SENIOR FIVE TERM 3

TOPIC 2/4: Random Variables

Competency: The learner models and analyses the outcomes of random phenomena through determining probabilities and expected values for prediction of uncertainties in real life.

Contents

Competency: The learner models and analyses the outcomes of random phenomena through determining probabilities and expected values for prediction of uncertainties in real life. 1

Definition of Random Variables 2

Discrete probability distribution 2

Properties of discrete probability density functions 3

Expectation of x, E(x) or mean of discrete random 8

Variance, Var(x) of discrete probability distribution 13

Properties of the variance for discrete random variable 16

Cumulative distribution function F(X) for discrete random variables 18

Continuous probability distribution 21

Finding Probabilities of continuous random variable 23

Finding p.d.f from a sketch graph of a continuous random variable 28

Expectation or mean of X of continuous random variable 31

Variance of X of continuous random variables 41

Median of continuous random variables 47

Cumulative distribution function, F(x) of continuous random variables 49

Finding f(x) from F(X) of continuous random variables 57

Uniform or rectangular distribution 60

Definition of Random Variables

A random variable is a function that assigns a numerical value to each outcome of a random event, linking the outcomes of chance to a measurable quantity.
For example, when rolling a die, a random variable can assign the numbers 1 through 6 to each face of the die, which allows for mathematical and statistical analysis.

Types of Random Variables

1. Discrete Random Variable

Takes **countable values** (like integers).

Examples

- Number of heads when flipping 3 coins.
- Number of students absent in a class.
- Rolling a die → possible values: 1, 2, 3, 4, 5, 6.
- the number of phone calls received in a 15-minute period.

2. Continuous Random Variable

Takes **uncountably infinite values** within an interval.

Examples

- Height of students in a school.
- Time taken to run a marathon.
- Temperature measured during the day.
- the exact time it takes to complete an exam.

Discrete probability distribution

It a function or table that assigns **probabilities** to each possible value of a discrete random variable.

Focus: It describes the *likelihood* of each outcome.

For example tossing a fair coin, P(H) = P(T) = ½

Differences between discrete random variable and discrete probability distribution

Aspect	Discrete Random Variable	Discrete Probability Distribution
Meaning	Represents possible outcomes of an experiment	Assigns probabilities to those outcomes
Nature	Variable (values only)	Function/table (values + probabilities)
Example	Number of heads in 3 coin tosses	Probability of getting 0, 1, 2, or 3 heads
Role	Defines the sample space	Quantifies likelihood of each outcome

Properties of discrete probability density functions

(i) $\sum P(X = x) = 1 \; or \; \sum f(x) = 1$

(ii) P(X=x)≥0

Examples 1

A discrete random variable has a probability function $P(X = x) = \begin{cases} cx^2 & x = 0, 1, 2, 3, 4 \\ 0 & otherwise \end{cases}$

Find the value of c and draw the graph of P(X = x)

Solution

$\sum P(X = x) = 1$

$c(0^2) + c(1^2) + c(2^2) + c(3^2) + c(4^2) = 1$

c + 4c + 9c + 16c = 2

c = $\frac{1}{30}$

Example 2

A discrete random variable has probability function

$f(x) = \begin{cases} kx, & x = 1, 2,3,4 \\ 0, & otherwise \end{cases}$, find the value of k and draw the graph of f(x)

Solution

$\sum f(x) = 1$

k + 2k + 3k + 4k = 1

k = $\frac{1}{10}$

Example 3

A random variable X of a discrete probability distribution given by

P(X=1) = 0.2, P(X=2) = P(X=3) = 0.1, P(X=4) = P(X=5) = c

Find the value of the constant c and draw the graph of P(X = x)

Solution

$\sum P(X = x) = 1$

0.2 + 0.1 + 0.1 + c + c = 2; c = 0.3

Example 4

A discrete random variable has a probability function

$$P(X=x)=\begin{cases} k\left(\frac{2}{3}\right)^x, & x=1,2,3,\dots \\ 0, & otherwise \end{cases}$$

Find the value of k

Solution

$$k\left(\frac{2}{3}\right)^0 + k\left(\frac{2}{3}\right)^1 + k\left(\frac{2}{3}\right)^2 + k\left(\frac{2}{3}\right)^3 + \cdots . = 1$$

$$k\left(1 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots .\right) = 1$$

Sum to infinity = $S_\infty = \frac{a}{1-r}$

⇨ $k\left(\frac{1}{1-\frac{2}{3}}\right)$ = 1; k = $\frac{1}{3}$

Finding probabilities of discrete random variable

Example 5

A discrete random variable has a probability distribution

y	-3	-2	-1	0	1
P(Y=y)	0.1	0.25	0.3	0.15	a

Find

(i) value of a (ii) P(-3≤ $Y < 0$) (iii) P(Y>-1) (iv) P(-1<Y<1) (v) mode

Solution

(i) $\sum P(Y=y) = 1$

0.1 + 0.25 + 0.3 + 0.15 + a = 1; a = 0.2

(ii) P(-3≤ $Y < 0$) = P(Y= -3) + P(Y = -2) + P(Y = -1) = 0.1 + 0.25 + 0.3 = 0.65

(iii) P(Y> -1) = P(Y = 0) + P(Y = 1) = 0.15 + 0.2 = 0.35

(iv) P(-1<Y<1) = P(Y = 0) = 0.15

(v) mode is the value y with the highest probability, mode = -1

Example 6

A discrete random variable X has a probability distribution

X	1	2	3	4	5
P(X = x)	0.15	0.20	0.15	c	0.1

Find

(i) the value of x (ii) P(X<4) (iii) P(X≤4 (iv) P(2≤ $X \leq 4$) (v) P$\left(\frac{X>2}{X\leq 4}\right)$ (vi) mode

Solution

(i) $\sum P(X = x) = 1$

0.15 + 0.20 + 0.15 + c + 0.1 = 1; c = 0.4

(ii) P(X<4) = P(X=1) + P(X= 2) + P(X = 3) = 0.15 + 0.20 + 0.15 = 0.5

(iii) P(X<4) = P(X=1) + P(X= 2) + P(X = 3) + P(X= 4) = 0.15 + 0.20 + 0.15 + 0.4 = 0.9

(iv) P(2≤ $X \leq 4$) = P(X = 2) + P(X = 3) + P(X = 4) = 0.20 + 0.15 + 0.4 = 0.75

(v) P$\left(X > 2/_{X \leq 4}\right)$ = $\frac{P(X>2,\ X\leq 4)}{P(X\leq 4)}$ = $\frac{P(X=3)+P(X=4)}{\text{P(X=1) + P(X= 2) + P(X = 3) + P(X= 4)}} = \frac{0.15\ +\ 0.4}{0.9}$ = 0.6111

(vi) the mode is a value with highest probability = 4

Example 7

A discrete random variable X has a probability function

$$f(x) = \begin{cases} kx, & x = 1, 2,3,4,5 \\ 0, & otherwise \end{cases}$$

Find (i) the value of k (ii) P(X = 3) (iii) P(X≥3) (iv) P(X≤ 3) (v) P(1< X ≤3) (vi) P$\left(\frac{X\geq 1}{X<4}\right)$

Solution

(i) $\sum P(X = x) = 1$

k + 2k + 3k + 4k + 5k = 1; k = $\frac{1}{15}$

(ii) P(X=3) = 3k = $\frac{3}{15} = \frac{1}{5}$

(iii) P(X≥3)= P(X=3) + P(X=4) + P(X = 5) = 3k + 4k + 5k = 12k = $\frac{12}{15} = \frac{4}{5}$

(iv) P(X≤ 3)= P(X= 1) + P(X=2) + P(X = 3) = k + 2k + 3k =$\frac{6}{15} = \frac{2}{5}$

(v) P(1< X ≤3) = P(X=2) + P(X = 3) =2k + 3k =$\frac{5}{15} = \frac{1}{3}$

(vi) P$\left(X \geq 2/_{X < 4}\right) = \frac{P(X\geq 2, X<4)}{P(X<4)} = \frac{P(X=2)+P(X=3)}{P(X=1)+P(X=2)+P(X=3)} = \frac{2k+3k}{k+2k+3k} = \frac{5k}{6k} = \frac{5}{6}$

Revision exercise 1

1. A discrete random variable X has probability distribution

x	1	2	3	4	5
P(X=x)	0.2	0.25	0.4	a	0.05

Find (i) value of a = 0.1 (ii) P(1$\leq x \leq 3$)=0.85 (iii) P(X>2) =0.55 (iv) P(2<X<5) = 0.5 (v) mode =3

2. A random variable x of a discrete pdf is given by P(X x) = kx, x = 12, 13, 14
Write the probability distribution and find the value of k

x	12	13	14
P(X=x)	12k	13k	14k

k = $\frac{1}{39}$

3. A random variable Y of discrete probability distribution is given by
P(Y = 3) = 0.1, P(Y = 5) = 0.05, P(Y=6) = 0.45 P(Y = 8) = 3P(Y = 10). Find P(Y=10) = 0.1

4. A discrete random variable has a distribution

x	1	2	3	4	5
P(X=x)	0.1	0.3	k	0.2	0.05

Find
(i) value of k = $\frac{7}{20}$ (ii) P(X≥4) = 0.25 (iii) P(X<1)=0 (iv) P(2$\leq x < 4$) = $\frac{13}{20}$

5. Write out the probability distribution for each of these variables
(a) The number of heads X obtained when two fair coins are tossed

x	0	1	2
P(X=x)	0.25	0.5	0.25

(b) The number of tails, X obtained when three fair coins are tossed.

x	0	1	2	3
P(X=x)	0.125	0.375	0.375	0.125

6. A drawer contains 8 brown socks and 4 blue shocks. A sock is taken from the drawer at random, its colour is noted and it is then replaced. The procedure is performed twice more. X is the random variable for the number of brown socks taken. Find the probability distribution for X.

x	0	1	2	3
P(X=x)	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

7. The discrete random variable R has a p.d.f is given by P(R=r) = c(3 – r), r = 0, 1, 2, 3
Find (i) value of c = $\frac{1}{6}$ (ii) P(1≤R<3) = 0.5

8. A discrete random variable has probability function

$P(X = x) = \begin{cases} k\left(\frac{4}{5}\right)^x, & x = 1, 2, 3, \ldots \\ 0, & otherwise \end{cases}$, find the value k = 0.2.

Solutions to revision exercise

5 Write out the probability distribution for each of these variables

(a) The number of heads X obtained when two fair coins are tossed

S =(TT, TH, HT, HH)

P(X= 0) = $\frac{1}{4} = 0.25$, P(X= 1) = $\frac{2}{4} = 0.50$, P(X= 2) = $\frac{1}{4} = 0.25$

Probability distribution table

x	0	1	2
P(X=x)	0.25	0.5	0.25

(b) The number of tails, X obtained when three fair coins are tossed.

S = (TTT, TTH, THT, HTH, THH, HTH, HHT, HHH)

Probability distribution table

number of heads, x	0	1	2	3
P(X=x)	$\frac{1}{8}$ = 0.125	$\frac{3}{8}$ = 0.375	$\frac{3}{8}$=0.375	$\frac{1}{8}$ = 0.125

6. A drawer contains 8 brown socks and 4 blue shocks. A sock is taken from the drawer at random, its colour is noted and it is then replaced. The procedure is performed twice more. X is the random variable for the number of brown socks taken. Find the probability distribution for X.

Let X' represent blue shocks

P(X = 0) = P(X'∩X'∩X') = $\frac{4}{12}x\frac{4}{12}x\frac{4}{12} = \frac{1}{27}$

P(X=1) = P(X∩X'∩X')+ P(X'∩X∩X') + P(X'∩X'∩X) = $\frac{8}{12}x\frac{4}{12}x\frac{4}{12}+\frac{4}{12}x\frac{8}{12}x\frac{4}{12}+\frac{4}{12}x\frac{4}{12}x\frac{8}{12}=\frac{2}{9}$

P(X=2) = P(X∩X∩X')+ P(X'∩X∩X) + P(X∩X'∩X) =$\frac{8}{12}x\frac{8}{12}x\frac{4}{12}+\frac{4}{12}x\frac{8}{12}x\frac{8}{12}+\frac{8}{12}x\frac{4}{12}x\frac{8}{12}=\frac{4}{9}$

P(X = 3)= P(X∩X∩X)=$\frac{8}{12}x\frac{8}{12}x\frac{8}{12}=\frac{8}{27}$

Probability distribution table

x	0	1	2	3
P(X=x)	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

7. The discrete random variable R has a p.d.f is given by P(R=r) = c(3 – r), r = 0, 1, 2, 3

Find (i) value of c

$\sum P(X = x) = 1$

3c +2c + c = 3

c= $\frac{1}{6}$

(ii) P(1≤R<3) = 2c + c = 3c = 3 x $\frac{1}{6}$ =0.5

8. A discrete random variable has probability function

$$P(X = x) = \begin{cases} k\left(\frac{4}{5}\right)^x, & x = 1, 2, 3, \dots \\ 0, & otherwise \end{cases}$$, find the value k.

Solution

$$k\left(\frac{4}{5}\right)^0 + k\left(\frac{4}{5}\right)^1 + k\left(\frac{4}{5}\right)^2 + k\left(\frac{4}{5}\right)^3 + \cdots = 1$$

$$k\left(1 + \left(\frac{4}{5}\right)^1 + \left(\frac{4}{5}\right)^2 + \left(\frac{4}{5}\right)^3 + \cdots\right) = 1$$

Sum to infinity = $S_\infty = \frac{a}{1-r}$

⇨ $k\left(\frac{1}{1-\frac{4}{5}}\right)$ = 1; k = $\frac{1}{5}$ =0.2

Expectation of x, E(x) or mean of discrete random

The expected value of x is given by E(x) = $\sum xP(X = x)$

Example 8

A discrete random variable has a probability distribution

x	-2	-1	0	1	2
P(X = x)	0.3	0.1	0.15	0.4	0.05

Find expectation, E(x)

Solution

E(X) = (-2 x 0.3) + (-1 x 0.1) + (0 x 0.15) + (1 x 0.4) + (2 x 0.05) = -0.2

Example 9

The discrete random variable Y has a probability distribution is given by

P(Y = y) = cy, y = 1 2,3,

P(Y = y) = c(8-y), y = 4, 5, 6, 7

Find (i) the value of c (ii) mean, μ

Solution

y	1	2	3	4	5	6	7
P(Y = y)	c	2c	3c	4c	3c	2c	c

(i) $\sum P(Y = y) = 1$

c + 2c + 3c + 4c + 3c +2c + c = 1

c = $\frac{1}{16}$

(ii) E(Y) = $\sum y(Y = y)$ = 1 x c + 2 x 2c + 3 x 3c + 4 x 4c + 5 x 3c + 6 x 2c + 7x c = 64c

= 64 x $\frac{1}{16}$ = 4

Example 10

A fair coin is tossed three times write out the probability distribution for the number of heads, X, obtained and hence obtain the expected number of heads

Solution

S = (TTT, TTH, THT, HTH, THH, HTH, HHT, HHH)

Probability distribution table

number of heads, x	0	1	2	3
P(X=x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

E(X) = $\sum x(X = x)$ = (0 x $\frac{1}{8}$) $+ 1x\ \frac{3}{8} + 2\ x\ \frac{3}{8} + 3\ x\ \frac{1}{8} = \frac{12}{8} = 1.5$

Example 11

A family plans to have 4children. Given that X is the number of girls in the family. Find the expected number of girls

Solution

S = (BBBB, BBBG, BBGB, BGBB, GBBB, BBGG, BGGB, BGBG, GGBB, BGGB, GBBG, BGGG, GBGG, GGBG, GGGB GGGG)

Probability distribution table

Number of girls, x	0	1	2	3	4
P(X=x)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

E(X) = $\sum x(X = x)$ = $\left(0\ x\ \frac{1}{16} + 1\ x\ \frac{4}{16} + 2\ x\ \frac{6}{16} + 3\ x\ \frac{4}{16} + 4\ x\ \frac{1}{16}\right)$ = 2

Example 12

A box A contains 4 red sweets and 3 green sweets. Box B contains 5 red sweets and 6 green sweets. Box A is twice more likely to be picked as Box B. If a box is chosen at random and two sweets are removed from it, one at a time without replacement.

(a) Find the probability that two sweets removed are of the same colour.

P(same colour) = $P(A\cap R_1\cap R_2) + P(A\cap G_1\cap G_2) + P(B\cap R_1\cap R_2) + P(B\cap B_1\cap B_2)$

$= \frac{2}{3} x \frac{4}{7}\ x \frac{3}{6} + \frac{2}{3} x \frac{3}{7}\ x \frac{2}{6} + \frac{1}{3} x \frac{5}{11}\ x \frac{4}{10} + \frac{1}{3} x \frac{6}{11}\ x \frac{5}{10}$

$=\frac{24}{126}+\frac{12}{126}+\frac{20}{330}+\frac{30}{330}=\frac{42}{126}+\frac{50}{330}=0.4372$

(b) (i) construct a probability distribution table for the number of red sweets removed

Let x = number of red sweets removed

P(X =0) = P(A∩G_1∩G_2) + P(B∩G_1∩G_2) = $\frac{2}{3}x\frac{3}{7}\ x\frac{2}{6}+\frac{1}{3}x\frac{6}{11}\ x\frac{5}{10}$ = 0.1861

P(X=1) = P(A∩R_1∩G_2) + P(A∩G_1∩R_2) + P(B∩R_1∩G_2) + P(B∩G_1∩R_2)

$=\frac{2}{3}x\frac{4}{7}\ x\frac{3}{6}+\frac{2}{3}x\frac{3}{7}\ x\frac{4}{6}+\frac{1}{3}x\frac{5}{11}\ x\frac{6}{10}+\frac{1}{3}x\frac{6}{11}\ x\frac{5}{10}$

$=\frac{24}{126}+\frac{24}{126}+\frac{30}{330}+\frac{30}{330}=\frac{48}{126}+\frac{60}{330}$

= 0.5628

P(X=2) = P(A∩R_1∩R_2)+ P(B∩R_1∩R_2) =$\frac{2}{3}x\frac{4}{7}\ x\frac{3}{6}+\frac{1}{3}x\frac{5}{11}\ x\frac{4}{10}=\frac{24}{126}+\frac{20}{330}$ = 0.2511

Probability distribution table

x	0	1	2
P(X= x)	0.1861	0.5628	0.2511

(ii) find the mean number of red sweets removed

Mean = $\sum x(X=x)$ = 0 x0.1861 + 1 x 0.5628 + 2 x 0.2511 = 1.065

Revision exercise 2

1. A discrete random variable X has a probability distribution.

x	0	1	2	3	4
P(X= x)	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{6}$

Find E(X) = 2.25

2. A discrete random variable X ha probability distribution

x	5	6	7	8	9
P(X=x)	$\frac{3}{11}$	$\frac{2}{11}$	$\frac{1}{11}$	$\frac{2}{11}$	$\frac{3}{11}$

Find the mean = 7

3. A discrete random variable X has a probability distribution

4. A discrete random variable has a probability distribution

x	0	1	2	3
P(X=x)	c	c^2	c^2+c	$3c^2$ + 2c

Find (i) the value of c= 0.2 (ii) expectation of c=2.08

5. Find the expected number of heads when two fair coins are tossed (E(x) 1)
6. A family plans to have 3 children. Given that x is the number of boys in the family. Find the expected number of boys (=1.5)

7. If X is a random variable for the product of the scores on two tetrahedral dice, where the score is the number on which the die lands, find the expected score for the throw (=6.25)
8. A bag contains 5 black counters and 6 red counters. Two counters are drawn at random, one at a time without replacement. Find the expected number of red counters. (= $\frac{12}{11}$)
9. An unbiased tetrahedral die is tossed once. If it lands on a face marked 1, the player has to pay 10,000/=. If it lands on marked with 2 or 4 the player wins 5000/= and if it lands on a 3, the player wins 3000/=. Find the expected gain in one throw.
10. A discrete random variable X can take on values 10 and 20 only. If E(X) = 16. Write out the probability distribution for X (P(X=x) = 0.4 and P(X=x) = 0.6)
11. A discrete random variable X can take on values 0, 1, 2, and 3 only. If E(X) = 1.4, P(X$\leq$ 2) = 0.9 and P(X$\leq$1) = 0.5. Find (i) P(X=1) = 0.3 (ii) P(X =0) = 0.2
12. The discrete random variable Y has a probability distribution is given by
    P(Y=y) = cy, y = 1, 2, 3, 4
    Find (i) value of c = 0.1 (ii) E(X) = 3
13. A discrete random variable has p.d.f
    $$P(X=x)=\begin{cases} k2^x, & x=0,1,2,3,4,5,6 \\ 0, & otherwise \end{cases},$$
    Find (i) value of k = $\frac{1}{127}$, (ii) mean = 5
14. A discrete random variable X has a probability distribution

x	0	1	2	3	4	5
P(X= x)	0.11	0.17	0.2	0.13	p	0.09

Find (i) the value of p = 0.3 (ii) Expected value of X (= 2.6)

Solutions to revision exercise 2

10. A discrete random variable X can take on values 10 and 20 only. If E(X) = 16. Write out the probability distribution for X
    Let P(X=10) = a and P(X=20) = b
    a+b = 1
    a = (1-b)………………. (i)
    10a + 20b = 16 ……. (ii)
    Eqn. (i) and eqn. (ii)
    10(1-b) + 20b = 16
    10 + 20b = 16
    b = 0.6
    a = 1- 0.6 = 0.4

Probability distribution:(P(X=x) = 0.4 and P(X=x) = 0.6)

11. A discrete random variable X can take on values 0, 1, 2, and 3 only. If E(X) = 1.4, P(X$\leq$ 2) = 0.9 and P(X$\leq$1) = 0.5. Find (i) P(X=1) (ii) P(X =0)
    Let P(X = 0) = a, P(X=1)=b, P(X=2)= c p(X = 3) = d
    a + b + c + d = 1 ……………. (i)
    P(X$\leq$ 2) = a + b+ c = 0.9 … (ii)

Eqn. (i) and eqn. (ii)
d = 0.1
P(X≤1) = a + b = 0.5 ……. (iii)
Eqn. (i) and eqn. (iii)
0.5 + c + 0.1 = 1
c = 0.4
E(X) = 0 x a + 1 x b + 2 x c + 3 x 0.1 = 1.4
= b + 2c + 0.3 = 1.4
b + 2c = 1.1
b + 2 x 0.4 = 1.1
b = 0.3
a = 0.5 – 0.3 = 0.2
Hence, (i) P(X=1) = 0.3 (ii) P(X =0) = 0.2

12. The discrete random variable Y has a probability distribution is given by

P(Y=y) = cy, y = 1, 2, 3, 4

Find (i) value of c = 0.1 (ii) E(X) = $\frac{11}{3}$

(i) $\sum P(X = x) = 1$

c + 2c + 3c + 4c = 1

10c = 1; c= 0.1

(ii) E(X) =$\sum xP(X = x)$= 1 x 0.1 + 2 x 0.2 + 3 x 0.3 + 4 x 0.4 =3

13. A discrete random variable has p.d.f

$$P(X = x) = \begin{cases} k2^x, & x = 0,1,2,3,4,5,6 \\ 0, & otherwise \end{cases},$$

Find

(i) (i) value of k = $\frac{1}{127}$,

$\sum P(X = x) = 1$

k($2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6$) = 1

k = $\frac{1}{127}$

(ii) Mean = $\sum Px(X = x) = \frac{1}{127}$(0 x 1 + 2 x 2 + 3 x 8 + 4 x 16 + 4 x 32 + 6 x 64) = 5.01

14. A discrete random variable X has a probability distribution

x	0	1	2	3	4	5
P(X= x)	0.11	0.17	0.2	0.13	p	0.09

Find

(i) the value of p

$\sum P(X = x) = 1$

0.11 + 0.17 + 0.2 + 0.13 + p + 0.09 = 1

p = 0.3

(ii) Expected value of X

E(X) =$\sum Px(X = x)$ = 0.11x0 + 0.17x1 + 0.2x2 + 0.13x3 + 0.3 x 4 + 0.09x5= 2.61

Properties of the mean

(i) E(a) = a
(ii) E(ax) = aE(x)
(iii) E(ax + b) = aE(x) + b
(iv) E(ax - b) = aE(x) - b

Example 13

A random variable X of discrete probability distribution is given by

x	1	2	3	4
P(X=x)	0.1	0.2	0.3	0.4

Find

(i) E(x) = $\sum Px(X = x)$ = 1 x 0.1 + 2 x 0.2 + 3 x 0.3 + 4 x 0.4 = 3
(ii) E(3x) = 3E(x) = 3 x 3 = 9
(iii) E(4x + 6) =4E(x) + 6 = 4 x 3 + 6 = 18

Example 14

A random variable X of discrete probability distribution is given by

x	-1	0	1	2
P(X= x)	0.25	0.10	0.45	0.20

Find

(i) P(-1$\leq$ X <1) = P(X= -1) + p(X = 0) = 0.25 + 0.10 = 0.35
(ii) E(X) =$\sum Px(X = x)$ = -1 x 0.25 + 0 x 0.10 + 1 x 0.45 + 2 x 0.20 = 0.6
(iii) E(6x – 2) = 6E(X) – 2 = 0.6 x 6 – 2 = 1.6

Variance, Var(x) of discrete probability distribution

Var(x) = E(X^2) = [E(x)]^2 where E(X^2) = $\sum x^2 P(X = x)$

Example 15

A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X=x)	0.1	0.3	0.2	0.3	0.1

Find

(i) The mean = $\sum Px(X = x)$ = 1 x 0.1 + 2 x 0.3 + 3 x 0.2 + 4 x 0.3 + 5 x 0.1= 3
(ii) Var(x)
E(X^2) =$\sum Px^2(X = x)$ = 1^2 x 0.1 + 2^2 x 0.3 + 3^2 x 0.3 + 4^2 x 0.3 + 5^2 x 0.1 = 10.4

Var(X) = 10.4 – $(3)^2$ = 1.4

Example 16

The discrete random variable Y has a probability distribution is given by P(Y =y), y= -3, -2, -1, 0, 1, 2,3

Find: (i) value of c (ii) mean (iii) standard deviation

Solution

y	-3	-2	-1	0	1	2	3
P(Y= y)	3c	2c	c	0	c	2c	3c

(i) $\sum P(X = x) = 1$

3c +2c + c+ c + 2c + 3c = 1; c = $\frac{1}{12}$

(ii) Mean = $\sum Px(X = x)$ = -3 x 3c + -2 x 2c + -1 x c + 0 x 0 + 1 x c + 2 x 2c + 3 x 3c = 0

(iii) $E(X^2) = (-3)^2 \times 3c + (-2)^2 \times 2c + (-1)^2 \times c + (0)^2 \times 0 + (1)^2 \times c + (2)^2 \times 2c + (3)^2 \times 3c = 72 \times \frac{1}{12} = 6$

Var (x) = $E(X^2) - (E(x))^2 = 6 - (0)^2 = 6$

S.D = $\sqrt{Var\ (X)} = \sqrt{6} = 2.45$

Example 17

Two marbles are drawn without replacement from a box containing 3 red marbles and 4 white marbles. The marbles are randomly drawn. If X is the random variable for the number of red marble drawn find

(i) Expected number of red marbles

P(X= 0) = P(W∩W) = $\frac{4}{7}\ x\ \frac{3}{6} = \frac{2}{7}$

P(X= 1) = P(W∩R) + P(R∩W) = $\frac{4}{7}\ x\ \frac{3}{6} + \frac{3}{7}\ x\ \frac{4}{6} = \frac{4}{7}$

P(X= 2) = P(R∩R) = $\frac{3}{7}\ x\ \frac{2}{6} = \frac{1}{7}$

The probability distribution table

x	0	1	2
P(X=x)	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

E(x) = $\sum Px(X = x) = \frac{2}{7}\ x\ 0 + \frac{4}{7}\ x\ 1 + \frac{1}{7}\ x\ 2 = \frac{6}{7}$

(ii) Standard deviation of X

$E(x^2) = \sum Px^2(X = x) == \frac{2}{7}\ x\ 0 + \frac{4}{7}\ x\ 1^2 + \frac{1}{7}\ x\ 2^2 = \frac{8}{7}$

Var (x) = $E(x^2) - (E(X))^2 = \frac{8}{7} - \left(\frac{6}{7}\right)^2 = \frac{20}{49}$

S.D = $\sqrt{Var\ (X)} = \sqrt{\frac{20}{49}} = 0.6389$

Example 18

A vendor stocks 12 copies of a magazine each week and the probability for each possible total number of copies sold is shown below

Number of copies	9	10	11	12
probability	0.2	0.35	0.30	0.15

(a) Estimate the mean and variance of the number of copies

Mean = $\sum Px(X = x)$ = 9 x 0.2 + 10 x 0.35 + 11 x 0.3 + 12 x 0.15 = 10.4

$E(X^2)$ = 9^2 x 0.2 + 10^2 x 0.35 + 11^2 x 0.3 + 12^2 x 0.15 =109.1

Var(x) = 109.1 – $(10.4)^2$ = 0.94

(b) The vendor buys the magazine at 8,500/= and sells at 14,500/=. Any copies not sold are destroyed. Construct a probability distribution table for vendor's weekly profit and hence find the expected weekly profit

Profit = S.P – C.P

Profit for 9 copies = 9 x 14,500 – 12 x 8500 = 28500

Profit for 10 copies= 10 x 14,500 – 12 x 8500 =43000

Profit for 11 copies= 11 x 14,500 – 12 x 8500 = 57500

Profit for 12 copies= 12 x 14,500 – 12 x 8500 = 72000

y	28500	43000	57500	72000
P(Y=y)	0.2	0.35	0.30	0.15

E(Y) = 0.2 x 28500 + 0.35 x 43000 + 0.30 x 57500 + 0.15 x 72000 = 48000/=

Example 19

The table below shows the number of red and green balls put in three identical boxes A, B and C.

Boxes	A	B	C
Red balls	4	6	3
Green balls	2	7	5

A box is chosen at random and two balls are then drawn from it successively without replacement. If the random variable X is "the number of green balls drawn".

(a) Draw a probability distribution table for X (06marks)

Using combination

$$P(X = 0) = \frac{1}{3}\left[\frac{^4C_2}{^6C_2} + \frac{^6C_2}{^{13}C_2} + \frac{^3C_2}{^8C_2}\right]$$

$$= \frac{1}{3}\left[\frac{2}{5} + \frac{5}{26} + \frac{2}{28}\right] = \frac{1273}{5460}$$

$$P(X = 1) = \frac{1}{3}\left[\frac{^2C_1\ x\ ^4C_1}{^6C_2} + \frac{^7C_1\ x\ ^6C_1}{^{13}C_2} + \frac{^3C_1\ x\ ^3C_1}{^8C_2}\right]$$

$$= \frac{1}{3}\left[\frac{8}{15} + \frac{7}{13} + \frac{15}{28}\right] = \frac{8777}{16380}$$

$$P(X = 2) = \frac{1}{3}\left[\frac{^2C_2}{^6C_2} + \frac{^7C_2}{^{13}C_2} + \frac{^5C_2}{^8C_2}\right]$$

$= \frac{1}{3}\left[\frac{1}{15} + \frac{7}{26} + \frac{5}{14}\right] = \frac{946}{4095}$

x	0	1	2
P(X =x)	$\frac{1273}{5460}$	$\frac{8777}{16380}$	$\frac{946}{4095}$

(b) Calculate the mean and variance of X (06marks)

1	0	1	2
P(X =x)	$\frac{1273}{5460}$	$\frac{8777}{16380}$	$\frac{946}{4095}$
xP(X =x)	0	$\frac{8777}{16380}$	$\frac{1892}{4095}$
x^2P(X =x)	0	$\frac{8777}{16380}$	$\frac{3784}{4095}$

E(X) = $\frac{8777}{16380} + \frac{1892}{4095}$ = 0.9979

E(X^2) = $\frac{8777}{16380} + \frac{3784}{4095}$ = 1.4599

Var(X) = 1.4599 – 0.9979

= 0.4642

Properties of the variance for discrete random variable

(i) Var(a) = 0

(ii) Var(aX) = a^2Var(X)

(iii) Var(aX + b) = a^2Var(X)

(iv) Var(aX – b) = a^2Var(X)

Example 20

A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X = x)	0.2	0.25	0.4	0.1	0.05

Find

(i) Mean = $\sum Px(X = x)$ = 1 x 0.2 + 2 x 0.25 + 3 x 0.4 + 4 x 0.1 + 5 x 0.05 = 2.55

(ii) The variance

E(X^2) = 1^2 x 0.2 + 2^2 x 0.25 + 3^2 x 0.4 + 4^2 x 0.1 + 5^2 x 0.05 = 7.65

Var(x) = E(X^2) – $(E(X))^2$ = 7.65 – $(2.55)^2$ = 1.148

(iii) Var (3x -2) = 3^2Var(x) = 9 x 1.148 = 10.332

Example 21

A random variable X of a discrete probability distribution given by

x	10	20	30
P(X = x)	0.2	0.3	0.5

Find

(i) E(X) = 10 x 0.2 + 20 x 0.3 + 30 x 0.5 =22

(ii) Var (X) = $E(X^2) - (E(X))^2$

$E(X^2) = 10^2$ x 0.2 + 20^2 x 0.3 + 30^2 x 0.5 = 520

Var(x) = 520 – 22^2 = 36

(iii) Var(4X + 3) = 4^2Var(x) = 16 x 36 = 576

Revision exercise 3

1. A random variable X of discrete probability distribution is given by

x	1	2	3
P(X = x)	0.2	0.3	0.5

Find (i) E(X) = 2.3 (ii) $E(X^2)$ = 5.9 (iii) Var (X) = 0.61

2. A random variable X of discrete probability distribution is given by

x	-1	0	1	2
P(X = x)	0.25	0.1	0.45	0.2

Find: (i) P(-1≤ X < 2) = 0.8 (ii) E(X) = 0.6 (iii) E(2x + 3) = 4.2

3. A random variable X of a discrete probability distribution
P(X = 0) = 0.05, P(X = 1) = 0.45 P(X = 2) = 0.5
Find: (i) E(X) = 1.45, (i) $E(X^2)$ = 2.45 (iii) Var (X) = 0.348
4. A random variable X of discrete probability distribution is given by P(X = 1) 0.1, P(X= 2) = 0.2, P(X = 3) = 0.3, P(X =4)= 0.4
Find (i) E(X) = 3 (ii) Var (X) = 1 (iii) P(X = 2/X≥2) = $\frac{2}{9}$
5. The discrete random variable Y has a probability distribution P(Y = y) = k =1, 2, 3, 4, 5,6
Find (i) mean, μ = 3.5 (ii) E(3X +4) = $15\frac{1}{6}$ (iii) $E(X^2)$=14.5 (iv) standard deviation = 1.708
6. The discrete random variable R has a probability distribution is given by
P(R = r) = $\frac{3r+1}{22}$; r = 0, 1, 2, 3
Find (i) mean, μ= $\frac{24}{11}$, E(R2) = $\frac{61}{11}$ (iii) E(3R-2) = $\frac{50}{11}$
7. The discrete random variable R has a probability distribution given by

$$P(R=r) = \begin{cases} \frac{2r+1}{20}; r = 0, 1, 2, 3 \\ \frac{11-r}{20}, \quad r = 4, 5 \end{cases}$$

Find (i) E(R) = 2.55, (ii) Var(R) = 1.45

8. The discrete random variable X has a probability distribution given by

$$P(X=x) = \begin{cases} kx, & x = 1, 2, 3, 4, 5 \\ k(10-x), & x = 6, 7, 8, 9 \end{cases}$$

Find (i) constant, k = 0.04, (i) E(X) = 5 (iii) Var(X) = 4

9. The discrete random variable X has a probability distribution is given by

P(X =x) = kx, x = 1, 2, 3,…..n ; where k is a constant

Show that k = $\frac{2}{n(n+1)}$, hence find in terms of n the mean X = $\frac{1}{3}(2n+1)$

10. A random variable X of a discrete probability distribution given by

P(X = 0) = P(X=1) = 0.1, P(X = 2) = 0.2, P(X= 3) = P(X = 4) = 0.3. Find Var (X) = 1.64

11. A random variable X of a discrete probability distribution given by

P(X = 2) = 0.1; P(X= 4) = 0.3; P(X = 6) = 0.5; P(X = 8) = 0.1. Find Var (X) 2.56

Cumulative distribution function F(X) for discrete random variables

F(X) is given by FX = $\sum P(X=x)$

Note F(+∞) = 1 where +∞ is the upper limit.

Example22

A discrete random variable has a probability distribution

x	1	2	3	4	5
P(X = x)	0.2	0.25	0.4	0.1	0.05

Find the cumulative distribution function

Solution

x	1	2	3	4	5
F(X))	0.2	0.45	0.85	0.95	1

Example 23

The random variable X has a cumulative function below

X	-1	0	1	2
F(X)	0.25	0.35	0.80	1

Find the probability distribution function

X	-1	0	1	2
F(X)	0.25	0.1	0.45	0.2

Example 24

A discrete random variable has a cumulative distribution

x	1	2	3	4	5
F(X)	0.2	0.32	0.67	0.91	1

Find (i) probability distribution function

x	1	2	3	4	5
F(X)	0.2	0.12	0.35	0.24	0.09

(ii) P(X = 3) =0.35

(iii) P(X>2) = P(X= 3) + P(X =4) + P(X=5) = 1- 0.12 = 0.68

Example 25

The random variable X has a cumulative function

X	1	2	3	4
F(X)	0.1	0.5	0.8	1

Find (i) mean (ii) Var(X) (iii) mode

Solution

X	1	2	3	4
P(X = x)	0.1	0.4	0.3	0.2

(i) Mean = $\sum xP(X = x)$ = 1 x 0.1 + 2 x 0.4 + 3 x 0.3 + 4 x 0.2 = 2.6

(ii) Var(X)

$E(X^2) = 1^2 \times 0.1 + 2^2 \times 0.4 + 3^2 \times 0.3 + 4^2 \times 0.2 = 5.92$

$Var(X) = E(X^2) - (E(X))^2 = 5.92 - (2.6)^2 = 0.84$

(iii) Mode 2

Revision Exercise 4

1. A discrete random variable has a cumulative distribution

x	0	1	2	3	4
F(X)	01	0.3	0.6	0.8	1

Find (i) E(X) = 15.2 (ii) Var(X)=1.56 (iii) Var(6X + 2) = 56.16

2. The random variable X has a cumulative function below

x	1	2	3	4
F(X)	0.13	0.54	0.75	1

Find (i) P(X=2) = 0.41 (ii) P(X>1) = 0.87 (iii) P(X ≥ 3) = 0.46 (iv) P(X<2) = 0.13 (V)E(X) = 2.58

3. A discrete random variable X has a cumulative distribution

x	3	4	5	6	7
F(X)	0.01	0.23	0.64	0.85	1

Find (i) probability distribution function (ii) Var (X) = 0.9724)

4. A discrete random variable has a cumulative probability function F(X) = $\frac{x^2}{9}$, x = 1, 2, 3.

   Find (1) F(2) = $\frac{4}{9}$ (ii) P(X = 2)= $\frac{1}{3}$ (iii) E(2X -3) = $\frac{17}{9}$

5. A discrete random variable has a cumulative probability function. F(X) = k, x = 1, 2,3

   Find the

   (i) constant k = $\frac{1}{3}$

   (ii) P(X<3) = $\frac{2}{3}$

   (iii) Standard deviation, σ = 0.816

6. A discrete random variable has a cumulative probability function

   F(X) = $1 - \left(1 - \frac{x}{4}\right)^x$ x = 1, 2, 3, 4

   Find the

   (i) F(3) = $\frac{63}{64}$

   (ii) F(2) =$\frac{3}{4}$

   (iii) Var (X) = 0.547

Continuous probability distribution

A probability density function (p.d.f) is continuous if it takes on values between an interval.

Properties of a continuous probability density functions

(i) $\int f(x)dx = 1$

(ii) f (x) ≥ 0

Example 26

A random variable X of continuous p.d.f is given by $f(x) = \begin{cases} kx & 0 \le x \le 5 \\ 0 & elsewhere \end{cases}$

Find the value of k

Solution

$\int_0^5 kxdx = 1$

$k\left[\frac{x^2}{2}\right]_0^5 = 1$

$k\left(\frac{5^2}{2} - \frac{0^2}{2}\right) = 1$

$k\frac{25}{2} = 1$

$k = \frac{2}{25}$

Example 27

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx & 0 \le x \le 2 \\ 2k(x-1), & 2 \le x \le 4 \\ 0, & elsewhere \end{cases}$

Solution

$\int_0^2 kxdx + \int_2^4 2k(x-1)dx = 1$

$k\left[\frac{x^2}{2}\right]_0^2 + 2k\left[\frac{x^2}{2} - x\right]_2^4 = 1$

$\left(\frac{2^2}{2} - \frac{0^2}{2}\right) + 2k\left\{\left(\frac{4^2}{2} - 4\right) - \left(\frac{2^2}{2} - 2\right)\right\} = 1$

2k + 8k = 1; k= $\frac{1}{10}$

Sketching f(x)

- find the initial and final points of f(x)
- join the initial and final points of f(x) using a line or curve.

Note

- A line is in the form of y = mx + c
- A curve has a power of x being 2 and above or fractional power e.g. $y = x^2$.
- A curve has a positive coefficient of x^2 has a minimum turning point while a curve with a negative coefficient has a maximum turning point

Example 28

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} kx & 0 \le x \le 3 \\ 0, & elsewhere \end{cases}$

Find the value of the constant k and sketch f(x)

Solution

$\int_0^3 kxdx = 1$

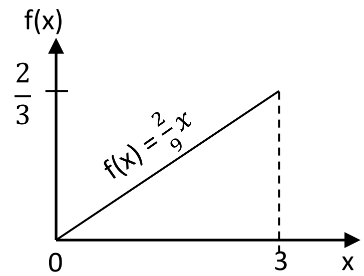
$k\left[\frac{x^2}{2}\right]_0^3 = 1$

$k\left(\frac{3^2}{2} - \frac{0^2}{2}\right) = 1$

k = $\frac{2}{9}$

When x = 0, f(x) = $\frac{2}{9}\ x\ 0 = 0$

When x = 3, f(x) = $\frac{2}{9}\ x\ 3 = \frac{2}{3}$



Example 29

A random variable X of continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 3 \\ k(6-x), & 3 \le x \le 6 \\ 0, & elsewhere \end{cases}$

Find the value of the constant k and sketch x

Solution

$\int_0^3 kxdx + \int_3^6 k(6-x)dx = 1$

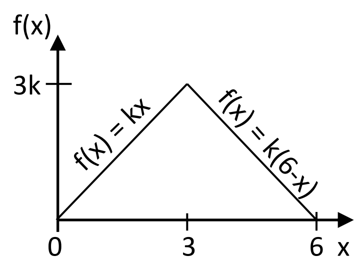
$k\left[\frac{x^2}{2}\right]_0^3 + k\left[6x - \frac{x^2}{2}\right]_3^6 = 1$

k = $\frac{1}{9}$

When x = 0, f(x) = k(0) = 0

When x = 3, f(x) = k(3) = 3k

When x = 6, f(x) = k(6-6) = 0



Example 30

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} k(x+2), & -2 \le x \le 0 \\ k(2-x), & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$

Find the value of k and sketch f(x)

$\int_{-2}^0 k(x+2)dx + \int_0^2 k(2-x)dx = 1$

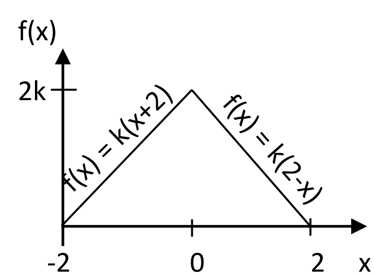
$k\left[\frac{x^2}{2} + 2x\right]_{-2}^0 + k\left[2x - \frac{x^2}{2}\right]_0^2 = 1$

k = $\frac{1}{4}$

When x = -2, f(x) = k(-2+2) = 0

When x = 0, f(x) = k(0 +2) = 2k

When x = 2, f(x) = k(2-2) = 0



Finding Probabilities of continuous random variable

Example 31

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx & 0 \le x \le 6 \\ 0, & elsewhere \end{cases}$

Find

(i) the value of k (ii) P(X > 4) (iii) P(X< 3) (iv) P(1< x < 3) (v) P$\left(X > 2/_{X \le 4}\right)$

Solution

(i) $\int_0^6 kxdx = 1$

$k\left[\frac{x^2}{2}\right]_0^6 = k\left[\frac{6^2}{2} - \frac{0^2}{2}\right] = 1$

k = $\frac{1}{18}$

(ii) P(X > 4) = $\frac{1}{18}\int_4^6 xdx = 1$

= $\frac{1}{18}\left[\frac{x^2}{2}\right]_4^6$ = $\frac{1}{18}\left[\frac{6^2}{2} - \frac{4^2}{2}\right] = \frac{5}{9} = 0.5556$

(iii) P(X < 3) = $\frac{1}{18}\int_0^3 xdx = 1$

= $\frac{1}{18}\left[\frac{x^2}{2}\right]_0^3$ = $\frac{1}{18}\left[\frac{3^2}{2} - \frac{0^2}{2}\right] = \frac{1}{4} = 0.25$

(iii) 1 < x < 3) = $\frac{1}{18}\int_1^3 xdx = 1$

= $\frac{1}{18}\left[\frac{x^2}{2}\right]_1^3$ = $\frac{1}{18}\left[\frac{3^2}{2} - \frac{1^2}{2}\right]$

$= \frac{2}{9} = 0.2222$

(iv) P$\left(X > 2/_{X \le 4}\right)$ = $\frac{P(X>2 \cap X\le 4}{P(X\le 4)}$ = $\frac{P(2<X<4)}{P(X\le 4)} = \frac{\frac{1}{18}\int_2^4 xdx=1}{\frac{1}{18}\int_0^4 xdx=1} = \frac{3}{4}$

Example 32

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx(6-x) & 0 \le x \le 6 \\ 0, & elsewhere \end{cases}$

Find the (i) value of k and sketch f(x) (ii) P(X ≥ 5)

(i) $\int_0^6 kx(6-x)dx = 1$

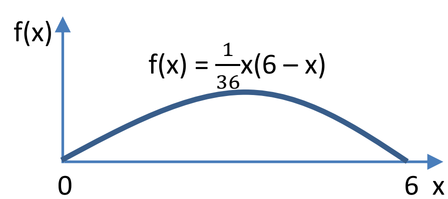
$k\left[3x^2 - \frac{x^3}{3}\right]_0^6 = k\left[\left(3x\ 6^2 - \frac{6^3}{3}\right) - \left(3\ x\ 0^2 - \frac{0^3}{3}\right)\right] = 1$

k = $\frac{1}{36}$

When x = 0, f(x) = $\frac{1}{36}(0)(6-0) = 0$

When x = 6, f(x) = $\frac{1}{36}(6)(6-6) = 0$

Sketch



(ii) P(X ≥ 5) = $\frac{1}{36}\int_5^6 x(6-x)dx$

= $\frac{1}{36}\left[3x^2 - \frac{x^3}{3}\right]_5^6 = \frac{1}{36}\left[\left(3x\ 6^2 - \frac{6^3}{3}\right) - \left(3\ x\ 5^2 - \frac{5^3}{3}\right)\right]$ = 0.074

Example 33

A random variable of continuous p.d.f is given by $f(x) = \begin{cases} kx^2 & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$

Find (i) value of k and sketch f(x) (ii) P(1 ≤ x ≤ 3)

Solution

(i) $\int_0^4 kx^2 dx = 1$

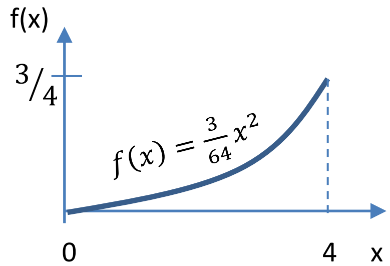
$k\left[\frac{x^3}{3}\right]_0^4 = k\left[\frac{4^3}{3} - \frac{0^3}{3}\right] = 1$

k = $\frac{3}{64}$

When x = 0, f(x) = $\frac{3}{64}0^2 = 0$

When x = 4, f(x) = $\frac{3}{64}4^2 = \frac{3}{4}$

Sketch



(ii) P(1 ≤ x ≤ 3)= $\frac{3}{64}\int_1^3 kx^2 dx = 1$

=$\frac{3}{64}\left[\frac{x^3}{3}\right]_1^3 = \frac{3}{64}\left[\frac{3^3}{3} - \frac{1^3}{3}\right] = 0.4063$

Example 34

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x^2+1) & 0 \le x \le 3 \\ 0, & elsewhere \end{cases}$

Find (i) value of k and sketch f(x) (ii) P(1 ≤ x ≤ 3)

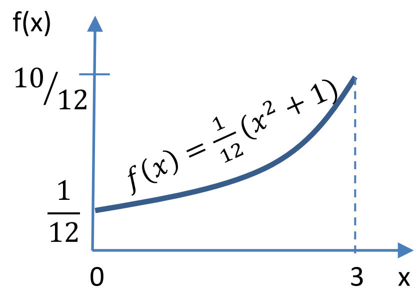
(i) $\int_0^3 k(x^2+1)dx = 1$

$k\left[\frac{x^3}{3} + x\right]_0^3 = k\left[\left(\frac{3^3}{3} + 3\right) - \left(\frac{0^3}{3} + 0\right)\right] = 1$

k = $\frac{1}{12}$

When x = 0, f(x) = $\frac{1}{12}(0^2 + 1) = \frac{1}{12}$

When x = 3, f(x) = $\frac{1}{12}[3^2 + 1] = \frac{10}{12}$



(ii) P(1 ≤ x ≤ 3)

$\frac{1}{12}\int_1^3 (x^2+1)dx = \frac{1}{12}\left[\frac{x^3}{3} + x\right]_1^3 = \frac{1}{12}\left[\left(\frac{3^3}{3} + 3\right) - \left(\frac{1^3}{3} + 1\right)\right] = 0.8889$

Example 35

A random variable X of continuous p.d.f is given by $f(x) = \begin{cases} k, & 0 \le x \le 2 \\ k(2x-3), & 2 \le x \le 3 \\ 0, & elsewhere \end{cases}$

Find (i) value of k and sketch f(x) (ii) P(X<1) (ii) P(X > 2.5) (iv) $\left(0 \le X \le 2/_{X \ge 1}\right)$

Solution

$\int_0^2 kdx + \int_2^3 k(2x-3)dx = 1$

$k[x]_0^2 + k[x^2-3x]_2^3 = 1$

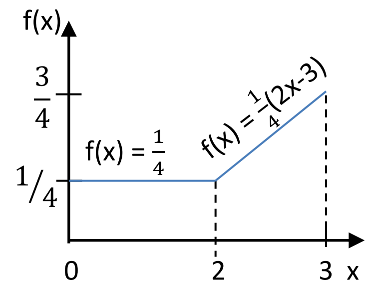
k = $\frac{1}{4}$

When x = 0, f(x) = k = $\frac{1}{4}$

When x = 2, f(x) = k = $\frac{1}{4}$

When x = 3, f(x) = $\frac{1}{4}$(2 x3 - 3)

= $\frac{3}{4}$



(ii) P(X<1) $=\frac{1}{4}\int_0^1 dx = \frac{1}{4}[x]_0^1 = \frac{1}{4}$

(iii) P(X > 2.5) $= \frac{1}{4}\int_{2.5}^3 (2x-3)dx = \frac{1}{4}[x^2-3x]_{2.5}^3 = 0.3125$

(iv) $P\left(0 \le X \le 2/_{X \ge 1}\right) = \frac{P(0\le X\le 2)}{P(X\ge 1)} = \frac{P((0\le X\le 2)\cap(X\ge 1))}{P(X\ge 1)} = \frac{\frac{1}{4}\int_1^2 dx}{\frac{1}{4}\int_1^2 dx + \frac{1}{4}\int_2^3 (2x-3)dx} = \frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{2}} = \frac{1}{3}$

Example 36

A random variable X of continuous p.d.f is given by $f(x) = \begin{cases} k(x+2)^2, & -2 \le x \le 0 \\ 4k, & 0 \le x \le \frac{4}{3} \\ 0, & elsewhere \end{cases}$

Find

(i) the value of the constant k and sketch f(x)

(ii) P(-1 < x< 1) (iii) P(X > 1)

Solution

$\int_{-2}^0 k(x+2)^2 dx + \int_0^2 4kdx = 1$

$k\left[\frac{(x+2)^3}{3}\right]_{-2}^0 + 4k[x]_0^2 = 1$

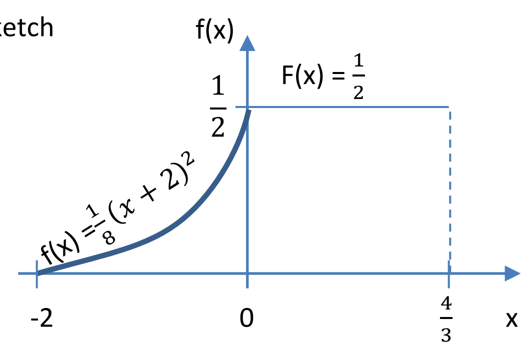
k = $\frac{1}{8}$

When x = -2, f(x) = $\frac{1}{8}$(-2 + 2)^2 = 0

When x = 0, f(x) = $\frac{1}{8}$ (0+2)^2 = $\frac{1}{2}$

When x = $\frac{4}{3}$, f(x) =4 x $\frac{1}{8}$ = $\frac{1}{2}$

Sketch



(ii) P(-1 < x< 1) $=\int_{-1}^0 k(x+2)^2 dx + \int_0^1 4kdx$

$= \frac{1}{8}\left[\frac{(x+2)^3}{3}\right]_{-12}^{0} + 4x\frac{1}{8}[x]_0^1 = \frac{7}{24} + \frac{1}{2} = \frac{19}{24}$

(iii) P(X > 1) = $\int_0^{\frac{4}{3}} 4kdx$ = $4x\frac{1}{8}[x]_1^{4/3}$ = $\frac{1}{6}$

Finding the constant k from a sketch graph

Example 37

A random variable X of continuous p.d.f is given by $f(x) = \begin{cases} kx & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$

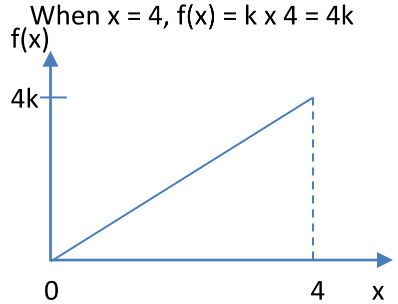
(a) Sketch and find the value of constant k

(b) Find (i) P(X ≤ 1) (ii) P(1 < x < 2)

Solution

(a) When x = 0, f(x) = k x 0 = 0

When x = 4, f(x) = k x 4 = 4k



Area under the curve = $\frac{1}{2}\ x\ 4\ x\ 4k = 1$

k = $\frac{1}{8}$

(b)(i) P(X ≤ 1) = $\frac{1}{8}\int_0^1 xdx$ = $\frac{1}{8}\left[\frac{x^2}{2}\right]_0^1$

= $\frac{1}{8}\left(\frac{1^2}{2} - \frac{0^2}{2}\right)$ = $\frac{1}{16}$

(ii) P(1 < x < 2) = $\frac{1}{8}\int_1^2 xdx$ = $\frac{1}{8}\left[\frac{x^2}{2}\right]_1^2$

= $\frac{1}{8}\left(\frac{2^2}{2} - \frac{1^2}{2}\right)$ = $\frac{3}{16}$

Example 38

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 2 \\ k(4-x), & 2 \le x \le 4 \\ 0, & elsewhere \end{cases}$

(a) Sketch f(x) and find the value of k

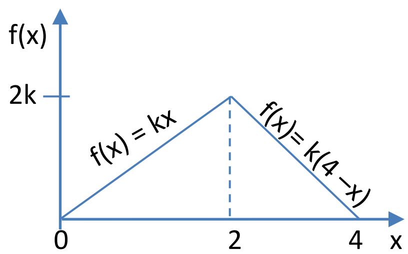
(b) Find (i) P(X< 1) (ii) P(X >3) (iii) P(1 ≤ x≤ 3) (iv) $P\left(X \ge 1/_{X \le 3}\right)$

Solution

When x = 0, f(x) = $k(0)$ = 0

When x = 2, f(x) = $k\ x\ 2 = 2k$

When x = 4, f(x) =k (4 – 4) = 0



Area under the curve = $\frac{1}{2}\ x\ 4\ x\ 2k = 1$

k = $\frac{1}{4}$

(b)(i) P(X< 1) = $\frac{1}{4}\int_0^1 xdx = \frac{1}{4}\left[\frac{x^2}{2}\right]_0^1$

$= \frac{1}{4}\left(\frac{1^2}{2} - \frac{0^2}{2}\right) = \frac{1}{8}$

(ii) P(X >3) = $\frac{1}{4}\int_3^4 (4-x)dx$

$= \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_3^4$ = 0.125

(iii) P(1 ≤ x≤ 3)= $\frac{1}{4}\int_1^2 xdx + \frac{1}{4}\int_2^3 4 - xkdx$

$= \frac{1}{4}\left[\frac{x^2}{2}\right]_1^2 + \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_2^3 = \frac{3}{4}$

(iv) $P\left(X \geq 1/_{X \leq 3}\right) = \frac{X\geq 1 \cap X\leq 3}{X\leq 3} = \frac{P(1\leq x \leq 3)}{X\leq 3} = \frac{3/_4}{\frac{1}{4}\int_0^2 xdx + \frac{1}{4}\int_2^3 4 - xkdx} = \frac{3}{4}\Big/\frac{7}{8} = \frac{6}{7}$

Example 39

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k, & 2 \leq x \leq 3 \\ k(x-2), & 3 \leq x \leq 4 \\ 0, & elsewhere \end{cases}$

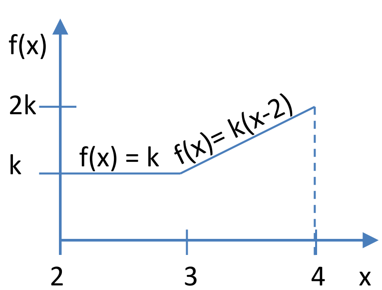
Find (i) the value of k and sketch the graph (ii) P($|X - 2.5| > 0.5$) (iii) P($|X - 2.5| < 0.5$)

Solution

(i) When x = 2, f(x) = k

When x = 3, f(x) = k

When x = 4, f(x) =k (4 – 2) = 2k



Area under the curve = 1 x k + $\frac{1}{2}(k + 2k)x\ 1 = 1$

k = $\frac{2}{5}$

(ii) P($|X - 2.5| > 0.5$) = P(-0.5 < X-2.5<0.5)

= P(2 < X <3)

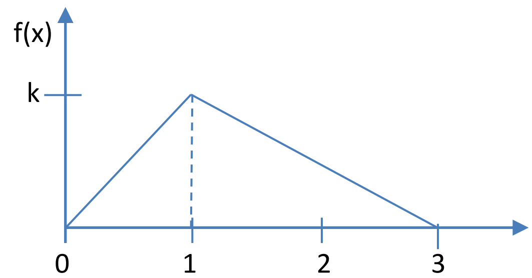
$= \frac{2}{5}\int_2^3 dx = [x]_2^3$

$= \frac{2}{5}$

Finding p.d.f from a sketch graph of a continuous random variable

Example 40

A random variable X of a continuous p.d.f is given by



(a) Area = 1 = $\frac{1}{2}\ x\ 3\ x\ k$

K = $\frac{2}{3}$

(b) Find f(x)

Let f(x) = y

For interval: $0 \le x\ \le 1$ coordinates are (0, 0) and (1, k)

grad = $\frac{y-0}{x-0}$ = $\frac{\frac{2}{3}-0}{1-0}$

y = $\frac{2}{3}x$

For interval 1≤ x≤ 3

Coordinates are (3,0) and (1, k)

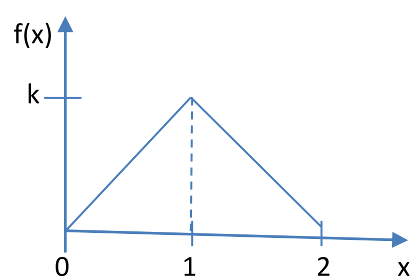
grad = $\frac{y-0}{x-3}$ = $\frac{\frac{2}{3}-0}{1-3}$

y = $-\frac{1}{3}(x-3)$

$$f(x) = \begin{cases} \frac{2}{3}x, & 0 \le x \le 1 \\ \frac{1}{3}(x-3), & 1 \le x \le 3 \\ 0, & elsewhere \end{cases}$$

Example 41

A continuous random variable X has a probability density function (p.d.f) f(x) as shown in the graph below



(a) Find the
   (i) value of k
   (ii) expression for the probability density function
(b) Calculate the
   (i) The mean
   (ii) P(X<1.5/X> 0.5)

Solution

(i) Area under the graph = 1

$\frac{1}{2}\ x\ 2\ x\ k = 1;$ k = 1

For interval: $1 \le x\ \le 2$ coordinates are

(1, k) and (2, 0)

grad = $\frac{y-1}{x-1}$ = $\frac{0-1}{2-1}$

y = $2-x$

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ (2-x), & 1 \le x \le 2 \\ 0, & elsewhere \end{cases}$$

(b)(i) E(X) =$\sum xf(x)$

$= \int_0^1 x.xdx + \int_1^2 x(2-x)dx$

$=\left[\frac{x^3}{3}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2$

$=\left(\frac{1}{3}-0\right) + \left[\left(4-\frac{8}{3}\right) - \left(1-\frac{1}{3}\right)\right]$

$=\frac{1}{3} + \frac{4}{3} - \frac{2}{3} = 1$

(ii) Let f(x) = y

For interval: $0 \le x \le 1$ coordinates are (0, 0) and (1, k)

grad = $\frac{y-0}{x-0} = \frac{1-0}{1-0}$

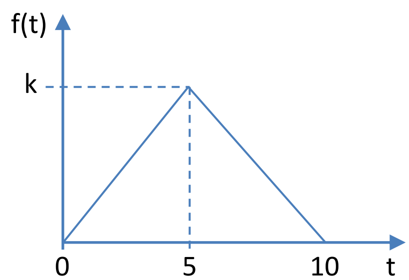
y = x

(b)(ii) P(X<1.5/X> 0.5) = $\frac{P(x<1.5 \cap x>0.5)}{P(X>0.5)} = \frac{P(0.5<x\ 1.5)}{P(X>0.5)} = \frac{\int_{0.5}^{1} xdx + \int_{1}^{1.5}(2-x)dx}{1-\int_{0}^{0.5} xdx}$

$$= \frac{\left[\frac{x^2}{2}\right]_{0.5}^{1} + \left[2x - \frac{x^2}{2}\right]_{1}^{1.5}}{1-\left[\frac{x^2}{2}\right]_{0}^{0.5}} = 0.8751$$

Example 42

The departure time T of pupils from a certain day primary school can be modelled as in the diagram below, where t is the time in minutes after the final bell at 5.00pm



Determine the

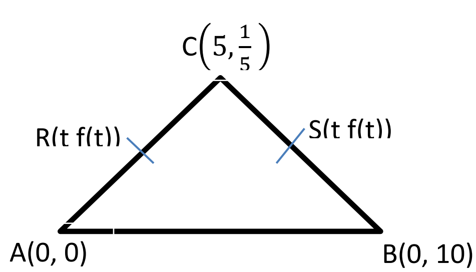
(i) value of k

Area under the curve = 1

$\frac{1}{2}\ x\ 10\ x\ k = 1$

$k = \frac{1}{5}$

(ii) equation of the p.d.f



Gradient of $\overline{AC}$ = Gradient of $\overline{AR}$

$\frac{\frac{1}{5}-0}{5-0} = \frac{f(x)-0}{t=0}$

$\frac{1}{25} = \frac{f(x)}{t}$

f(x) =$\frac{1}{25}t$

Gradient of $\overline{BC}$ = Gradient of $\overline{BS}$

$\frac{\frac{1}{5}-0}{5-10} = \frac{f(x)-0}{t=10}$

$-\frac{1}{25} = \frac{f(x)}{t-10}$

f(x) =$\frac{10-t}{25}$

Hence $f(x) = \begin{cases} \frac{1}{25}t, & 0 \leq x \leq 5 \\ \frac{1}{25}(10-t), & 5 \leq x \leq 10 \\ 0, & elsewhere \end{cases}$

(iii) E(T): since the graph is symmetrical about t = 5; Hence E(T) = 5

(iv) Probability that a pupil leaves between 4 and 7 minutes after the bell

P(4 < t < 7) = $\frac{1}{25}\int_4^5 tdx + \frac{1}{25}\int_5^7 (10-t)kdx$

$=\frac{1}{25}\left[\frac{t^2}{2}\right]_4^5 + \frac{1}{25}\left[10t - \frac{t^2}{2}\right]_5^7$ = 0.5

Revision exercise 5

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx^2 & 0 \leq x \leq 2 \\ 0, & elsewhere \end{cases}$

(a) Find the value of the constant k (=$\frac{3}{8}$) and sketch f(x)

(b) Find (i) P(X≥1) = $\frac{3}{8}$ (ii) P(0.5≤ x ≤ 1.5) = $\frac{13}{32}$

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k & -2 \leq x \leq 3 \\ 0, & elsewhere \end{cases}$

(i) Sketch f(x)

(ii) Find the value of the constant k = $\frac{1}{5}$.

(iii) Find P(-1.6 ≤ x ≤ 2.1) = 0.74

3. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(4-x) & 1 \leq x \leq 3 \\ 0, & elsewhere \end{cases}$

(i) Sketch f(x)

(ii) Find the value of the constant k = $\frac{1}{4}$.

(iii) Find P(1.2 ≤ x ≤ 2.4) = 0.66

4. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x+2)^2 & 0 \leq x \leq 2 \\ 0, & elsewhere \end{cases}$

(a) Sketch f(x)

(b) Find the value of the constant k = $\frac{1}{56}$.

(c) Find (i) P(0 ≤ x ≤ 1) = $\frac{19}{56}$ (ii) P(X ≥ 1) =$\frac{37}{56}$

5. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x)^3 & 0 \le x \le c \\ 0, & elsewhere \end{cases}$

   Given that P(X ≤ 0.5) = $\frac{1}{16}$

   (i) Find the value of k and c (k = 1 and k = 4)

   (ii) Sketch f(x)

6. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$

   (i) Sketch f(x)

   (ii) Find the value of the constant k = $\frac{1}{8}$.

   (iii) Find P(1 ≤ x ≤ 2.5) = 0.328

7. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k, & 0 \le x \le 2 \\ k(2x-3), & 2 \le x \le 3 \\ 0, & elsewhere \end{cases}$

   (i) Sketch f(x)

   (ii) Find the value of the constant k = $\frac{1}{4}$.

   (iii) Find (i) P(X >1) =$\frac{1}{4}$ (ii) P(X > 2.5) = 0.3125 (iii) P(1 ≤ x ≤ 2.3) = 0.3475

8. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} a, & 0 \le x \le 1.5 \\ \frac{a}{2}(2-x), & 2 \le x \le 3 \\ 0, & elsewhere \end{cases}$

   Find (i) value of a= $\frac{16}{25}$ (ii) P(X < 1.6) = 0.9744

Expectation or mean of X of continuous random variable

Example 43

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx^2 & 0 \le x \le 3 \\ 0, & elsewhere \end{cases}$

Find the

(i) value of the constant k and sketch f(x)

(ii) the mean, μ

(iii) P(X ≤ μ)

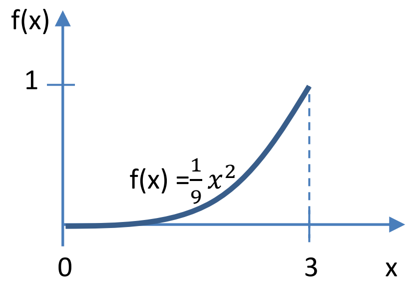
Solution

(i) $\int_0^3 kx^2dx = 1$

$k\left[\frac{x^3}{3}\right]_0^3 = 1$, k = $\frac{1}{9}$

When x = 0, f(x) =$\frac{1}{9}(0)^2 = 0$

When x = 3, f(x) =$\frac{1}{9}(3)^2 = 1$



(ii) E(X) = $\int_0^3 x.x^2dx$

=$\frac{1}{9}\left[\frac{x^4}{4}\right]_0^3$ = 2.25

(iii) P(X ≤ μ) =$\frac{1}{9}\int_0^{2.25} x^2dx$

= $\frac{1}{9}\left[\frac{x^4}{4}\right]_0^{2.25}$

= 0.42

Example 44

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx^3 & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$

Find (i) the value of the constant k

$\int_0^3 kx^3dx = 1$

$k\left[\frac{x^4}{4}\right]_0^2 = 1$, k = $\frac{1}{4}$

(ii) mean

E(X) = $\frac{1}{4}\int_0^3 x.x^3dx = \frac{1}{4}\left[\frac{x^5}{5}\right]_0^2$ = 1.6

(iii) P(X ≤ 1) =$\frac{1}{4}\int_0^1 x^3dx = \frac{1}{4}\left[\frac{x^4}{4}\right]_0^1 = 0.0625$

Example 45

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(4x - x^2), & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$

Find

(i) the value of constant k

$\int_0^2 k(4x - x^2)dx = 1$

$k\left[2x^2 - \frac{x^3}{3}\right]_0^2$, k = $\frac{3}{16}$

(ii) E(X)

$\frac{3}{16}\int_0^2 x(4x - x^2)dx$ = $\frac{3}{16}\left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_0^2 = 0.25$

(iii) P(X ≤ 1) = $\frac{3}{16}\int_0^1 (4x - x^2)dx$ = $\frac{3}{16}\left[2x^2 - \frac{x^3}{3}\right]_0^1$ = 0.3125

Example 46

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} 3x^k, & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$

(i) Find the value of k

$3\int_0^1 x^k dx = 1$

$3\left[\frac{x^{k+1}}{k+1}\right]_0^1 = 1$

$3\left[\frac{1^{k+1}}{k+1} - \frac{0^{k+1}}{k+1}\right] = 1$

$\frac{3}{k+1} = 1$

k = 2

(ii) Find the mean

E(X) = $\int_0^1 x(3x^2)dx$ = 3 $\left[\frac{x^4}{4}\right]_0^1$ = 0.75

(iii) Find the value of a such that P(X ≤ a) = 0.5

P(X ≤ a) = 3 $\int_0^a x^2 dx$ = 0.5

= 3 $\left[\frac{x^3}{3}\right]_0^a$ = $a^3 - 0^3$ = 0.5

= a^3 = 0.5; a = 0.794

Example 47

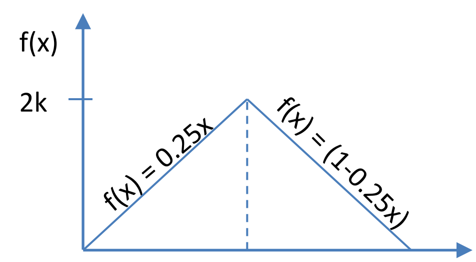
A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{4}x, & 0 \le x \le 2 \\ \left(1 - \frac{1}{4}x\right), & 2 \le x \le 4 \\ 0, & elsewhere \end{cases}$

(i) Sketch f(x)

When x = 0, f(x) = $\frac{1}{4}\,x\,(0) = 0$

When x = 2, f(x) = $\frac{1}{4}\,x\,(2) = 0.25$

When x = 4, f(x) = $\left(1 - \frac{1}{4}(4)\right)$ = 0



(ii) Mean

E(X) = $\frac{1}{4}\int_0^2 x.xdx + \int_2^4 x\left(1 - \frac{1}{4}x\right)dx$

$\frac{1}{4}\left[\frac{x^3}{3}\right]_0^2 + \left[\frac{x^2}{2} - \frac{x^3}{12}\right]_2^4$ = 2

(iii) P(X > 3) = $\int_3^4 \left(1 - \frac{1}{4}x\right)dx$

= $\left[x - \frac{x^2}{8}\right]_3^4$ = 0.125

Example 48

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x+2), & -1 \le x \le 0 \\ 2k(1-x), & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$

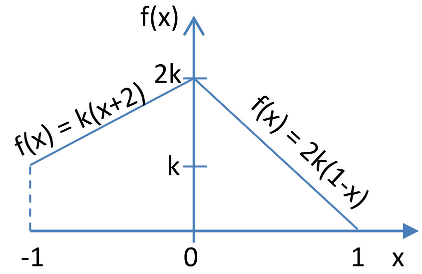
(i) Sketch f(x)

When x = -1, f(x) = k(-1 + 2) = k

When x = 0, f(x) = k(0 + 2) = 2k

When x = 1, f(x) =2k(1-1) = 0

Sketch



(ii) value of k

Area under the graph = 1

$\frac{1}{2}\ x\ 1(k + 2k) + \frac{1}{2}\ x\ 1\ x\ 2k = 1$

$k = \frac{2}{5}$

(iii) $P\left(0 < x < 0.5 / X > 0\right)$

$P\left(0 < x < 0.5 / X > 0\right) = \frac{P(0<x<0.5)}{P(X>0)} = \frac{\frac{4}{5}\int_0^{0.5}(1-x)dx}{\frac{4}{5}\int_0^{1}(1-x)dx} = \frac{\left[x-\frac{x^2}{2}\right]_0^{0.5}}{\left[x-\frac{x^2}{2}\right]_0^{1}} = \frac{3/8}{1/2} = 0.75$

(iv) Mean

E(X) = $\frac{2}{5}\int_{-1}^{0} x(x+2)dx + \frac{4}{5}\int_0^1 x(1-x)dx$

$= \frac{2}{5}\left[\frac{x^3}{3} + x^2\right]_{-1}^{0} + \frac{4}{5}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = -\frac{2}{15}$

Properties of the mean of continuous random variables

(i) E(a) = a

(ii) E(ax) = a.E(x)

(iii) E(ax + b) = aE(x) + b

(iv) E(ax - b) = aE(x) – b

Where a and b are constants

Example 49

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{20}(x+3), & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$

(i) Sketch f(x)

(ii) Find E(X)

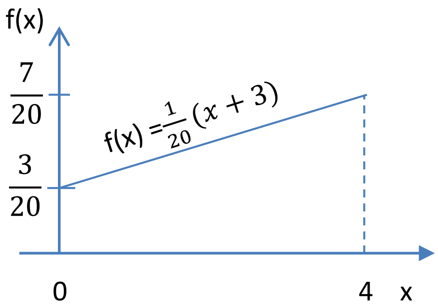
(iii) Find E(2X + 5)

Solution

(i) When x = 0, f(x) = $\frac{1}{20}(0+3)$ = $\frac{3}{20}$

When x = 4, f(x) = $\frac{1}{20}(4+3)$ =$\frac{7}{20}$

Sketch



(ii) E(X) =$\frac{1}{20}\int_0^4 x(x+3)dx$

=$\frac{1}{20}\left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^4$

= 2.266

(iii) E(2X + 5) = 2 x 2.266 + 5 = 9.533

Example 50

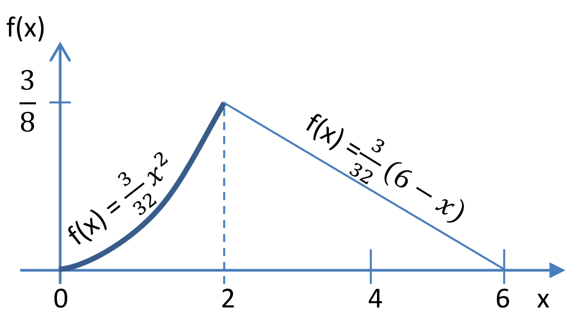
A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{3}{32}x^2, & 0 \le x \le 2 \\ \frac{3}{32}(6-x), & 2 \le x \le 6 \\ 0, & elsewhere \end{cases}$

(i) Sketch f(x)

When x = 0, f(x) = $\frac{3}{32}(0)^2$ = 0

When x = 2, f(x) =$\frac{3}{32}(2)^2$ = $\frac{3}{8}$

When x =6, f(x) =$\frac{3}{32}(6-6)$ = 0



(ii) Find P(X< 4)

P(X<4) =$\frac{3}{32}\int_0^2 x^2dx + \frac{3}{32}\int_2^4 (6-x)dx$

=$\frac{3}{32}\left[\frac{x^3}{3}\right]_0^2$+$\frac{3}{32}\left[6x - \frac{x^2}{2}\right]_2^4$ = $\frac{13}{16}$

(iii) find the mean

E(X) =$\frac{3}{32}\int_0^2 x.x^2dx + \frac{3}{32}\int_2^4 x(6-x)dx$

=$\frac{3}{32}\left[\frac{x^4}{4}\right]_0^2$+$\frac{3}{32}\left[3x^2 - \frac{x^3}{3}\right]_2^6$

= 2.875

(iv) Find E(100x -20)

E(100X -20) = 100 x 2.875 – 20 = 267.50

Revision exercise 6

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx^2, & 0 \leq x \leq 4 \\ 0, & elsewhere \end{cases}$

   (i) Sketch f(x) (ii) Find E(x) = 3 (iii) find E(2X + 5) = 11

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx^2(10 - x), & 0 \leq x \leq 4 \\ 0, & elsewhere \end{cases}$

   (i) Find value of k = $\frac{3}{2500}$ (ii) Find E(x) = 6 (iii) find E(3X - 4) = 14

3. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 5 \leq x \leq 10 \\ 0, & elsewhere \end{cases}$

   (i) Sketch f(x) (ii) Find value of k = $\frac{2}{75}$ (iii) Find E(x) = $\frac{70}{9}$ (iii) find P(X >8) = 0.48

4. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k[1 - (x-2)^2], & 1 \leq x \leq 3 \\ 0, & elsewhere \end{cases}$

   (i) Find value of k = $\frac{3}{4}$ (ii) sketch f(x) (iii) find E(X) = 2

5. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx(5 - x), & 0 \leq x \leq 5 \\ 0, & elsewhere \end{cases}$

   (i) Find value of k = $\frac{6}{125}$ (ii) sketch f(x) (iii) find E(X) = 2.5

6. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(1 - cosx), & 0 \leq x \leq \pi \\ 0, & elsewhere \end{cases}$

   (i) Find value of k = $\frac{1}{\pi}$ (ii) sketch f(x) (iii) find mean of x = 0.9342

7. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{k}{3}x, & 0 \leq x \leq 3 \\ k, & 3 \leq x \leq 4 \\ 0, & elsewhere \end{cases}$

   (i) Sketch f(x) (ii) find k = $\frac{2}{5}$ (iii) find E(X) = 2.6

   (iv) find value of c such that P(X>c) = 0.85; c = 1.5

8. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x - \frac{1}{a}), & 1 \leq x \leq 3 \\ 0, & elsewhere \end{cases}$

   Given that P(X> 1) = 0.8,
   Find (i) values of a and k ($\frac{2}{15}, -1$) (ii) probability between 0.5 and 2.5 = 0.6667 (iii) E(X) =1.8

9. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x+2) & -1 \leq x \leq 0 \\ 2k, & 0 \leq x \leq 1 \\ \frac{k}{2}(5 - x) & 1 \leq x \leq 3 \\ 0, & elsewhere \end{cases}$

   (a) Sketch the function f(x)
   (b) Find the value of k (=$\frac{2}{13}$) and the mean (= $\frac{12}{13}$)

10. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} 2kx, & 0 \leq x \leq 1 \\ k(3 - x) & 1 \leq x \leq 2 \\ 0, & elsewhere \end{cases}$

    (a) Sketch f(x)
    (b) Find the value of k (=$\frac{2}{5}$) and the mean = $\frac{17}{15}$

11. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \alpha(1 - cosx), & 0 \leq x \leq \frac{\pi}{2} \\ \alpha sinx, & \frac{\pi}{2} \leq x \leq \pi \\ 0, & elsewhere \end{cases}$

(i) Find value of α (= $\frac{2}{\pi}$) (ii) mean, μ (= 1 +$\frac{\pi}{4}$) (iii) $P\left(\frac{\pi}{3} < x < \frac{3\pi}{4}\right)$ = 0.6982

12. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k_1x, & 1 \leq x \leq 3 \\ k_2(4 - x), & 3 \leq x \leq 4 \\ 0, & elsewhere \end{cases}$

(a) Show that $k_2 = 3k_1$
(b) Find (i) values of k_1 and k_2 (ii) mean, μ

13. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{y+1}{4} & 1 \leq y \leq k \\ 0, & elsewhere \end{cases}$

Find

(i) Value of k =2
(ii) Expectation Y = 1.6667
(iii) P(1≤ y≤ 1.5) = 0.2813

Solutions to revision exercise 6

8. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x - \frac{1}{a}), & 1 \leq x \leq 3 \\ 0, & elsewhere \end{cases}$

Given that P(X> 1) = 0.8,
Find

(i) values of a and k ($\frac{2}{15}, -1$)

$\int_0^3 k\left(x - \frac{1}{a}\right)dx = 1$

$k\left[\frac{x^2}{2} - \frac{x}{a}\right]_0^3 = 1$

$k\left(\frac{9}{2} - \frac{3}{a}\right) = 1$

(9a-6)k = 2a .... (i)

Given P(X>1) = 0.8

⇨ $\int_1^3 k\left(x - \frac{1}{a}\right)dx = 0.8$

$k\left[\frac{x^2}{2} - \frac{x}{a}\right]_1^3 = 0.8$

$k\left[\left(\frac{9}{2} - \frac{3}{a}\right) - \left(\frac{1}{2} - \frac{1}{a}\right)\right] = 1$

(8a - 4)k = 1.6a ..... (ii)

(ii) probability between 0.5 and 2.5

P(0.5 < x < 2.5) =$\frac{2}{15}\int_{0.5}^{2.5}(x + 1)dx$

= $\frac{2}{15}\left[\frac{x^2}{2} - \frac{x}{a}\right]_{0.5}^{2.5}$ = 0.6667

(iii) mean

E(X) = $\frac{2}{15}\int_0^3 x(x + 1)dx$

=$\frac{2}{15}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^3$ = 1.8

Eqn.(i) and (ii), a = -1, k = $\frac{2}{15}$

9. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x+2) & -1 \le x \le 0 \\ 2k, & 0 \le x \le 1 \\ \frac{k}{2}(5-x) & 1 \le x \le 3 \\ 0, & elsewhere \end{cases}$

(a) Sketch the function f(x)

For $-1 \le x \le 0$, f(x) = $k(x+2)$

When x = -1, f(x) = k

When x = 0, f(x) =2k

For $0 \le x \le 1$, f(x) = $2k$,

When x = 0, f(x) = 2k

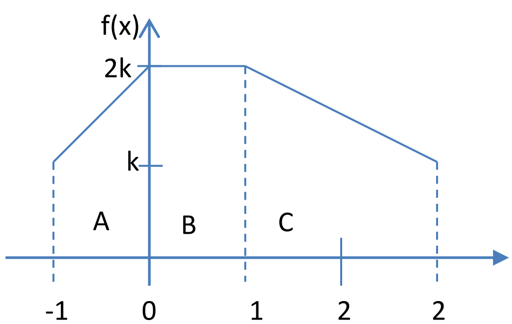
When x = 1, f(x) =2k

For $1 \le x \le 3$, f(x) = $\frac{k}{2}(5-x)$

When x = 1, f(x) = $\frac{k}{2}(5-1)$ =2k

When x = 3, f(x) =$\frac{k}{2}(5-3)$ =k

Sketch



(b)(i) find value of k

Area under the graph = 1

$\frac{1}{2} x\ 1\ (k+2k) + 1\ x\ 2k + \frac{1}{2} x\ 2\ (k+2k)$= 1

k = $\frac{2}{13}$

or

$k\int_{-1}^{0}(x+2)dx$ +$2k\int_{0}^{1}dx + \frac{k}{2}\int_{1}^{3}(5-x)dx = 1$

$k\left[\frac{x^2}{2}+2x\right]_{-1}^{0} + 2k[x]_0^1 + \frac{k}{2}\left[5x - \frac{x^2}{2}\right]_1^3 = 1$

k = $\frac{2}{13}$

(b) (ii) Find the mean

E(X) =$\frac{2}{13}\int_{-1}^{0}x(x+2)dx$ +$\frac{4}{13}\int_{0}^{1}xdx + \frac{1}{13}\int_{1}^{3}x(5-x)dx$

=$\frac{2}{13}\left[\frac{3}{3}+x^2\right]_{-1}^{0} + \frac{4}{13}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{13}\left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_1^3 = \frac{12}{13}$

10. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} 2kx, & 0 \le x \le 1 \\ k(3-x) & 1 \le x \le 2 \\ 0, & elsewhere \end{cases}$

(a) Sketch f(x)

For $0 \le x \le 1$, f(x) = 2kx

When x = 0, f(x) = 2k(0) = 0

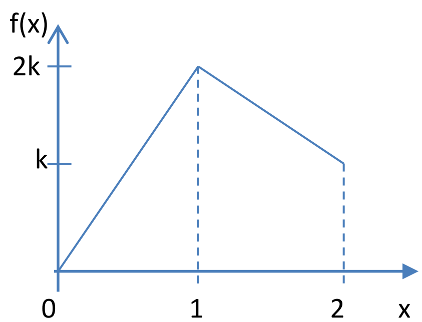
When x = 1, f(x) = 2k(1) =2k

For $1 \le x \le 2$, f(x) = k(3-x)

When x= 1, f(x) = k(3 – 1) = 2k

When x =3, f(x) = k(3 -2) = k

Sketch



(b) Find value of k

Area under the graph = 1

$\frac{1}{2}\ x1\ x\ 2k + \frac{1}{2}\ x\ 1\ (k + 2k) = 1$

k = $\frac{2}{5}$

Alternatively

$2k\int_0^1 xdx + k\int_1^2 (3-x)\,dx = 1$

$2k\left[\frac{x^2}{2}\right]_0^1 + k\left[3x - \frac{x^2}{2}\right]_1^2 = 1$

k = $\frac{2}{5}$

(b) Find the mean

E(X) = $\frac{4}{5}\int_0^1 x^2dx + k\int_1^2 x(3-x)\,dx = 1$

= $\frac{4}{5}\left[\frac{x^3}{3}\right]_0^1 + \frac{4}{5}\left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_1^2 = \frac{17}{15}$ =1.133

11. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \alpha(1-cosx), & 0 \le x \le \frac{\pi}{2} \\ \alpha sinx, & \frac{\pi}{2} \le x \le \pi \\ 0, & elsewhere \end{cases}$

(i) Find value of α

$\alpha\int_0^{\frac{\pi}{2}} (1-cosx)dx + \alpha\int_{\frac{\pi}{2}}^{\pi} sinxdx$ = 1

$\alpha[x - sinx]_0^{\frac{\pi}{2}} + \alpha[-cosx]_{\frac{\pi}{2}}^{\pi}$ =1

α = $\frac{2}{\pi}$

(ii) mean, μ

E(X) = $\frac{2}{\pi}\int_0^{\frac{\pi}{2}} x(1-cosx)dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} xsinxdx$

= $\frac{2}{\pi}\int_0^{\frac{\pi}{2}}(x - xcosx)dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} xsinxdx$

= $\frac{2}{\pi}\left[\left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} xcosxdx\right] + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} xsinxdx$

$=\frac{2}{\pi}\left[\left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} - [xsinx + \cos x]_0^{\frac{\pi}{2}}\right] + \frac{2}{\pi}[-xcosx + sinx]_{\frac{\pi}{2}}^{\pi}$

$=\frac{2}{\pi}\left[\frac{x^2}{2} - (xsinx + cosx)\right]_0^{\frac{\pi}{2}} = 1 + \frac{\pi}{4}$

(iii) $P\left(\frac{\pi}{3} < x < \frac{3\pi}{4}\right)$

$P\left(\frac{\pi}{3} < x < \frac{3\pi}{4}\right) = \frac{2}{\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(1 - cosx)dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} sinxdx$ = 1

$\alpha[x - sinx]_{\frac{\pi}{3}}^{\frac{\pi}{2}} + \alpha[-cosx]_{\frac{\pi}{2}}^{\frac{3\pi}{4}}$ = 0.6982

12. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k_1x, & 1 \leq x \leq 3 \\ k_2(4-x), & 3 \leq x \leq 4 \\ 0, & elsewhere \end{cases}$

(a) Show that $k_2 = 3k_1$

For$1 \leq x \leq 3$, f(x) = k_1(x)

f(3) = $3k_1$ ………………………………(i)

For$3 \leq x \leq 4$, f(x) =$k_2(4-x)$

f(3) = k_2

Eqn. (i) and eqn. (ii)

$k_2 = 3k_1$

(b) Find (i) values of k_1 and k_2

$k_1\int_1^3 xdx + 3k_1\int_3^4 (4-x)dx = 1$

$k_1\left[\frac{x^2}{2}\right]_1^3 + 3k_1\left[4x - \frac{x^2}{2}\right]_3^4 = 1$

$k_1 = \frac{2}{11}$

$k_2 = \frac{6}{11}$

(c) mean, μ

E(X) $=\frac{2}{11}\int_1^3 x^2dx + \frac{6}{11}\int_3^4 x(4-x)dx$

$\frac{2}{11}\left[\frac{x^3}{3}\right]_1^3 + 3k_1\left[2x^2 - \frac{x^3}{3}\right]_3^4$ = 2.485

13. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{y+1}{4} & 1 \leq y \leq k \\ 0, & elsewhere \end{cases}$

Find

(a) The value of k (06marks)

$\int_0^k \frac{(y+1)}{4}dy = \frac{1}{4}\left[\frac{y^2}{2} + y\right]_0^k = 1$

$\frac{1}{4}\left[\left(\frac{k^2}{2} + k\right) - 0\right] = 1$

$k^2 + 2k - 8 = 0$
(k + 4)(k-2) = 0
Either
k + 4 = 0; k = -4
Or
k-2 = 0; k = 2
$\therefore k = 2$ (since k is greater than zero)

(b) The expectation of Y (03marks)

E(Y) = $\int_0^2 y dy$

$= \int_0^2 y \left[\frac{y+1}{4}\right] dy$

$= \int_0^2 \left(\frac{y^2+y}{4}\right) dy$

$= \frac{1}{4}\left[\frac{y^3}{3} - \frac{y^2}{2}\right]_0^2$

$= \frac{1}{4}\left[\left(\frac{8}{3} - \frac{4}{2}\right) - 0\right] = \frac{7}{6} = 1.166$

(c) P(1≤ $Y \leq 1.5$) (03marks)

P(1≤ $Y \leq 1.5) = \int_1^{1.5} \left[\frac{y+1}{4}\right] dy$

$= \frac{1}{4}\left[\frac{y^2}{2} + y\right]_1^{1.5}$

$= \frac{1}{4}\left[\left(\frac{(1.5)^2}{2} + 1.5\right) - \left(\frac{1}{2} + 1\right)\right]$

$= \frac{1}{4}(2.625 - 1.5)$

= 0.28125

Variance of X of continuous random variables

For a continuous random variable with p.d.f, f(x)

Var(X) = $EX^2 - [E(X)]^2$ or Var(X) = $E(X^2) - \mu^2$

Where $E(X^2) = \int x^2(x)dx$ and μ = mean

Properties of variance

(i) Var(a) = 0
(ii) Var(ax) = a^2Var(x)
(iii) Var(ax + b) = a^2Var(x)
(iv) Var(ax – b) = a^2Var(X)

Where a and b are constants

Example 52

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} k(1 - x^2), & 0 \leq x \leq 1 \\ 0, & elsewhere \end{cases}$

Find

(i) the value of k

$k\int_0^1 (1-x^2)\,dx = 1$

$k\left[x-\frac{x^3}{3}\right]_0^1 = 1$

k = 1.5

(ii) E(X)

E(X) =1.5 $\int_0^1 x(1-x^2)\,dx$

=1.5 $\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1 = \frac{3}{8}$

(iii) Var(X)

E(X2) = 1.5 $\int_0^1 x^2(1-x^2)\,dx$

= 1.5 $\left[\frac{x^3}{3}-\frac{x^5}{5}\right]_0^1 = \frac{1}{5}$

Var(X) = $EX^2 - [E(X)]^2$

$=\frac{1}{5}-\left(\frac{3}{8}\right)^2 = \frac{19}{320}$

Example 53

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{8}x, & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$

Find

(i) E(X)

E(X) $=\frac{1}{8}\int_0^4 x.x\,dx = \frac{1}{8}\left[\frac{x^3}{3}\right]_0^4 = 2.667$

(ii) Var(X)

$E(X^2) = \frac{1}{8}\int_0^4 x^2.x\,dx = \frac{1}{8}\left[\frac{x^4}{4}\right]_0^4 = 8$

Var(X) = $EX^2 - [E(X)]^2$

$= 8 - (2.667)^2 = 0.887$

(iii) Standard deviation

s.d = $\sqrt{Var(X)}$

$=\sqrt{0.887} = 0.942$

(iv) Var(3x + 2) = 0.887 x 3 = 7.983

Example 54

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{4}{25}(5-2x), & 0 \le x \le 2.5 \\ 0, & elsewhere \end{cases}$

Find

(i) Mean

E(X) $=\frac{4}{25}\int_0^{2.5} x(5-2x)\,dx = \frac{4}{25}\left[\frac{5x^2}{2}-\frac{2x^3}{3}\right]_0^{2.5} = 0.833$

(ii) Standard deviation

$E(X^2) = \frac{4}{25}\int_0^{2.5} x^2(5-2x)\,dx = \frac{4}{25}\left[\frac{5x^3}{3}-\frac{2x^4}{4}\right]_0^{2.5} = 1.041$

Var(X) = $EX^2 - [E(X)]^2 = 1.041 - (0.5625)^2 = 0.347$

s.d = $\sqrt{Var(X)} = \sqrt{0.347} = 0.59$

Example 55

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{3}{4}(1+x^2), & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$

Find

(i) Mean

$E(X) = \frac{3}{4}\int_0^1 x(1+\ x^2)\,dx = \frac{3}{4}\left[\frac{x^2}{2} + \frac{x^4}{4}\right]_0^1 = 0.5625$

(ii) Standard deviation

$E(X^2) = \frac{3}{4}\int_0^1 x^2(1+\ x^2)\,dx = \frac{3}{4}\left[\frac{x^3}{3} + \frac{x^5}{5}\right]_0^1 = 0.4$

Var(X) = 0.4 – (0.525)2 = 0.835

s.d = $\sqrt{0.0835}$ = 0.289

(iii) $P(|X\ -\ \mu| < \sigma)$

$P(|X\ -\ \mu| < \sigma) = P(|X - 0.5625\ | < x < 0.289)$

= P(0.2735 < x < 0.8515

$\frac{3}{4}\int_{0.2735}^{0.8515}(1+\ x^2)\,dx = \frac{3}{4}\left[x + \frac{x^3}{3}\right]_{0.2735}^{0.8515} = 0.583$

Revision exercise 7

1. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} kx^2, & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$
   (a) Sketch f(x)
   (b) Find (i) value of k (=$\frac{3}{64}$) (ii) E(X) = 3 and var (X) = 0.6 (iii) P(1<X<2) = $\frac{7}{64}$
2. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 1 \\ k(2-x) & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$
   Find (i) constant k = 1 (ii) E(X) = 1 (iii) var(X) =$\frac{1}{6}$ (iv)P(0.75 < X < 1.5) = $\frac{19}{32}$ (v) mode =1
3. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{27}x^2, & 0 \le x \le 3 \\ \frac{1}{3} & 3 \le x \le 5 \\ 0, & elsewhere \end{cases}$
   (a) Sketch f(x)
   (b) Find (i) E(X) = 3417 (ii) standard deviation = 1.008
4. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{k}{x(4-x)}, & 1 \le x \le 3 \\ 0, & elsewhere \end{cases}$
   (i) Show that k = $\frac{3}{Inx}$
   (ii) Find (i) E(X) = 2 (ii) Var(X) = 4-$\frac{4}{Inx}$
5. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} k(ax-x^2), & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$

(i) Show that k = $\frac{8}{6a-8}$

(ii) Given that E(X) = 1, find the values of a (=2) and k(=0.75)

(iii) For the above values of a and k, find Var(X) = 0.2

6. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} 12(x^2 - x^3), & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$

   Find the (i) mean = 0.6 (ii) standard deviation = 0.2

7. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{k}{\beta}, & 0 \le x \le \beta \\ 0, & elsewhere \end{cases}$

   Find (i) value of k (=1) (ii) mean = $\frac{\beta}{2}$ (iii) standard deviation = $\frac{\beta}{2\sqrt{3}}$

8. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{8}(x+1), & 2 \le x \le 4 \\ 0, & elsewhere \end{cases}$

   Find (i) mean = $\frac{37}{12}$ (ii) var(X) = $\frac{47}{144}$ (iii) P(2.5 < x< 3) = 0.234

9. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} k(1-x)^2, & 2 \le x \le 4 \\ 0, & elsewhere \end{cases}$

   Find (i) constant k = $\frac{3}{26}$ (ii) mean = $\frac{1}{4}$ (iii) standard deviation = 0.94

10. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 2 \\ k(4-x) & 2 \le x \le 4 \\ 0, & elsewhere \end{cases}$

    Find (i) value of k = $\frac{1}{4}$ (ii) E(X) = 2 (iii) Var(X) = $\frac{2}{3}$ (iv) P(X< 1) = $\frac{1}{8}$ (iv) P(X<X<3) = $\frac{3}{8}$

Mode of continuous random variables

This is the value of f(x) is maximum in the given range of x.

(i) The mode is obtained from $\frac{d}{dx}(fx) = 0$

The maximum value is confirmed if $\frac{d^2}{dx^2}(fx) =$ negative

(ii) When a sketch of f(x) is drawn, the value of x for which f(x) is maximum gives the mode.

Note: for any line the mode can be determined from a sketch of f(x)

Example 56

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} k(2+x)(4-x), & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$

Find

(i) Value of k

$k\int_0^4 (2+x)(4-x)dx = 1$

$k\int_0^4 (8+2x-x^2)dx = 1$

$\left[8x + x - \frac{x^3}{3}\right]_0^4 = 1;$ k =$\frac{3}{80}$

(ii) Mode

$\frac{d}{dx}(fx) = 0$

$\frac{d}{dx}\frac{3}{80}(8 + 2x - x^2) = 0$

$\frac{3}{80}(2 - 2x) = 0$; x = 1

∴mode = 1

Example 57

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{108}x(6-x)^2, & 0 \le x \le 6 \\ 0, & elsewhere \end{cases}$

Find

(i) Mean

E(X) = $\int_0^6 \frac{1}{108}x^2(6-x)^2 dx$

$= \frac{1}{108}\int_0^6 (36x^2 - 12x^3 + x^4)dx$

$= \frac{1}{108}\left[12x^3 - 3x^4 + \frac{x^5}{5}\right]_0^6$ = 2.4

(ii) Standard deviation

E(X^2) = $\int_0^6 \frac{1}{108}x^3(6-x)^2 dx$

$= \frac{1}{108}\int_0^6 \frac{1}{108}(36x^3 - 12x^4 + x^5)dx$

$= \frac{1}{108}\left[9x^4 - \frac{12x^5}{5} + \frac{x^6}{6}\right]_0^6$ = 7.2

s.d = $\sqrt{7.2 - (2.4)^2}$ = 1.2

(iii) mode

$\frac{d}{dx}(fx) = 0$

$\frac{d}{dx}\frac{1}{108}x(6-x)^2 = 0$

$\frac{d}{dx}\frac{1}{108}(36x - 12x^2 + x^3)$=0

$\frac{1}{108}(36 - 24x + 3x^2)$ = 0

$(6-x)(2-x) = 0$

x = 6 or x = 2

∴ mode = 2 or 6

Example 58

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} ksinx, & 0 \le x \le \pi \\ 0, & elsewhere \end{cases}$

Find

(i) value k

$\int_0^\pi ksinx\,dx = 1$

$k[-cosx]_0^\pi = 1$

k[-cosπ- -cos0) = 1

k = $\frac{1}{2}$

(ii) P(X ≥ $\frac{\pi}{3}$)

(iii) P$\left(\ge \frac{\pi}{3}\right)$ =$\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} sinx\,dx$ =$k[-cosx]_{\frac{\pi}{3}}^{\pi} = \frac{3}{4}$

(iv) Mean

E(x) = $\frac{1}{2}\int_0^\pi xsinx\,dx$

Sign	derivative	integral sign
+	x	sinx
-	1	-cosx
+	0	-sinx

⇨ E(x) = $\frac{1}{2}\int_0^\pi xsinx\,dx$

=$\frac{1}{2}[-xcosx + sinx]_0^\pi$

= $\frac{\pi}{2}$

(v) Var (X)

$E(X^2) = \frac{1}{2}\int_0^{\pi} x^2 sinx\ dx$

Sign	Derivative	Integral sign
+	x^2	sinx
-	2x	-cosx
+	2	-sinx
-	0	cosx

⇨ $E(X^2) = \frac{1}{2}\int_0^{\pi} x^2 sinx\ dx = \frac{1}{2}[-x^2cosx + 2xsinx + 2cosx]_0^{\pi} = \frac{\pi^2-4}{2}$

∴Var(X) = $\frac{\pi^2-4}{2}$

(vi) Mode

$\frac{d}{dx}\left(\frac{1}{2}sinx\right) = 0$

$\frac{1}{2}cosx = 0$

$x = 90^0$

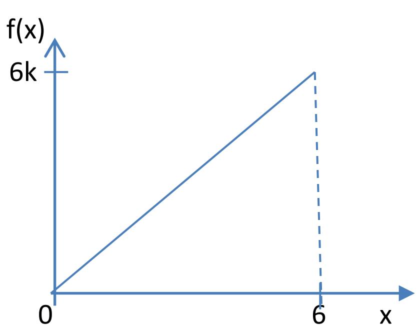
∴mode = $\frac{\pi}{2}$

Example 59

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 6 \\ 0, & elsewhere \end{cases}$

(a) Sketch f(x)

When x= 0, f(x) = k(0) = 0
When x= 6, f(x) = k(6) = 6k



(b) value of k

Area under the graph = 1

$\frac{1}{2}\ x\ k\ x\ 6\ x\ 6 = 1$

k = $\frac{1}{18}$

(c) mode = 6

Median of continuous random variables

This is the value of f(x) for which $\int_a^m f(x) = 0.5$; where m is the median, and a is the lower limit.

Example 60

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{8}x, & 0 \leq x \leq 4 \\ 0, & elsewhere \end{cases}$

Find the median

$\int_0^m \frac{1}{8}xdx = 0.5$

$\left[\frac{1}{16}x^2\right]_0^m = 0.5$

$\frac{m^2}{16} = 0.5$; m = $\sqrt{8} = \pm 2.828$

Median = 2.828 (since it falls in the range)

Example 61

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{2}{5}(x+2), & -1 \leq x \leq 0 \\ \frac{4}{5}(1-x) & 0 \leq x \leq 10 \\ 0, & elsewhere \end{cases}$

Find the median

Solution

We need to first integrate the first interval to check if it is ≥ 0.5. if not the median lies in the second interval

$\int_{-1}^0 \frac{2}{5}(x+2)dx = \frac{2}{5}\left[\frac{x^2}{2} + 2x\right]_{-1}^0 = 0.6$

It shows that the median lies in the first interval

Then $\int_{-1}^m \frac{2}{5}(x+2)dx = \frac{2}{5}\left[\frac{x^2}{2} + 2x\right]_{-1}^m = 0.5$

m = -0.129 or m = -3.871

the median = -0.129 since it lies in the range

Example 62

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{2}{3}(x+1), & -1 \leq x \leq 0 \\ \frac{1}{3}(2-x) & 0 \leq x \leq 2 \\ 0, & elsewhere \end{cases}$

Find the median

We need to first integrate the first interval to check if it is ≥ 0.5. if not the median lies in the second interval

$\int_{-1}^{0} \frac{2}{3}(x+1)dx = \frac{2}{5}\left[\frac{x^2}{2}+x\right]_{-1}^{0} = \frac{1}{3}$

It shows that the median lies in the second interval

Then $\frac{1}{3} + \frac{1}{3}\int_{0}^{m}(2-x)dx = \frac{1}{2}$

$\frac{1}{3}\left[2x - \frac{x^2}{2}\right]_{0}^{m} = \frac{1}{6}$; m =0.268

Revision exercise 8

1. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} kx(4-x^2), & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$

   Find
   (i) value of the constant =0.25 (iii) mean = 1.067
   (ii) median x =2.613 (iv) standard deviation = 0.442
2. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 1 \\ k(2-x) & 1 \le x \le 2 \\ 0, & elsewhere \end{cases}$

   Find
   (i) constant k = 1 (ii) median = 1 (iii) mode = 1
3. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} kx(4-x^2), & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$

   Find
   (i) value of the constant =$\frac{1}{4}$ (iii) mean = 1.0667
   (ii) median x =2.6131 (iv) standard deviation = 0.4422
4. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \alpha, & 2 \le x \le 3 \\ \alpha(x-2) & 3 \le x \le 4 \\ 0, & elsewhere \end{cases}$

   (a) sketch f(x)
   (b) find (i) constant α = 0.4 (ii) median, m = 3.225 (iii) P(2.5 < x < 3.5) =0.65
5. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \beta, & 0 \le x \le 2 \\ \beta(3-x) & 2 \le x \le 3 \\ 0, & elsewhere \end{cases}$

   Find (i) value of β =0.4 (ii) mean = $\frac{19}{15}$ (iii) standard deviation = $\frac{5}{4}$ (iv) P(X<μ –σ) =0.207
6. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ \frac{1}{2} & 1 \le x \le k \\ 0, & elsewhere \end{cases}$

   (i) Sketch f(x)
   (ii) Find (i) value of k = $\frac{7}{3}$ (ii) mean = $\frac{49}{36}$ (iii) median = $\frac{4}{3}$

7. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \leq x \leq 1 \\ k & 1 \leq x \leq 2 \\ 0, & elsewhere \end{cases}$

(i) Sketch f(x)

(ii) Find (i) value of k = $\frac{2}{3}$ (ii) mean = $\frac{49}{36}$ (iii) median = 1.25 (iv) P($|X - m|$>0.5) = $\frac{17}{48}$

8. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} 2k(x+1), & -1 \leq x \leq 0 \\ k(2-x) & 0 \leq x \leq 2 \\ 0, & elsewhere \end{cases}$

(i) Sketch f(x)

(ii) Find (i) value of k = $\frac{1}{3}$ (ii) mean = $\frac{1}{3}$ (iii) Var(X) = $\frac{5}{18}$ (iv) mode =0

Cumulative distribution function, F(x) of continuous random variables

The cumulative distribution function F(x) is defined by F(x) = $\int_a^x fx)dx$

Steps in finding F(x)

- For each interval, integrate its function from lower limit to x with respect to x.
- Substitute the upper limit in the integral and carry it forward to the next interval
- Continue the process until when the last upper limit has been substituted to get a 1.

Example 63

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{6}(x+1), & 1 \leq x \leq 3 \\ 0, & elsewhere \end{cases}$

Find F(x)

Solution

F(x) = $\frac{1}{6}\int_1^x (x+1)dx = \frac{1}{6}\left[\frac{x^2}{2} + x\right]_1^x = \frac{1}{6}\left\{\left(\frac{x^2}{2} + x\right) - \left(\frac{1^2}{2} + 1\right)\right\}$

F(x) = $\frac{1}{6}\left(\frac{x^2}{2} + x - \frac{3}{2}\right)$

F(3) =$\frac{1}{6}\left(\frac{3^2}{2} + 3 - \frac{3}{2}\right)$ =1

$\therefore$ F(x) = $\begin{cases} 0 & x \leq 1 \\ \frac{1}{6}\left(\frac{x^2}{2} + x - \frac{3}{2}\right), & 1 \leq x \leq 3 \\ 1, & x \geq 3 \end{cases}$

Example 64

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{3}{26}(1-x)^2, & 2 \leq x \leq 4 \\ 0, & elsewhere \end{cases}$

Find F(x)

F(x) = $\frac{3}{26}\int_2^x (1-x)^2 dx$= $\frac{3}{26}\int_2^x (1-2x+x^2)dx$ = $\frac{3}{26}\left[x - x^2 + \frac{x^3}{3}\right]_2^x$

= $\frac{3}{26}\left\{\left(x - x^2 + \frac{x^3}{3}\right) - \left(2 - 2^2 + \frac{2^3}{3}\right)\right\}$ =$\frac{3}{26}\left(x - x^2 + \frac{x^3}{3} - \frac{2}{3}\right)$

F(4) =$\left(4 - 4^2 + \frac{4^3}{3} - \frac{2}{3}\right)$=1

$\therefore$ F(x) = $\begin{cases} 0 & x \leq 2 \\ \frac{3}{26}\left(x - x^2 + \frac{x^3}{3} - \frac{2}{3}\right), & 2 \leq x \leq 3 \\ 1, & x \geq 4 \end{cases}$

Example 65

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ (2-x), & 1 \leq x \leq 2 \\ 0, & elsewhere \end{cases}$

Find F(x)

For $0 \leq x \leq 1$, F(x) = $\int_o^x xdx$ = $\left[\frac{x^2}{2}\right]_0^x$ = $\left(\frac{x^2}{2} - \frac{0^2}{2}\right) = \frac{x^2}{2}$

F(1) = $\frac{1^2}{2}$ = $\frac{1}{2}$

For $1 \leq x \leq 2$; F(x) = $\frac{1}{2} + \int_1^x (2-x)dx = \frac{1}{2} + \left[2x - \frac{x^2}{2}\right]_1^x$ =$\frac{1}{2} + \left\{\left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1^2}{2}\right)\right\}$

=$\left(2x - \frac{x^2}{2}\right) - 1$

F(x) = $\left(2x2 - \frac{2^2}{2}\right) - 1$= 1

$$\therefore F(x) = \begin{cases} 0 & x \leq 1 \\ \frac{x^2}{2}, & 0 \leq x \leq 1 \\ \left(2x - \frac{x^2}{2}\right) - 1, & 1 \leq x \leq 2 \\ 1, & x \geq 2 \end{cases}$$

Example 66

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{2}{5}, & 0 \leq x \leq 2 \\ \frac{2}{5}(3-x), & 2 \leq x \leq 3 \\ 0, & elsewhere \end{cases}$

Find F(x)

For $0 \leq x \leq 2$, F(x) =$\frac{2}{5}\int_0^x dx = \frac{2}{5}[x]_0^x = \frac{2}{5}\{x - 0\} = \frac{2}{5}x$

F(2) = $\frac{2}{5}x2 = \frac{4}{5}$

For $2 \leq x \leq 3$, F(X) =$\frac{4}{5} + \frac{2}{5}\int_0^x (3-x)dx = \frac{4}{5} + \frac{2}{5}\left[3x - \frac{x^2}{2}\right]_2^x$=$\frac{4}{5} + \frac{2}{5}\left(3x - \frac{x^2}{2}\right) - \left(3x2 - \frac{2^2}{2}\right)$

F(x) =$\frac{2}{5}\left(3x - \frac{x^2}{2}\right) - \frac{4}{5}$

F(3) =$\frac{2}{5}\left(3x3 - \frac{3^2}{2}\right) - \frac{4}{5}$ = 1

$$\therefore F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{2}{5}x, & 0 \leq x \leq 2 \\ \frac{2}{5}\left(3x - \frac{x^2}{2}\right) - \frac{4}{5}, & 2 \leq x \leq 3 \\ 1, & x \geq 3 \end{cases}$$

Finding the median, quartiles and probability from F(x)

- The median is the value of m for which F(m) = 0.5
- The lower quartile is the value q_1 for which F(q_1) = 0.25
- The upper quartile is the value q_3 for which F(q_3) = 0.75

Example 67

The continuous random variable X has a cumulative distribution function given below

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^2}{16} & 0 \leq x \leq 4 \\ 1 & x \geq 4 \end{cases}$$

Find

(i) P(0.3≤ X ≤ 1.8)

P(0.3≤ X ≤ 1.8) = F(1.8) – F(0.3) = $\frac{1.8^2}{16} - \frac{0.3^2}{16}$ = 0.197

(ii) Median, m

F(m) = 0.5

$\frac{m^2}{16} = 0.5; m = \pm 2.828$

$median = 2.828\ (since\ it\ is\ within\ the\ range$

(iii) Interquartile range

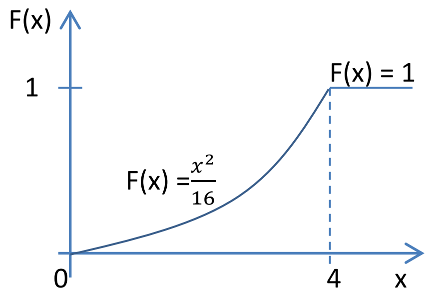
F(q_1) = 0.25

$\frac{q_1^2}{16} = 0.25;\ q_1 = 2$

F(q_3) = 0.75

$\frac{q_3^2}{16} = 0.75;\ q_3 = 3.464$

Interquartile range = 3.464 – 2 = 1.464



Example 68

The continuous random variable X has a c.d.f given by $F(x) = \begin{cases} 0 & x \le 0 \\ 2x - x^2 & 0 \le x \le 1 \\ 1 & x \ge 4 \end{cases}$

Find

(i) F(X≤0.5)

F(X≤0.5) = F(0.5) – F(0) =$(2(0.5) - (0.5)^2) - (2(0) - (0)^2)$=0.75

(ii) Median, m

F(m) = 0.5

$(2(m) - (m)^2) = 0.5$

$m^2 - 2m + 0.5 = 0$

m = 1.71 or m = 0.293

m = 0.293 (since it is in the range)

(iii) Interquartile range

$F(q_1) = 0.25$

$2q_1 - q_1^2 = 0.25;\ q_1 = 0.134$

$F(q_3) = 0.75$

$2q_3 - q_3^2 = 0.75;\ q_3 = 0.5$

Interquartile range = 0.5 – 0.134 = 0.366

Example 69

The cumulative distribution function is given by $F(x) = \begin{cases} 0 & x \le 0 \\ \frac{x^2}{6} & 0 \le x \le 2 \\ -\frac{x^2}{3} + 2x - 2 & 2 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$

Find

(i) P(1 ≤ x ≤ 2.5)

P(1 ≤ x ≤ 2.5) = P(2.5) – P(1)

$-\frac{2.5^2}{3} + 2x\ 2.5 - 2 - \frac{1^2}{6}$ =0.75

(ii) Median, m

P(0 ≤ x ≤ 2) = F(2) –F(0)

$= \frac{2^2}{6} - \frac{0^2}{6} = \frac{2}{3}$

Since $\frac{2}{3} > 0.5$ the median lies between $0 \le x \le 2$

F(m) =0.5

$\frac{m^2}{6} = 0.5$

$m = \pm 1.73$

Median = 1.73

Revision exercise 9

1. The random variable X has a probability density function f(x) =$\begin{cases} \frac{3}{8}x^2 & 0 \le x \le 2 \\ 0 & elsewhere \end{cases}$

   Find

   (i) Sketch F(X)

   (ii) Cumulative distribution function; F(x) =$\begin{cases} 0 & x < 2 \\ \frac{1}{8}x^3 & 0 \le x \le 2 \\ 1 & x \ge 0 \end{cases}$

   (iii) Median, m = 1.59

2. The random variable X has a probability density function f(x) =$\begin{cases} \frac{1}{4}(4 - x) & 1 \le x \le 3 \\ 0 & elsewhere \end{cases}$

   Find

   (i) cumulative mass function; F(x) =$\begin{cases} 0 & x \le 1 \\ \frac{1}{8}(8x - x^2 - 7) & 1 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$

   (ii) P(1.5≤x≤2)=$\frac{9}{32}$

   (iii) median, m = 1.764

   (iv) sketch F(x)

3. The random variable X has a probability density function f(x) =$\begin{cases} k & 1 \le x \le 6 \\ 0 & elsewhere \end{cases}$

   Find

   (i) Value of k = $\frac{1}{5}$

   (ii) Cumulative function, F(x) =$\begin{cases} 0 & x \le 1 \\ \frac{1}{5}(x - 1) & 1 \le x \le 6 \\ 1 & x \ge 6 \end{cases}$

   (iii) Interquartile range =2.5

4. The random variable X has probability density function f(x) =$\begin{cases} \frac{1}{4} & 0 \le x \le 2 \\ \frac{1}{4}(2x - 3) & 2 \le x \le 3 \\ 0 & elsewhere \end{cases}$

   Find

(i) Cumulative function, F(x) = $\begin{cases} 0 & x \le 1 \\ \frac{x}{4} & 0 \le x \le 2 \\ \frac{1}{4}(x^2 - 3x + 4) & 2 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$

(ii) Median, m= 2

(iii) Sketch F(x)

5. The random variable X has a cumulative distribution function, F(x) =$\begin{cases} 0 & x \le 0 \\ x^4 & 1 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$

Find

(i) P(0.3 ≤ x ≤ 0.6) =0.1215

(ii) Median, m = 0.841

(iii) The value of a such that P(X>a) = 0.88

6. The random variable X has a probability density function f(x) =$\begin{cases} \frac{1}{3} & 0 \le x \le 3 \\ 0 & elsewhere \end{cases}$

Find (i) E(x) = 1.5 (ii) Var (X) = 0.75 (iii) P(X> 1.8) =0.4 (iv) P(1.1 < x <1.7) =0.2

(v) cumulative distribution function, F(x) =$\begin{cases} 0 & x \le 0 \\ \frac{1}{3}x & 0 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$

7. The random variable X has a probability density function f(x) =$\begin{cases} kx^2 & 1 \le x \le 2 \\ 0 & elsewhere \end{cases}$

Find

(i) Value of k = $\frac{3}{7}$ (ii) standard deviation = 0.272 (iii) median, m = 1.65

(ii) Cumulative mass function, F(x) =$\begin{cases} 0 & x \le 1 \\ \frac{1}{7}(x^3 - 1) & 1 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$

8. The continuous random variable X has a p.d.f given by f(x) =$\begin{cases} k(4 - x^2) & 0 \le x \le 2 \\ 0 & elsewhere \end{cases}$

Find (i) constant k (= $\frac{3}{16}$) (ii) E(x) = $\frac{3}{4}$ (iii) Var(X) = $\frac{19}{80}$ (iv) median = 0.695

(v) cumulative distribution function, F(X) = $\begin{cases} 0 & x \le 0 \\ \frac{3}{4}x - \frac{1}{16}x^3 & 0 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$

(vi) = P(0.69 ≤ x ≤ 0.7)= 0.007

9. The continuous random variable X has a p.d.f given by f(x) =$\begin{cases} \frac{1+x}{6} & 1 \le x \le 3 \\ 0 & elsewhere \end{cases}$

(i) Sketch f(x)

(ii) Find the mean = $\frac{19}{9}$

(iii) Find m such that P(X≤ m) 0.5; m = 2.16

(iv) Determine cumulative function, F(X) and sketch it

F(X) = $\begin{cases} 0 & x \leq 0 \\ \frac{1}{5}x + \frac{1}{12}x^2 - \frac{1}{4} & 1 \leq x \leq 3 \\ 1 & x \geq 3 \end{cases}$

10. A factory is supplied with flour at the beginning of each week. The weekly demand, X thousand tones for flour from this factory is a continuous random variable having a probability density function f(x) = $\begin{cases} k & 1 \leq x \leq 3 \\ 0 & elsewhere \end{cases}$

    Find

    (i) Value of k = 5

    (ii) Mean of x = $\frac{1}{6}$

    (iii) Variance of x = $\frac{5}{252}$

11. The continuous random variable X has a p.d.f given by f(x) = $\begin{cases} \frac{1}{4} & 0 \leq x \leq 1 \\ \frac{x^3}{5} & 1 \leq x \leq 2 \\ 0 & elsewhere \end{cases}$

    Find

    (i) Cumulative mass function, F(x) and sketch it F(x) = $\begin{cases} 0 & x < 0 \\ \frac{1}{4}x & 0 \leq x \leq 1 \\ \frac{1}{5} + \frac{x^4}{20} & 1 \leq x \leq 2 \\ 1 & x \geq 2 \end{cases}$

    (ii) Median, m=1.565 (iii) interquartile range = 0.821

12. The continuous random variable X has a p.d.f given by f(x) = $\begin{cases} k(x+3) & -3 \leq x \leq 3 \\ 0 & elsewhere \end{cases}$

    (a) Show that k = $\frac{1}{18}$

    (b) Find (i) E(x) =1, (ii) Var (x) = 2 (iii) Lower quartile, q_1 = 0

    (c) Given that E(ax + b) = 0 and Var(ax+b) = 1, find the values of a and b where a>0 (a = b) = $\frac{1}{\sqrt{2}}$

13. The continuous random variable X has a p.d.f given by f(x) = $\begin{cases} kx & 0 \leq x \leq 8 \\ 8k & 8 \leq x \leq 9 \\ 0 & elsewhere \end{cases}$

    (a) Sketch f(x)

    (b) Find value of k = 0.025 (ii) P(X>6) =0.55

    (c) Find F(X) == $\begin{cases} 0 & x < 0 \\ 0.0125x & 0 \leq x \leq 8 \\ 0.2x - 0.8 & 8 \leq x \leq 9 \\ 1 & X \geq 9 \end{cases}$

14. The continuous random variable X has a p.d.f given by f(x) = $\begin{cases} ax - bx^2 & 0 \leq x \leq 2 \\ 0 & elsewhere \end{cases}$

    If E(X) =1, find

    (i) values of a and b (a= 1.5, b= 0.75) (ii) Var(x) = 0.2

    (ii) F(X) = $\begin{cases} 0 & x < 0 \\ 0.75x^2 - 0.25x^3 & 0 \leq x \leq 2 \\ 1 & x \geq 2 \end{cases}$

15. The continuous random variable X has a p.d.f given by f(x) = $\begin{cases} \frac{k}{x} & 1 \leq x \leq 9 \\ 0 & elsewhere \end{cases}$

    Find (i) value of k =0.455, (ii) median =3 (iii) mean =3.64 (iv) Var(X) =4.95

(v) F(X) = $\begin{cases} 0 & x < 1 \\ \frac{1}{In\,9} Inx & 1 \le x \le 9 \\ 1 & x \ge 9 \end{cases}$

16. The continuous random variable X has a p.d.f given by f(x) = $\begin{cases} \frac{20}{5^5}\, w^3(5-w) & 0 \le w \le 5 \\ 0 & elsewhere \end{cases}$

    Find (i) P(2<w<5) =0.5 (ii) mean = 3.33 (iii) Var(X) = 0.794 (iv) mode = 3.5

    (v) F(X) = = $\begin{cases} 0 & w < 0 \\ \frac{w^4}{5^5}(25-w) & 0 \le x \le 5 \\ 1 & w \ge 5 \end{cases}$

17. The continuous random variable X has a p.d.f given by f(x) = $\begin{cases} kx & 0 \le x \le 1 \\ k(4-x^2) & 1 \le x \le 2 \\ 0 & elsewhere \end{cases}$

    Find (i) value of k = $\frac{6}{13}$ (ii) E(X) = 1.1923 (iii) Var(x) =0.1399

    (iv) F(x) = $\begin{cases} 0 & x < 0 \\ \frac{3}{13}x & 0 \le x \le 1 \\ \frac{1}{13}(24x - 2x^3 - 19) & 1 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$

18. The probability density function f(x) of a random variable x takes on the form shown in the diagram below

Find

(i) Expression for f(x)

(ii) F(x), cumulative distribution function

(iii) Mean =$\frac{2}{3}$ and Var(x) =$\frac{2}{9}$

Finding f(x) from F(X) of continuous random variables

f(x) can be obtained from; f(x) = $\frac{d}{dx}F(X)$

Example 70

The continuous random variable X has a c.d.f F(X) = $\begin{cases} 0 & x < 0 \\ \frac{x^3}{27} & 0 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$

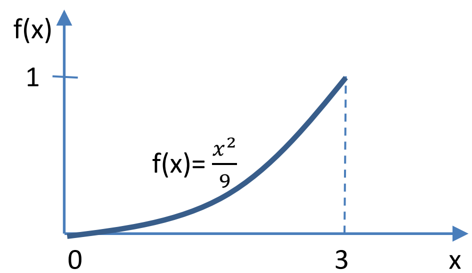
Find the probability density function f(x) and sketch f(x)

f(x) = $\frac{d}{dx}F(X) = \frac{d}{dx}\left(\frac{x^3}{27}\right) = \frac{3x^2}{27} = \frac{x^2}{9}$

f(x) = $\begin{cases} \frac{x^2}{9} & 0 \le w \le 3 \\ 0 & elsewhere \end{cases}$

When x = 0, f(x) = $\frac{0^2}{9} = 0$

When x = 3, f(x) = $\frac{3^2}{9} = 1$



Example 42

The continuous random variable X has a c.d.f F(X) = $\begin{cases} 0 & x < 0 \\ kx^3 & 0 \le x \le 4 \\ 1 & x \ge 4 \end{cases}$

Find

(i) Value of k

F(4) – F(0) = 1

K(4^3) = 1; k = $\frac{1}{64}$

(ii) Probability density function, f(x)

f(x) = $\frac{d}{dx}F(X) = \frac{d}{dx}\frac{x^3}{64} = \frac{3x^2}{64}$

f(x) = $\begin{cases} \frac{3x^2}{64} & 0 \le w \le 4 \\ 0 & elsewhere \end{cases}$

Example 43

The continuous random variable X has a c.d.f F(X) = $\begin{cases} 0 & x < 0 \\ 2x - 2x^2 & 0 \le x \le 0.25 \\ a + x & 0.25 \le x \le 0.5 \\ b + 2x^2 - x & 0.5 \le x \le 0.75 \\ 1 & x \ge 0.75 \end{cases}$

Find

(i) Value of constants a and b

For $0 \le x \le 0.25$, F(X) =$2x - 2x^2$

F(0.25) =$2x0.25 - 2(0.25)^2$ = 0.375

For $0.25 \le x \le 0.5$; F(X) = a + x

F(0.25) = a + 0.25 = 0.375

a = 0.125

For $0.5 \le x \le 0.75$; F(X) = $b + 2x^2 - x$

F(0.75) = b + 2(0.75)^2 -0.75 = 1; b = 0.625

(ii) Probability density function f(x)

f(x) = $\frac{d}{dx}F(X)$

$$f(x) = \begin{cases} 2 - 4x & 0 \le x \le 0.25 \\ 1 & 0.25 \le x \le 0.5 \\ 4x - 1 & 0.5 \le x \le 0.75 \\ 0 & elsewhere \end{cases}$$

(iii) Mean = 0.375

Revision exercise 10

1. The continuous random variable X has cumulative distribution function

$$F(X) = = \begin{cases} 0 & x < 2 \\ 0.25x - 0.5 & 2 \le x \le 6 \\ 1 & x \ge 6 \end{cases}$$

Find the

(i) probability density function f(x); $f(x)= \begin{cases} \frac{1}{4} & 2 \le x \le 6 \\ 0 & elsewhere \end{cases}$

(ii) E(X) = 4 (iii) interquartile range = 2 (iv) sketch f(x)

2. The continuous random variable X has cumulative distribution function

$$F(X) = = \begin{cases} 0 & x < 0 \\ x^3 & 0 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$$

Find (i) median (m=0.794) (ii) mean (μ =0.75)

3. The continuous random variable X has cumulative distribution function

$$F(X) = \begin{cases} 0 & x < 0 \\ x - kx^2 & 0 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$$

Find the (i) value of k= 0.25, (ii) median (m= 0.586) (iii) variance of x (Var(x) = $\frac{2}{9}$)

(iv) probability density function; $f(x) = \begin{cases} 1 - 0.5x & 0 \le x \le 2 \\ 0 & elsewhere \end{cases}$

4. The continuous random variable X has cumulative distribution function

$$F(X) = \begin{cases} 0 & x < 0 \\ \frac{2x}{3} & 0 \le x \le 1 \\ \frac{x}{3} + k & 1 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$$

Find (i) value of k = $\frac{1}{3}$ (ii) mean ($\mu = \frac{5}{6}$) (iii) standard deviation =0.5528

(iv) $P(|\mu - \sigma| < \sigma)$ =0.608

(v) p.d.f; f(x) = $\begin{cases} \frac{2}{3} & 0 \leq x \leq 2 \\ \frac{1}{3} & 1 \leq x \leq 2 \\ 0 & elsewhere \end{cases}$ (vi) sketch f(x)

5. The continuous random variable X has cumulative distribution function

$$F(X) = \begin{cases} 0 & x < 1 \\ \frac{(x-1)^2}{12} & 1 \leq x \leq 3 \\ \frac{(14x - x^2 - 25)}{24} & 3 \leq x \leq 7 \\ 1 & x \geq 7 \end{cases}$$

Find

(i) probability density function, f(x) = $\begin{cases} \frac{1}{6}(x-1) & 1 \leq x \leq 3 \\ \frac{1}{12}(7-x) & 3 \leq x \leq 7 \\ 0 & elsewhere \end{cases}$

(ii) sketch f(x) (iii) mean of X ($\mu = \frac{11}{3}$) (iv) Var (x) = $\frac{14}{9}$ (v) median of X (m= 3.45)

(vi) P(2.8 < x < 5.2) = 0.595

6. The continuous random variable X has cumulative distribution function

$$F(X) = \begin{cases} 0 & x < -1 \\ \frac{x+1}{8} & -1 \leq x \leq 0 \\ \frac{3x+1}{8} & 0 \leq x \leq 2 \\ \frac{x+5}{8} & 2 \leq x \leq 3 \\ 1 & x \geq 3 \end{cases}$$

Find (i) probability density function, f(x) (ii) P($3 \leq 2x \leq 5$) (iii) mean and variance

7. The continuous random variable X has cumulative distribution function

$$F(X) = \begin{cases} 0 & x < 0 \\ \alpha x & 0 \leq x \leq 1 \\ \frac{x}{3} + \beta & 1 \leq x \leq 1 \\ 1 & x \geq 2 \end{cases}$$

Find (i) values of α and β ($\alpha = \frac{2}{3}$; $\beta = \frac{1}{3}$) (ii) mean ($\mu = \frac{5}{6}$) (iii) Var(X) = $\frac{19}{36}$

(iv) $P\left(X < 1.5 / X > 1\right)$= 0.4998 (v) probability density function, f(x) and sketch it

8. The continuous random variable X has cumulative distribution function

$$F(X) = \begin{cases} 0 & x < 1 \\ \frac{x^2-1}{2} - x & 1 \leq x \leq 2 \\ 3x - \frac{x^2}{2} & 2 \leq x \leq 3 \\ 1 & x \geq 3 \end{cases}$$

Find

(i) Probability density function, f(x) and sketch it

(ii) P(1.2 < x< 2.4) = 0.8

(iii) Mean (μ = 2)

9. The continuous random variable X has cumulative distribution function

$$F(X) = \begin{cases} 0 & x < 0 \\ \frac{k}{2}x^2 & 0 \le x \le 2 \\ k(6x - x^2 - 6) & 2 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$$

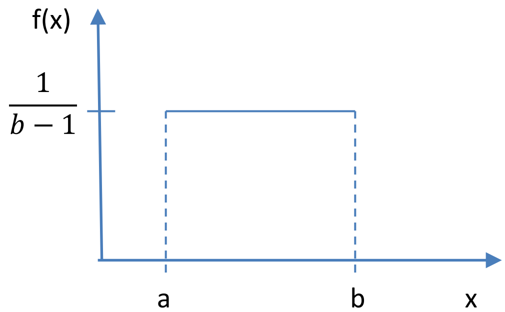
(a) Determine the value of k $= \frac{1}{3}$. Hence sketch graph of F(X)

(b) Find the probability density function.

Uniform or rectangular distribution

A continuous random variable X is said to be uniformly distributed over the interval a and b, if the p.d.f is given by $f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & elsewhere \end{cases}$

Graph of f(x)



Example 71

X is uniformly distributed between 6 and 9.

(i) Write the probability density function

$f(x) = \begin{cases} \frac{1}{9-6} & 6 \le x \le 9 \\ 0 & elsewhere \end{cases}$

(ii) Find P(7.2 < x < 8.4)

$P(7.2 < x < 8.4) = \int_{7.2}^{8.4} \frac{1}{3} dx = \frac{1}{3}[x]_{7.2}^{8.4} = 0.4$

Example 72

X is uniformly distributed between 0 and $\frac{\pi}{2}$.

(i) Write the probability density function

$f(x) = \begin{cases} \frac{1}{\frac{\pi}{2}-0} & 0 \le x \le \frac{\pi}{2} \\ 0 & elsewhere \end{cases}$

(ii) Find $P(\frac{\pi}{3} < x < \frac{\pi}{2})$

(iii) $P(\frac{\pi}{3} < x < \frac{\pi}{2} = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{2}{\pi} dx = \frac{2}{\pi}[x]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{1}{3}$

Expectation of X, E(x) of rectangular distribution

E(x)= $\int_a^b xf(x)dx = \int_a^b \frac{1}{b-a}xdx = \frac{1}{2(b-a)}[x^2]_a^b = \frac{1}{2(b-a)}(b^2-a^2) = \frac{(b+a)(b-a)}{2(b-a)} = \frac{(b+a)}{2}$

Variance of x, Var(X)

Var(x) = $\int_a^b x^2f(x)dx - [E(x)]^2 = \int_a^b \frac{1}{b-a}x^2dx - \left[\frac{(b+a)}{2}\right]^2 = \frac{1}{3(b-a)}[x^3]_a^b - \left[\frac{(b+a)}{2}\right]^2$

$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} - \left[\frac{(b+a)}{2}\right]^2 = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} - \frac{b^2+2ab+a^2}{4}$

$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12} = \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$

Example 73

X is a rectangular distribution between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

(i) Write the probability density function; f(x) = $\begin{cases} \frac{1}{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)} & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 0 & elsewhere \end{cases}$

(ii) Find the mean = $\frac{(b+a)}{2} = \frac{\left(\frac{\pi}{2}+\left(-\frac{\pi}{2}\right)\right)}{2} = 0$

(iii) Find the variance of x = $\frac{(b-a)^2}{12} = \frac{\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]^2}{12} = \frac{\pi^2}{12}$

Example 74

X is a rectangular distribution between over the interval $-3 \le x \le -1$

Find

(i) P(-2 ≤ X ≤ - 1.5) = $\int_{-2}^{-1.5} \frac{1}{2}dx = \frac{1}{2}(x)_{-2}^{-1.5} = \frac{1}{4}$

(ii) Mean = $\frac{(b+a)}{2} = \frac{(-1+(-3))}{2} = -2$

(iii) Var(x) = $\frac{(b-a)^2}{12} = \frac{(-1--3)^2}{12} = \frac{1}{3}$

Revision exercise 11

1. X follows a uniform distribution with probability density function f(x) = $\begin{cases} k & 3 \le x \le 6 \\ 0 & elsewhere \end{cases}$

   Find (i) value of k = $\frac{1}{3}$ (ii) E(X) = 4.5 (iii) var(X) = 0.75 (iv) P(X>5) = $\frac{1}{3}$
2. X is distributed uniformly over -5 ≤ x ≤ -2

   Find (i) P(-4.3 ≤ X≤ -2.8) = 0.5 (ii) E(X) = -2.5 (iii) standard deviation = 0.865
3. The continuous random variable has a probability density function f(x) = $\begin{cases} \frac{1}{4} & 1 \le x \le k \\ 0 & elsewhere \end{cases}$

   Find (i) value of k = 5 (ii) P(2.1 ≤ X≤ 3.4) =0.325 (iii) E(X) = 3 (jv) Var (X) = $1\frac{1}{3}$
4. The continuous random variable has a probability density function f(x) = $\begin{cases} \frac{1}{5} & 32 \le x \le 37 \\ 0 & elsewhere \end{cases}$

Find the probability that y lies within one standard deviation of the mean= 0.577

5. The continuous random variable X has cumulative distribution function

$$F(X) = \begin{cases} 0 & x < 2 \\ \frac{x-2}{5} & 2 \le x \le 7 \\ 1 & x \ge 7 \end{cases}$$

Find (i) E(X) = 4.5 (ii) Var(X) = $2\frac{1}{12}$

6. The continuous random variable X is uniformly distributed in the interval a ≤ x ≤ b. the lower quartile is 5 and the upper quartile is 9. Find
   (i) Values of a and b (a= 3, b = 11)
   (ii) P(6 ≤ X ≤ 7) = 0.125
   (iii) Cumulative distribution function; $F(X) = \begin{cases} 0 & x < 3 \\ \frac{x-3}{8} & 3 \le x \le 11 \\ 1 & x \ge 11 \end{cases}$

7. The number of patients visiting a certain hospital is uniformly distributed between 150 and 210
   (i) Write down the probability density function of the number of patients

   f(x) = $\begin{cases} \frac{1}{210-150} & 150 \le x210 \\ 0 & elsewhere \end{cases}$

   (ii) Find P(170< x< 194) = 0.4

Thank you

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