

(ii) the interest being added twice a year

Solution

A = $P\left(1 + \frac{r}{100}\right)^{2n}$

$= 500000\left(1 + \frac{5}{100}\right)^{20}$

A = 1,326,648.853

(b) Find the amount at the end of ten years when 500000 shillings is invested at 5% simple interest

A = $\frac{nPr}{100} = \frac{10\ x\ 500000\ x\ 5}{100}$ =750,000

Application of Geometrical progression

1: Finance & Economics

- **Compound Interest:** Money invested grows by a fixed percentage each period. Example: If you invest Sh.1000 at 10% interest annually, the amounts form a GP: 1000,1100,1210,1331,…1000, 1100, 1210, 1331, ……
- **Depreciation of Assets:** Value of a car decreasing by 20% each year follows GP.

2. Population Growth

- If a population multiplies by a fixed factor (e.g., doubling every 25 years), GP models the growth.

3. Physics & Engineering

- **Radioactive Decay:** The remaining mass of a substance decreases by a fixed ratio over equal time intervals.
- **Waves & Acoustics:** Harmonics in sound frequencies often follow GP patterns.

4. Computer Science

- **Algorithm Complexity:** Recursive algorithms (like binary search or divide-and-conquer) often involve GP in their step counts.
- **Data Structures:** Memory allocation sometimes grows geometrically (e.g., dynamic arrays doubling in size).

5. Architecture & Design

- **Fractals:** Patterns like Sierpinski triangles or geometric tiling use GP scaling.
- **Building Levels:** Skyscraper window sizes or spacing sometimes follow geometric scaling.

6. Daily Life

- **Phone Battery Drain/Charge:** Charging often follows exponential (GP-like) growth.
- **Medicine Dosage Decay:** Drug concentration in the body decreases geometrically over time.

Proof by induction

This is a mathematical technique that uses the reasoning that if a statement is true for a particular value say n = 1, then it must be true for n = 2, 3, 4, …. This involves the proof that the series on the LHS must be equal to the terms on the RHS

Example 24

Prove by induction that

1 + 2 + 3 + …+ n = $\frac{1}{2}n(n+1)$

Solution

Here we need to show that the above series agrees for all values of n = 1, 2, 3 .., q and q +1

Suppose n = 1

LHS = 1 [taking only the 1st number]

RHs = $\frac{1}{2}(1)((1)+1)$= 1[substituting for n = 1]

∴LHS = RHS => the series hold for n = 1

Suppose n = 2

LHS = 1+ 2 = 3 [taking first 2 numbers]

RHs = $\frac{1}{2}(2)((2)+1)$= 3[substituting for n = 2]

∴LHS = RHS => the series hold for n = 2

Suppose n = q

1 + 2 + 3 + …+ q = $\frac{1}{2}q(q+1)$

For n = q +1(i.e. adding k + 1 on both sides)

1 + 2 + 3 + ...+ q + (q+1) = $\frac{1}{2}q(q+1)$+ (q + 1)

=$(q+1)(\frac{1}{2}q+1)$

=$\frac{1}{2}(q+1)(q+2)$

The result is true for n = q + 1, hence true for all positive values of n

Example 25

Prove by induction

(a) $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$

Solution

For n = 1

LHS = 1^2 = 1;

RHS = $\frac{1}{6}(1)((1)+1)(2(1)+1)$ = 1

∴ LHS = RHS => the series holds for n = 1

For n = 2

LHS = $1^2 + 2^2$ = 5;

RHS = $\frac{1}{6}(2)((2)+1)(2(2)+1)$ = 5

∴ LHS = RHS => the series holds for n = 2

For n = k

$1^2 + 2^2 + 3^2 + ... + k^2 = \frac{1}{6}k(k+1)(2k+1)$

For n = k + 1

$1^2 + 2^2 + 3^2 + ... + k^2 + (k+1)^2$

$$= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$$

=$[k\ +1]\left[\frac{1}{6}k(2k+1) + (k+1)\right]$

=$[k\ +1]\left[\frac{1}{6}(2k^2+7+6)\right]$

=$\frac{1}{6}(k+1)(k+2)(2k+3)$

∴ LHS = RHS => the series holds for n = k+1 hence for all positive values of n

(b) $\sum_{r=1}^{n=r} r^3 = \frac{1}{4}n^2(n+1)^2$

Solution

For n = 1

LHS = $\sum_{r=1}^{r=1} r^3 = 1^3$ = 1

RHS =$\frac{1}{4}(1)^2(1+1)^2$= 1

∴ LHS = RHS => the series holds for n = 1

For n = 2

LHS = $\sum_{r=1}^{r=2} r^3 = 1^3 + 2^3$ = 1

RHS =$\frac{1}{4}(2)^2(2+1)^2$= 9

∴ LHS = RHS => the series holds for n = 2

For n = k

$$\sum_{r=1}^{n=k} r^3 = \frac{1}{4}k^2(k+1)^2$$

For n = k+1

$\sum_{r=1}^{n=k+1} r^3 = \sum_{r=1}^{n=k} r^3 + (k+1)^3$

=$\frac{1}{4}k^2(k+1)^2 + (k+1)^3$

=$(k+1)^2\left[\frac{1}{4}k^2 + k + 1\right]$

=$\frac{1}{4}(k+1)^2[k^2+4k+4]$

=$\frac{1}{4}(k+1)^2(k+2)^2$

(c) $p + pq + pq^2 + ... + pq^{n-1} = p\left(\frac{1-q^n}{1-q}\right)$

For n = 1,

LHS = p

RHS =$p\left(\frac{1-q^1}{1-q}\right)$ = p

∴the identity is true for n = 1

For n = 1,

LHS = p + pq =p(1+q)

RHS =$p\left(\frac{1-q^2}{1-q}\right) = p\left(\frac{(1+q)(1-q)}{1-q}\right)$=p(1+q)

∴the identity is true for n = 2

For n = k

$p + pq + pq^2 + ... + pq^{k-1} = p\left(\frac{1-q^k}{1-q}\right)$

For n = k+1

$p + pq + pq^2 + ... + pq^{k-1} + pq^k = p\left(\frac{1-q^k}{1-q}\right) + pq^k$

$= p\left(\frac{1-q^k+q^k+p^{k+1}}{1-q}\right)$

$= p\left(\frac{1-q^{k+1}}{1-q}\right)$

∴the identity is true for n = k+1, hence true for all positive values of n

Revision exercise 3

1. Five millions shillings is invested each year at a rate of 15% compound interest by a certain bank.
   (a) Find how much he will receive at the end of ten years [116.7464m]
   (b) How many years will it take to accumulate to more than 50m [6]
2. John opened an account in the bank and deposited200,000 shillings every month for ten months without withdrawing. Find how much money he accumulated after 10 months if the bank offered 10% compound interest per month. [3,506,233.412]
3. Peter deposited sh. 100,000 at the beginning of every year for 5 years; find how much he got at the end of the fifth year at the compound interest rate of 2% per annum. [530,812.1]
4. Prove by induction
   (i) $\sum_{r=1}^{n} 3^{r-1} = \frac{3^n-1}{2}$
   (ii) $1^3 + 2^3 + 3^3 + ... + n^3$
   $= \frac{1}{4}n^2(3n+1)(5n+1)$
   (iii) $\sum_{r-1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$
   (iv) $\frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n(n-1)} = 1 - \frac{1}{n}$
   (v) $\sum_{r=1}^{n} \frac{1}{r(1+1)} = \frac{n}{n+1}$
   (vi) $\sum_{r=1}^{n} ap^{r-1} = \frac{a(p^n-1)}{p-1}$
5. The sum of the first n terms of a Geometric Progression (G.P) is $\frac{4}{3}(4^n - 1)$. Find its nth term as an integral power of 2 [2^{2n}]
6. Prove by mathematical induction the $3^{2n} - 1$ is a multiple of 8 for all positive integers n
7. Use the method of induction to prove that 6n – 1is divisible by 5 for all positive integral values of n
8. Prove by induction that $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$ hence evaluate $\sum_{r=1}^{20} r(r+1)$

Binomial theorem

Pascal's triangle

The Pascal's triangle below and its further extension is used to determine the coefficients of the expansion of $(p + q)^n$

1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Observations

- The coefficients are symmetrical; they are the same irrespective of which side they are read from, for instance the coefficients of $(p + q)^6$ are
  1 6 15 20 15 6 1
- The coefficient of the 2nd term in the expansion is the index of a given expansion. E.g. in the expansion of $(p + q)^6$, the coefficient of the 2nd term is 6.
- The number of terms in the expansion exceeds the index by one, e.g. $(p + q)^4$ with index 4, has 5 terms
- The index of the first term of the expansion decreases by one, from the index given till zero, whereas, the index of the second term increases by one from zero to the given index
  For $(p + q)^3 = 1p^3q^0 + 3p^2q^1 + 3p^1q^2 + 1p^0q^3$
  $= p^3 + 3p^2q + 3pq^2 + q^3$
- The sum of indices in each term s constant and equal to the index of the expansion.

Example 1

Use Pascal's triangle to expand

(a) $(p + q)^4$

The coefficients are 1 4 6 4 1

Terms are $1p^4q^0 + 4p^3q^1 + 6p^2q^2 + 4p^1q^3 + 1q^4$

$= p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$

(b) $(2x + 3y)^3$

The coefficients are 1 3 3 1

Terms are

$1(2x)^3(3y)^0 + 3(2x)^2(3y)^1 + 3(2x)^1(3y)^2 + 1(2x)^0(3y)^3$

$= 8x^3 + 36x^2y + 54xy^2 + 27y^3$

(c) $(a + b + c)^2$

The terms can be grouped into two ways:

Either a and (b + c) or (a + b) and c

The coefficients are 1 2 1

Either

Terms are: $1a^2(b + c)^0 + 2a^1(b + c)^1 + 1a^0(b + c)^2$

$= a^2 + 2a(b + c) + (b + c)^2$

$= a^2 + 2ab + 2ac + b^2 + 2bc + c^2$

Or

$(a + b + c)^2$

$= 1(a + b)^2c^0 + 2(a + b)_1c^1 + 1c^2$

$= (a + b)^2 + 2c(a + b) + c^2$

$= a^2 + 2ab + b^2 + 2ac + 2bc + c^2$

Example 2

Expand $(a + b)^4$ using Pascal's triangle. Hence find $(1.996)^4$ correct to 3 decimal places

Solution

From Pascal's triangle the coefficients are
1 4 6 4 1

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(1.996)^4 = (2- 0.004)^4$

Substituting a = 2 and b = -0.004

$24 + 4(2)^3(-0.004) + 6(2)^2(-0.004)^2 + 4(2)(-0.004)^3 + (-0.004)^4$

$16 – 0.128 + 0.000384 = 15.872$ (3D)

Example 3

(a) Expand $(2 – 3x)^4$ using Pascal's triangle. Hence evaluate $(1.97)^4$ correct to 3 decimal places.

Solution

Coefficients: 1 4 6 4 1

$(2 – 3x)^4 = 1 (2)^4(-3x)^0 + 4(2)^3(-3x)^1 + 6(2)^2(-3x)^2 + 4(2)^1(-3x)^3 + 1(2)^0(-3x)^4$

$= 16 – 96x + 216x^2 – 216x^3 + 81x^4$

Now $(1.97)^4 = (2 – 0.03)^4 = (2 – 3(0.01))^4$

⇨ x = 0.01

$(1.97)^4 = 16 – 96 (0.01) + 216(0.01)^2 - 216(0.01)^3 + 81(0.01)^4 = 15.0613848$

= 15.061 (3d.p)

(b) Use Pascal's triangle to evaluate $(1.02)^3$ correct 5 significant figures

Solutions

$(1.02)^3 = (1 + 02)^3$

Coefficients are : 1 3 3 1

$(1 + 02)^3 = 1^3 + 3(1)^2(0.02) + 3(1)(0.02)^2 + (0.02)^3 = 1.061208$

= 1.0612 (5 significant figures)

The idea of factorial and combination can also be used to determine the coefficients of the expansions

Example 3

Expand

(a) $(p + 3q)^3$

Solution

The coefficients are $^3C_0$ $^3C_1$ $^3C_2$ $^3C_3$

Terms are p^3 $p^2(3q)$ $p(3q)^2$ $(3q)^3$

$(p + 3q)^3 = {}^3C_0p^3 + {}^3C_1p^2(3q) + {}^3C_2p(3q)^2 + {}^3C_3(3q)^3$

= $P^3 + 9P^2 3q + 27pq^2 + 27q^3$

(b) $(1 – x)^4$. Hence evaluate $(0.99)^4$ correct to four decimal places

Solution

Coefficients are $^4C_0$ $^4C_1$ $^4C_2$ $^4C_3$ $^4C_4$

Or simply $\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

$1 – x)^4 = {^4C_0}1^4(-x)^0 + {^4C_1}1^3(-x)^1 + {^4C_2}1^2(-x)^2 + {^4C_3}1^1(-x)^3 + {^4C_4}1^0(-x)4^0$

$= 1 – 4x + 6x^2 – 4x^3 + x^4$

Now 0.99 = 1 – 0.01) = x = 0.01

$(0.99)^4$

$= 1 – 4(0.01) + 6(0.01)^2 – 4(0.01)^3 + (0.01)^4$

= 0.96059601

= 0.9606 (4d.p)

The binomial theorem for positive integral index

Consider the expansion of $(1 + x)^4$, the result is

$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

Observations

(i) The indices of x increase by 1 from term to term

(ii) The index of the last term being the same as the power to which (1 + x) s raised

(iii) The coefficients of the terms of expansion are $^4C_0$ $^4C_1$ $^4C_2$ $^4C_3$ $^4C_4$

Hence the expansion of $(1 + x)^n$ is

$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + ... x^n$

The expansion of $(a + x)^n$ is

$(a + x)^n = a^n\left(1 + \frac{x}{a}\right)^n$

$= a^n\left[1 + \binom{n}{1}\left(\frac{x}{a}\right) + \binom{n}{2}\left(\frac{x}{a}\right)^2 + \cdots x\left(\frac{x}{a}\right)^n\right]$

$= a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \cdots x^n$

Example 4

Expand $(1 + 4x)^{14}$ in ascending power of x up to and include the 4^{th} term. Hence evaluate $(1.0004)^{14}$ correct to four decimal places

Solution

$(1 + 4x)^{14}$

$= 1 + \binom{14}{1}(4x) + \binom{14}{2}(4x)^2 + \binom{14}{3}(4x)^3$

$= 1 + 56x + 1456x^2 + 23296x^3$

Now $(1.0004)^{14} = (1 + 0004)^{14}$ => x = 0.0001

$(1 + 4x)^{14}$

$= 1 + 56(0.0001) + 1456(0.0001)^2 + 23296(0.0001)^3$

= 1.005614583

= 1.0056 (4d.p)

Note the next term in the expansion is

$\binom{14}{4}(4x)^4 = 256256x^4$

Its value = $256256(0.0001)^4 = 2.56256 \times 10^{-11}$

Which when added to the above answer there will negligible change in value

Example 5

Expand $(3 – 2x)^{12}$ in ascending powers of x up to and including the term x^3. Hence evaluate $(2.998)^{12}$ correct to the nearest whole number.

Solution

$(3 – 2x)^{12} = 3^{12} + \binom{12}{1}(3)^{11}(-2x)^1 + \binom{12}{2}(3)^{10}(-2x)^2 + ...$

$= 531441 – 4251528x + 1588936x^2$

Now $(2.998)^{12} = (3 – 0.002)^{12}$ => x = 0.001

$(2.998)^{12} = 531441 – 4251528(0.001) + 1588936(0.001)^2$

= 527205.0609 =527205 (nearest whole number)

Particular terms of binomial expansion

As earlier seen, the expansion of

$(a + x)^5 = a^5x^0 + 5a^4x^1 + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$

In general if r is the power of x in the expansion $(a + x)^n$, then

the U_{r+1} term of x = ${}^nC_ra^{n-r}x^r$

Example 6

Find the term of x^4 in the expression of

(i) $(1 + x)^9$

By using U_{n+1}; term of x = ${}^nC_ra^{n-r}x^r$

n = 9, r = 4, a = 1

$U_5 = {}^9C_4 1^{9-4}x^4 = {}^9C_4x^4 = 126x^4$

(ii) (3 + x)7

n = 7, a = 3, r = 4

$U_5 = {}^7C_4 3^{7-4}x^4 = {}^7C_4(3)^3x^4 = 945x^4$

(iii) $\left(2-\frac{x}{2}\right)^{12}$

$n = 12, a = 2$

$U_5 = {}^{12}C_4 2^{12-4}\left(-\frac{x}{2}\right)^4 = {}^{12}C_4(2)^8\left(-\frac{x}{2}\right)^4 = 7920x^4$

Example 7

Find the term indicated in expansion of the following expression

(i) $\left(3x-\frac{2}{x}\right)^5$ [x^3]

$\left(3x-\frac{2}{x}\right)^5 = 3^5x^5\left(1-\frac{2}{3x^2}\right)^5$

The term in $x^3 = 3^5x^5.{}^5C_1\left[1^4\left(-\frac{2}{3x^2}\right)^1\right]$

$= 3^5x^5.{}^5C_1\left(-\frac{2}{3x^2}\right) = -810x^3$

(ii) $\left(2x+\frac{5}{x}\right)^6$ [x^4]

$\left(2x+\frac{5}{x}\right)^5 = 2^6x^6\left(1+\frac{5}{2x^2}\right)^6$

The term in $x^3 = 2^6x^6.{}^6C_1\left[1^5\left(\frac{5}{2x^2}\right)^1\right]$

$= 2^6x^6.{}^6C_1\left(\frac{5}{2x^2}\right)$ $960x^4$

Finding terms independent of x

The term is said to be independent of x if the power of x is zero

Example 8

Find the term independent of x in the following expansion

(a) $\left(2x+\frac{1}{x^2}\right)^{12}$

Solution

$\left(2x+\frac{1}{x^2}\right)^{12} = 2^{12}x^{12}\left(1+\frac{1}{2x^3}\right)^{12}$

The term independent of x = $2^{12}x^{12}$ multiplied by the term in x^{-12} in the expansion $\left(2x+\frac{1}{x^2}\right)^{12}$

⇨ The term independent of x

$= 2^{12}x^{12}.{}^{12}C_4\left(\frac{1}{2x^3}\right)^4 = 2^{12}x^{12} \text{ x } {}^{12}C_4 \text{ x } \frac{1}{2^4x^{12}}$ 4

$=2^8 \text{ x } 495 = 126720$

Alternatively

By using $U_{r+1} = {}^nC_ra^{n-r}x^r$

The term independent of x is got by equating the index of x to zero

$U_{r+1} = {}^{12}C_r(2x)^{12-r}\left(\frac{1}{x^2}\right)^r$

$= {}^{12}C_r(2x)^{12-r}x^{-2r}$

$= {}^{12}C_r.\ 2^{12-r}.x^{12-3r}$

Equating the $x^{12-3r} = 1$

⇨ 12 – 3r = 0; r = 4

Term independent of x = $^{12}C_4 . 2^{12-4}$

$= {}^{12}C_4 . 2^8$

= 126720

(b) $\left(2x^2 + \frac{3}{x}\right)^{12}$

Solution

$\left(2x + \frac{3}{x}\right)^{12} = 2^{12}x^{12}\left(1 + \frac{3}{2x^2}\right)^{12}$

Un+ r = $^{12}C_r(2x^2)^{12-r}(3x^{-1})^r$

$= {}^{12}C_r(2x^2)^{12-r}(3^r)x^{-r}$

$= {}^{12}C_r . 2^{12-r}(3^r)x^{24-3r}$

Equating the $x^{12-3r} = 1$

⇨ 24 – 3r = 0
r = 8

Term independent of x = $^{12}C_8 . 2^{12-4}.3^8$

$= {}^{12}C_8 . 2^4.3^8$

= 51, 963,120

Binomial expansion of terms with fractional or negative powers

As noted earlier

$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots . x^n$

The following is noted

(i) For positive integral value of n, i.e. n ≥ 1, the series above terminates at the term x^n and its sum is $(1 + x)^n$.

(ii) For fractional or negative values of n, the series above does not terminate but instead converges to $(1 + x)^n$ as the limit of its sum only -1 < x< 1 or $|x| < 1$

Example 9

Expand $\sqrt{(1-2x)}$ up to the term x^3. Hence evaluate $\sqrt{0.98}$ correct to four decimal places.

Solution

$\sqrt{(1-2x)} = (1-2x)^{\frac{1}{2}}$

Using

$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots . x^n$

Comparing with $(1-2x)^{\frac{1}{2}}$ => x ≡ -2x and n = ½

$\sqrt{(1-2x)} = 1 + \frac{1}{2}(-2x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)(-2x)^2}{2!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-2x)^3}{3!}$

$= 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3$

Now $\sqrt{0.98} = \sqrt{(1-0.02)}$

Comparing with $\sqrt{(1-2x)}$; x = 0.01

Substituting

$\sqrt{0.98} = 1 - (0.01) - \frac{1}{2}(0.01)^2 - \frac{1}{2}(0.01)^3$

= 0.9899495

$\therefore \sqrt{0.98} = 0.9899$ (4d.p)

Example 10

Given that x is very small that its cube and higher powers can be neglected, show that

$\sqrt{\left(\frac{1-x}{1+x}\right)} = 1 - x + \frac{1}{2}x^2$

By putting x = $\frac{1}{8}$ show that $\sqrt{7} = \frac{339}{128}$

Solution

By rationalizing the denominator

$\sqrt{\left(\frac{1-x}{1+x}\right)\left(\frac{1-x}{1-x}\right)} = \sqrt{\frac{(1-x)^2}{(1-x^2)}}$

$= (1-x)(1-x^2)^{\frac{1}{2}}$

But $(1-x^2)^{\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-x^2) + \ldots..$

$= 1 + \frac{1}{2}x^2$

$\sqrt{\left(\frac{1-x}{1+x}\right)} = (1-\text{x})\left(1+\frac{1}{2}x^2\right)$

$= 1-\text{x} + \frac{1}{2}x^2$

Putting x = $\frac{1}{8}$

$\sqrt{\left(\frac{1-\frac{1}{8}}{1+\frac{1}{8}}\right)} = 1-\frac{1}{8} + \frac{1}{2}\left(\frac{1}{8}\right)^2$

$\sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{3} = 1-\frac{1}{8} + \frac{1}{128} = \frac{113}{128}$

$\sqrt{7} = \frac{339}{128}$

Example 11

Expand $\sqrt{\left(\frac{1+x}{1-x}\right)}$ up to the term x^3

Use your expansion to evaluate $\sqrt{23}$ correct to 3 decimal places taking x = $\frac{1}{24}$.

Solution

By rationalizing the denominator

$\sqrt{\left(\frac{1+x}{1-x}\right)\left(\frac{1+x}{1+x}\right)} = (1+x)(1-x^2)^{\frac{1}{2}}$

But$(1-x^2)^{\frac{1}{2}} = 1+\frac{1}{2}x^2+\cdots$

$\sqrt{\left(\frac{1+x}{1-x}\right)} = (1+x)\left(1+\frac{1}{2}x^2\right)$

$= 1+x+\frac{1}{2}x^2+\frac{1}{2}x^3$

Hence putting x = $\frac{1}{24}$

$\sqrt{\left(\frac{1+\frac{1}{24}}{1-\frac{1}{24}}\right)} = 1+\frac{1}{24}+\frac{1}{2}\left(\frac{1}{24}\right)^2+\frac{1}{2}\left(\frac{1}{24}\right)^3$

$\sqrt{\frac{25}{23}} = \frac{5}{\sqrt{23}} = \frac{28825}{27648}$

$\sqrt{23} = \frac{5\ x\ 27648}{28825} = 4.7958$

$\therefore \sqrt{23} = 4.7958$ (4d.p)

Example 12

Use the binomial theorem to expand $\sqrt[3]{(1-x)}$ up to x^3. Use your expansion to evaluate $\sqrt[3]{7}$ correct to four decimal places

Solution

$\sqrt[3]{(1-x)} = (1-x)^{\frac{1}{3}}$

$= 1+\frac{1}{3}(-x)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)(-x)^2}{2!}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(-x)^3}{3!}+\ldots$

$= 1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{1}{81}x^3$

Hence $\sqrt[3]{7} = \sqrt[3]{(8-1)} = \sqrt[3]{8\left(1-\frac{1}{8}\right)}$

$= 2\left(1-\frac{1}{8}\right)^{\frac{1}{3}}$ => x = $\frac{1}{8}$

$\sqrt[3]{7} = 2\left[1+\frac{1}{3}\left(\frac{1}{8}\right)-\frac{1}{9}\left(\frac{1}{8}\right)^2+\frac{1}{81}\left(\frac{1}{8}\right)^3\right]$

=1.9130(4d.p)

Example 13

Write down the expansion of $\sqrt{(1-x)}$ in ascending powers of x as far as the term x^4. Use your expansion to find $\sqrt{80}$ correct to four significant figures.

Solution

$\sqrt{(1-x)} = (1-x)^{\frac{1}{2}}$

For binomial expansion

$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots x^n$

Comparing $(1-x)^{\frac{1}{2}}$ with$(1 + x)^n$

n = $\frac{1}{2}$ and x = -x

$(1-x)^{\frac{1}{2}} = 1+\frac{1}{2}(-x)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)(-x)^2}{2!}+$

$\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^3}{3!}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-x)^4}{4!}$

$(1-x)^{\frac{1}{2}} = 1-\frac{1}{2}x-\frac{x^2}{8}-\frac{x^3}{6}-\frac{x^4}{24}$

Now $\sqrt{80} = \sqrt{81-1} = \sqrt{81\left(1-\frac{1}{81}\right)}$

$$= 9\left(1-\frac{1}{81}\right)^{\frac{1}{2}}$$

Comparing $\left(1-\frac{1}{81}\right)^{\frac{1}{2}}$ with $(1-x)^{\frac{1}{2}}$

⇨ x = $\frac{1}{81}$

Substituting for x

$\left(1-\frac{1}{81}\right)^{\frac{1}{2}}$

$=1 - \frac{1}{2}\left(\frac{1}{81}\right) - \frac{\left(\frac{1}{81}\right)^2}{8} - \frac{\left(\frac{1}{81}\right)^3}{6} - \frac{\left(\frac{1}{81}\right)^4}{24}$

$=\frac{52163}{52488}$

$\sqrt{80} = 8.944$ (4 S.F)

Not the termx3 has been neglected as it does not affect the answer to 4 significant figured

Example 14

John operates an account with a certain bank which pays a compound interest rate of 13.5% per annum. He opened the account at the beginning of the year with sh. 500,000 and deposits the same amount of money at the beginning of every year. Calculate how much he will accumulate at the end of 9 years. After how long will the money have accumulated to sh. 3.32 millions?

Solution

The 1st deposit will grow to

$500000\left(1+\frac{13.5}{100}\right)$= 500000 x 1.135

2nd deposit will grow to 500000 x 1.135^2

nth deposit will grow to 500000 x 1.135^n

9th deposit will grow to 500000 x 1.135^9

The total = 500,000[1.135 +1.135^2 + … + 1.135^9]

$=500000\left[a\left(\frac{r^n-1}{r-1}\right)\right]$

$=500000\left[1.135\left(\frac{1.135^9-1}{1.135-1}\right)\right]$

=8,936,381

Finding how long it will take the money to accumulate to sh. 3,320,000

$$500000\left[1.135\left(\frac{1.135^n-1}{1.135-1}\right)\right] = 3320000$$

n = 4.6years

Example 15

Expand $(1 + x)^{-2}$ in descending powers of x including the term x^{-4}. If x = 9 find the percentage error in using the first two terms of the expression.

Solution

From

$$(1+x)^n = 1 + nx + \frac{2(n-1)x^2}{2!} + \cdots + x^n$$

Now $(1 + x)^{-2} = x^{-2}\left(1+\frac{1}{x}\right)^{-2}$

$x^{-2}\left(1+\frac{1}{x}\right)^{-2}$

$$= x^{-2}\left[1 + (-2)\frac{1}{x} + \frac{(-2)(-3)}{2!}\left(\frac{1}{x}\right)^2\right]$$

$=x^{-2}\left[1-\frac{2}{x}+\frac{3}{x^2}\right]$

$=x^{-2}[1-2x^{-1}+3x^{-2}]$

$=x^{-2}-2x^{-3}+3x^{-4}$

If x = 9

$(1 + x)^{-2}=9^{-2}-2(9)^{-3}+3(9)^{-4}$

$=\frac{1}{81}-\frac{2}{729}$ (using the first 2 terms)

$=\frac{7}{729}$

The exact value is $(1 + 9)^{-2} = \frac{1}{100}$

Error = $\frac{1}{100} - \frac{7}{729} = \frac{29}{72900}$

%error = $\frac{29}{72900}\ x100\ x100$= 3.978%(3d.p)

Example 16

(a) Find the three terms of the expansion $(2 - x)^6$ and use it to find $(1.998)^6$ correct to two decimal places (06 marks)

$(2 - x)^6 = 2^6 + \binom{6}{1}2^5(-x)^1 + \binom{6}{2}4(-x)^2$

$= 64 - 192x + 240x^2$

$(1.998)^2 = (2 - 0.002)^2$

$= 64 - 192(0.002) + 240(0.002)^2$
$= 64 - 0.384 + 0.00096$
$= 63.61696$
$= 63.62\ (2D)$

Example 17

(a) Prove by induction

1.3 + 2.4 + … + n(n + 2) = $\frac{1}{6}n(n+1)(2n+7)$ for all values of n.
Suppose n = 1
L.H.S = 1 x 3 = 3
R.H.S = $\frac{1}{6}x1(1+1)(2+7) = 3$
L.H.S = R.H.S, hence the series holds for n =1
Suppose n = 2
L.H.S= 1 x 3 + 2 x 4 = 11
R.H.S = $\frac{1}{6}\ x\ 2(2+1)(4+7) = 11$
L.H.S = R.H.S, hence the series holds for n =2
Suppose n = k
1.3 + 2.4 + … + k(k + 2) = $\frac{1}{6}k(k+1)(2k+7)$
For n = k + 1
1.3 + 2.4 + … + k(k + 2), (k + 1)(k + 3) = $\frac{1}{6}k(k+1)(2k+7)$+(k +1)(k + 3)
=(k + 1)$\left[\frac{1}{6}k(2k+7) + (k+3\right]$
= $\frac{1}{6}(k+1)(2k^2+13k+18)$
= $\frac{1}{6}(k+1)(2k^2+4k+9k+18)$
= $\frac{1}{6}(k+1(k+2)(2k+9)$
= $\frac{1}{6}(k+1)(k+2)[2(k+1)+7]$
Which is equal to R.H.S when n = k + 1
It holds for n = 1, 2, 3 …, hence it holds for all integral values of n.

(b) A man deposits Shs. 150,000 at the beginning of every year in a micro finance bank with the understanding that at the end of the seven years he is paid back his money with 5% per annum compound interest. How much does he receive?

Using amount, A = P$\left(1 + \frac{r}{100}\right)^n$

=150000$\left(1\ + \frac{5}{100}\right)^7$=211,065.06

Alternatively
1st year

P = 150,000

He is paid back principal plus interest; P$\left(1 + \frac{5}{100}\right)$ = 150,000$\left(1 + \frac{5}{100}\right)$=157,500

2nd year

P = 157, 500

He is paid back principal plus interest; P$\left(1 + \frac{5}{100}\right)$ = 157500$\left(1 + \frac{5}{100}\right)$=165375

3rd year

P = 165375

Interest = $\frac{5}{10}\ x\ 165375 = 8268.75$

He is paid back principal plus interest; P$\left(1 + \frac{5}{100}\right)$ = 165375 $\left(1 + \frac{5}{100}\right)$=173643.75

4th year

P=173643.75

He is paid back principal plus interest; P$\left(1 + \frac{5}{100}\right)$ = 173643.75$\left(1 + \frac{5}{100}\right)$=182325.94

5th year

P =182325.94

His paid back principal plus interest; P$\left(1 + \frac{5}{100}\right)$ = 182325.94$\left(1 + \frac{5}{100}\right)$=191442.23

6th year

P = 191442.23

He is paid back principal plus interest; P$\left(1 + \frac{5}{100}\right)$ = 191442.23$\left(1 + \frac{5}{100}\right)$=201014.35

7th year

P = 201014.35

He is paid back principal plus interest; P$\left(1 + \frac{5}{100}\right)$ = 201014.35$\left(1 + \frac{5}{100}\right)$=211,065.06

∴ by the 7th year he has accumulated shs. **211,065.06**

Example 18

Expand $\sqrt{\left(\frac{1+2x}{1-x}\right)}$ up to the term x^2. Hence find the value of $\sqrt{\left(\frac{1.04}{0.98}\right)}$ to four significant figures. (12marks)

$$\sqrt{\left(\frac{1+2x}{1-x}\right)} = (1+2x)^{\frac{1}{2}}(1-x)^{\frac{-1}{2}}$$

$$using\ (1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots$$

$$\sqrt{\left(\frac{1+2x}{1-x}\right)} = \left(1 + x - \frac{1}{2}x^2\right)\left(1 + \frac{1}{2}x + \frac{3}{8}\ x^2\right)$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + x + \frac{1}{2}x^2 - \frac{1}{2}x^2$$

$$= 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

$$\therefore \sqrt{\left(\frac{1+2x}{1-x}\right)} \approx 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

$$substituting\ for\ x = 0.02$$

$$\sqrt{\left(\frac{1.04}{0.98}\right)} = \sqrt{\frac{1+2(0.02)}{1-0.02}}$$

$$= 1 + \frac{3}{2}(0.02) + \frac{3}{8}(0.02)^2 = 1.030$$

Revision exercise 5

1. Given that the ratio of the 3rd to the 4th term of the expansion (2+3x)n is 5:14, find the value of n when x = $\frac{2}{5}$. [n= 16]
2. Expand $(3 – 4x)^5$ in ascending order of x up to and including the term x^3. Hence evaluate $(4.96)^5$ correct to 2 d.p. [3001.98]
3. (a) Find the coefficient of x2 in the expansion of $(1 – 2x)^n$ is 24 [4]
   (b) Find the term independent of x in the expansion of $\left(2x^3 - \frac{1}{x}\right)^{20}$ [-496128]
   (c) Use the binomial expansion to expand $\sqrt[4]{(1+2x)}$ up to the term x^3. Hence evaluate $\sqrt[4]{83}$ correct to three decimal places [3.018]
4. Expand $\sqrt{\left(\frac{1+2x}{1-2x}\right)}$ up to and including the term x3. Hence find the value of $\sqrt{\frac{1.02}{0.98}}$ to four significant figures. Deduce the value of $\sqrt{51}$ to 3 significant figures [1.0202, 7.14]
5. Five millions shillings are invested each year at a rate of 15%. In how many years will it accumulate to more than 50millions? [6years]
6. Expand $\left(1 - \frac{x}{3}\right)^{\frac{1}{2}}$ as far as x3. Hence evaluate $\sqrt{8}$ correct to 3 decimal places [2.829]
7. A man deposits sh. 800,000 into his saving account on which interest is 15% per annum. If he makes no withdraws, after how many years will his balance exceed sh. 8millions? [16.5 years]
8. Determine the binomial expansion of $\left(1 + \frac{x}{2}\right)^4$. Hence evaluate $(2.1)^4$ correct to 2 decimal places.[19.45]
9. Determine the binomial expansion of $\left(1 - \frac{x}{2}\right)^5$. Hence evaluate $(0.875)^5$ correct to four decimal places [0.5129]
10. A financial credit society give a compound interest of 2% per annum to its members. If Bbosa deposits sh. 10000 at the beginning of every year. How much would he accumulate at the end of the fifth year if no withdraws within this period [sh. 530812]
11. Expand $\sqrt{\left(\frac{1+x}{1-x}\right)}$ in ascending powers of x up to a term x^2. $\left[1 + x + \frac{x^2}{x} + \cdots\right]$
12. (a) Using the expansion $(1 + x)^{\frac{1}{2}}$ up to the term x^3, find the value of $\sqrt{1.08}$ to four decimal places [1.0392]
    (b) Express $\sqrt{1.08}$ in the form $\frac{a}{b}\sqrt{c}$. Hence evaluate $\sqrt{3}$ correct to 3 significant figure $\left[\frac{3}{5}\sqrt{3}, 1.73\right]$
13. (a) obtain the first four non – zero terms of the binomial expansion in ascending powers of x of $(1 - x^2)^{-\frac{1}{2}}$ give that $|x| < 1$ $\left[1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16}\right]$
    (b) show that, when x = $\frac{1}{3}$; $(1 - x^2)^{-\frac{1}{2}} = \frac{3\sqrt{2}}{4}$
    (c) substitute x = $\frac{1}{3}$ into your expansion and hence obtain an approximation of $\sqrt{2}$, give your answer to five decimal places [1.41415]
14. (a) show that $\frac{1}{\sqrt{4-x}} = \frac{1}{2}\left(1 - \frac{x}{4}\right)^{-\frac{1}{2}}$
    (c) Write the first three terms in the binomial expansion of $\left(1 - \frac{x}{4}\right)^{-\frac{1}{2}}$ in ascending power of x stating the range for which this expansion is valid. $\left[1 + \frac{x}{8} + \frac{3x^2}{128};\ |x| < 4\right]$
    (d) Find the first three terms in the expansion of $\frac{2(1+x)}{\sqrt{4-x}}$ in ascending power

of x for small values of x. $\left[1 + \frac{9x}{8} + \frac{19x^2}{128};\right]$

Thank you

Dr. Bbosa Science