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SENIOR FIVE TERM 3

TOPIC 3/4: Probability Distributions

Competency: The learner demonstrates the ability to analyse and apply binomial, rectangular, and normal distributions to solve real-world problems.

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15. 1 Binomial distribution

A binomial distribution is a special type of a discrete random variable in which an experiment gives rise to only two outcomes either success or failure.

Conditions for binomial distribution

- (i) The experiment has a finite (repeated) number of trials, n
- (ii) The trials are independent
- (iii) The outcome of each trial is either a success or a failure
- (iv) The probability, p of successful outcome is constant for all trials

If a discrete random variable X is the number of **successful** outcomes in n trials and satisfies the above conditions, then X follows a binomial distribution written as $X \sim B(n, p)$ or $X \sim \text{Bin}(n, p)$

Formula for Binomial distribution

(a) If $X \sim B(n, p)$, the probability of obtaining, r success in n trials $P(X = r)$ where

$$P(X = r) = {}^n C_r p^r q^{n-r} \text{ for } r = 0, 1, 2, 3, \dots, n \text{ where } q = 1-p$$

$$= \frac{n!}{(n-r)!r!} p^r q^{n-r}$$

- (b) $E(X) = np$
- (c) $\text{Var}(X) = npq$ where $q = 1 - p$
- (d) $s.d = \sqrt{npq}$

Example 1

The random variable X is distributed $B(7, 0.2)$. find

- (i) $P(X=3)$
- (ii) $P(1 < X \leq 4)$
- (iii) $P(X > 1)$

Solution

- (i) $n = 7, p = 0.2, q = 1 - 0.2 = 0.8$
 $P(X=3) = {}^7 C_3 \times 0.2^3 \times 0.8^4$
 $= \frac{7!}{(7-3)!3!} \times 0.2^3 \times 0.8^4$
 $= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1)(3 \times 2 \times 1)} \times 0.2^3 \times 0.8^4 = 0.115$
- (ii) $P(1 < X \leq 4) = P(X=2) + P(X=3) + P(X=4)$
 $= {}^7 C_2 \times 0.2^2 \times 0.8^5 + {}^7 C_3 \times 0.2^3 \times 0.8^4 + {}^7 C_4 \times 0.2^4 \times 0.8^3$
 $= 0.275 + 0.115 + 0.029 = 0.419$
- (iii) $P(X > 1) = 1 - (P(X \leq 1) = [1 - P(X = 0) + P(X = 1)])$
 $= 1 - [{}^7 C_0 \times 0.2^0 \times 0.8^7 + {}^7 C_1 \times 0.2^1 \times 0.8^6]$
 $= [1 - (0.210 + 0.367)]$
 $= 0.423$

Example 2

At freedom city super market, 60% of the customers shop on Saturday. Find the probability that in a randomly selected sample of 10 customers

- (i) Exactly 2 shop on Saturday
 $n = 10, p = 0.6, q = 1 - 0.6 = 0.4$
 $P(X = 2) = {}^{10}C_2 \times 0.6^2 \times 0.4^8 = 0.011$
- (ii) More than 7 shop on Saturday
 $P(X > 7) = P(X = 8) + P(X = 9) + P(X = 10)$
 $= {}^{10}C_8 \times 0.6^8 \times 0.4^2 + {}^{10}C_9 \times 0.6^9 \times 0.4^1 + {}^{10}C_{10} \times 0.6^{10} \times 0.4^0$
 $= 0.121 + 0.040 + 0.006 = 0.167$

Example 3

The probability that a marble drawn from a box is red is 0.4. If a sample of 6 marbles is taken, find the probability that it will contain;

- (i) No red marble
 $n = 6, p = 0.4, q = 1 - 0.4 = 0.6$
 $P(X = 0) = {}^6C_0 \times 0.4^0 \times 0.6^6 = 0.047$
- (ii) $P(X = 5 \text{ or } 6) = P(X = 5) + P(X = 6) = {}^6C_5 \times 0.4^5 \times 0.6^1 + {}^6C_6 \times 0.4^6 \times 0.6^0$
 $= 0.037 + 0.004 = 0.041$
- (iii) Less than half red marbles
 $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$
 $= {}^6C_0 \times 0.4^0 \times 0.6^6 + {}^6C_1 \times 0.4^1 \times 0.6^5 + {}^6C_2 \times 0.4^2 \times 0.6^4$
 $= 0.047 + 0.187 + 0.331 = 0.545$

Example 4

A biased coin is such that the chance of a head appearing upper most when tossed is twice of the tail appearing uppermost. If the coin is tossed 10 times. Find the probability that

- (i) Exactly 6 heads will appear
 $P(H) + P(T) = 1$
 $2x + x = 1$
 $x = \frac{1}{3}$
 $n = 10, p = \frac{2}{3}; q = \frac{1}{3}$
 $P(X = 6) = {}^{10}C_6 \times \left(\frac{2}{3}\right)^6 \times \left(\frac{1}{3}\right)^4 = 0.228$
- (ii) $P(5 < X < 8) = P(X = 6) + P(X = 7) = {}^{10}C_6 \times \left(\frac{2}{3}\right)^6 \times \left(\frac{1}{3}\right)^4 + {}^{10}C_7 \times \left(\frac{2}{3}\right)^7 \times \left(\frac{1}{3}\right)^3$
 $= 0.228 + 0.260 = 0.488$

Example 5

A box contains a large number of pens. The probability that a pen is faulty is 0.1. How many pens would you need to select to be more than 95% certain of picking one faulty pen?

Solution

$$n? p = 0.1, q = 0.9$$

$$P(X \geq 1) = 1 - P(X = 0) > 0.95$$

$$= 1 - {}^n C_0 \times 0.1^0 \times 0.9^n > 0.95$$

$$= 0.05 > 0.9^n$$

$$n > \frac{\log_{10} 0.05}{\log_{10} 0.9}$$

$$n > 29$$

The least value $n = 29$

Example 6

In Binomial experiment, the probability of a success for n trials is 0.6. If the mean is 7.2, find the

(a) value of n . (02 marks)

$$\text{Mean} = np$$

$$7.2 = 0.6n$$

$$n = 12$$

(b) probability of obtaining 7 success. (03marks)

$$\begin{aligned} P(X = x) &= {}^n C_x p^x q^{(n-x)} \\ &= {}^{12} C_7 (0.6)^7 (0.4)^5 \\ &= 0.227 \end{aligned}$$

Example 7

It was observed that 3 seeds in every four seeds planted germinate. If 16 seeds were planted, calculate

(i) expected number of seeds that will germinate (03marks)

$$\text{Given } p = \frac{3}{4}, q = \frac{1}{4}, n = 16$$

Let X be the random variable that the seed germinates

$$X \sim B\left(16, \frac{3}{4}\right)$$

$$E(x) = np = 16 \times \frac{3}{4} = 12 \text{ seeds}$$

(ii) probability that exactly 14 seeds will germinate (02marks)

$$P(X = 14) = {}^{16} C_{14} \left(\frac{3}{4}\right)^{14} \left(\frac{1}{4}\right)^2 = 0.1336$$

Using Cumulative binomial probability table

The table give values of $P(X \geq x)$ for values of n and p .

- (i) $P(X \leq X) = 1 - (P(X \geq (X+1)))$
- (ii) $P(X = x) + P(X \geq x) - (PX \geq x+1)$

Example 8

The random variable is distributed $B(5, 0.3)$. Find

- (i) $P(X \geq 3)$ (ii) $P(X > 1)$ (iii) $P(X \leq 4)$ (iv) $P(X < 3)$ (v) $P(X = 2)$

Solution

$$n = 5, p = 0.3$$

- (i) $P(X \geq 3) = 0.1631$
- (ii) $P(X > 1) = P(X \geq 2) = 0.4718$
- (iii) $P(X \leq 4) = 1 - (P(X \geq 5)) = 1 - 0.0024 = 0.9976$
- (iv) $P(X < 3) = P(X \geq 2) = 1 - P(X \geq 3) = 1 - 0.1631 = 0.8369$
- (v) $P(X = 2) = P(X \geq 2) - P(X \geq 3) = 0.4718 - 0.1631 = 0.3087$

Example 9

An unbiased coin is tossed 15 times. Find th probability that

- (i) Exactly eight heads will appear upper most
- (ii) Between 6 and 10 heads will appear
- (iii) Between 6 and 10 heads inclusive will appear

Solution

$$n = 15, p = 0.5$$

- (i) $P(X = 8) = P(X \geq 8) - P(X \geq 9) = 0.5000 - 0.3036 = 0.1964$
- (ii) $P(6 < X < 10) = P(X \leq X \leq 9) = P(X \geq 7) - P(X \geq 10) = 0.6964 - 0.1509 = 0.5455$
- (iii) $P(X \leq X \leq 10) = P(X \geq 6) - P(X \geq 11) = 0.8491 - 0.0592 = 0.7899$

Example 10

A student attempts 20 objective questions by guest work. Each questions has got four possible alternatives out of which one is correct. Find the probability that he gets

- (i) Exactly 9 correct answers
- (ii) At least 12 correct answers
- (iii) At most 6 correct answers
- (iv) Between 6 and 14 correct answers inclusive
- (v) Exactly 7 correct answers.

Solution

For correct answers, $n = 20$, $p = 0.25$, $q = 0.75$

- (i) $P(X = 9) = P(X \geq 9) - P(X \geq 10) = 0.0409 - 0.0139 = 0.027$
- (ii) $P(X \geq 12) = 0.0009$
- (iii) $P(X \leq 6) = 1 - P(X \geq 7) = 1 - 0.2142 = 0.7858$
- (iv) $P(X \leq X \leq 14) = P(X \geq 6) - P(X \geq 15) = 0.3828 - 0.0000 = 0.3828$

For incorrect answers, $n = 20$, $P = 0.75$, $q = 0.25$

Revision exercise 1

1. A biased coin is such that a head is three times as likely to occur as a tail. The coin is tossed 5 times. Find the probability that at most two tails occur = 0.8965
2. Tom's chance of passing an examination is $\frac{2}{3}$. If he sits for four examinations, calculate the probability he passes
 - (i) Only two examinations = 0.2963
 - (ii) More than half of the examinations = 0.5926
3. A fair die is rolled 6 times, calculate the probability that
 - (i) A 2 or 4 appears on the first throw = $\frac{1}{3}$
 - (ii) Four 5's will appear in the six throws = 0.0080
4. Usain Bolt makes 5 practice ran in the 100m sprint. A run is successful if he runs it in less than 11 seconds. There are 8 chances out of 10 that he is successful. Find the probability that;
 - (i) He records at least no success at all = 0.0003
 - (ii) Exactly 5 games = 0.3277
5. The probability that Alex wins a chess game is $\frac{2}{3}$. He plays 8 games, what is the probability that he wins
 - (i) At least 7 games = 0.1951
 - (ii) Exactly 5 games = 0.2731
6. Usain Bolt makes 5 practice ran in the 100m sprint. A run is successful if he runs it in less than 11 seconds. There are 8 chances out of 10 that he is successful. Find the probability that;
 - (i) He records at least no success at all = 0.0003
 - (ii) He record at least 2 success = 0.9933
 - (iii) If he is successful in 5 practice runs, he makes two additional runs. The probability of success in either of the additional runs is 0.7. Determine the probability that Bolt will make 7 successful runs = 0.1606
7. In a test there are 10 objective questions each with a choice of five possible alternatives out of which only one is correct. If a student guesses each of the answers, find the probability that he gets at least two answers correct. = 0.6242
8. 30% of the students in the school are day scholars. From a sample of 10 students chosen at random, find the probability that
 - (i) Only 3 are day scholars = 0.267
 - (ii) Less than half are day scholars = 0.850

9. The probability that a shopper buys a cake is 0.25. Find the probability that in a random sample of 9 shoppers
- Exactly 3 buys a cake = 0.2334
 - More than 7 buy a cake = 0.0001
10. A bag contains counter books of which 40% are blue and the rest are black. A counter book is taken from the bag, its colour is noted then replaced. This is performed 8 times in all. Calculate the probability that
- Exactly 3 will be blue = 0.279
 - At least one will be blue = 0.983
 - More than half will be black = 0.594
11. At a certain school, the records taken from admission's office the ratio of male to female s. 5 applicants is 4:6. Basing on this experience, what is the probability that will be more female applicants in a random collection of a dozen applicants? = 0.665
12. The random variable X is B(6, 0.42). find
- $P(X = 6) = 0.00549$
 - $P(X = 4) = 0.157$
 - $P(X \leq 2) = 0.503$
13. An unbiased die is thrown seven times. find the probability of throwing at least 5 sixes. = 0.002
14. In a family a couple is likely to produce a girl or a boy. Find the probability that in a sample of 5 children there will be more boys than girls = 0.5
15. The probability that it will rain on any given day during examination period is 0.3. calculate the probability that in a given week during examination period, it will rain on;
- Exactly 2 days = 0.318
 - At most two days = 0.647
 - Exactly three days that are consecutive = 0.0324
16. A fair coin is tossed 6 times. find the probability of throwing at least four heads = 0.344
17. The random variable x is B(4, p) and $P(X = 4) = 0.0256$. find $P(X = 2) = 0.3456$
18. In agriculture lab, Silvia plants bean seeds and the probability that they germinate successfully is $\frac{1}{3}$.
- She takes 9 seeds. Find the probability that
 - More than five seeds germinate = 0.0424
 - At least three seeds germinate = 0.623
 - Find the number of seeds that she needs to take in order to 99% certain at least one germinate = 12
19. In a shooting competition, the probability of hitting the target with a single shot is 0.6, if 7 shots are taken; find the probability that the target is hit more than twice. = 0.9037
20. In mass production of shirts, it is found that 5% are defective. Shirts are selected at random of put into packets of 10.
- A packet is selected at random. Find the probability that it contains
 - Three defective shirt = 0.0105
 - Less than three defective shirts = 0.9885
 - Two packets are selected at random. Find the probability that there are no defective shirts in either packet = ${}^{10}C_0(0.05)^0(0.95)^{10} \times {}^{10}C_0(0.05)^0(0.95)^{10} = 0.358$
21. A biased coin is such that it is twice as likely to show a head as a tail. If the coin is tossed 5 times. find the probability that

- (i) Exactly three heads are obtained = 0.329
(ii) More than three heads are obtained = 0.3333
22. The probability that a target is hit 0.3. Find the probability the least number of times shots should be fired if the probability that the target is hit is at least once is greater than 0.95. =9
23. 1% of the light bulbs in a box are faulty. Find the largest sample size which can be taken if it required that the probability that there is no faulty bulb in the sample is greater than 0.5
 $(0.99)^n > 0.5$
 $n = 68$
24. In a test there are 10 multiple choice questions. Each question has got four possible alternatives out of which only one is correct. If a student guesses each of the answers, find the probability that he gets
(i) More than 7 correct answers = 0.0004
(ii) More than half correct answers = 0.0197
25. $X \sim B(n, 0.3)$. find the value of n such that $P(X \geq 1) = 0.8$:
26. The random variable X is $B(n, 0.6)$. find the value of n such that $P(X < 1) = 0.0256$

Solutions to revision exercise 1

1. A biased coin is such that a head is three times as likely to occur as a tail. The coin is tossed 5 times. Find the probability that at most two tails occur

Solution

$$n = 5$$

$$p + q = 1$$

$$x + 3x = 1, x = 0.25$$

$$p = 0.25, q = 0.75$$

$$\begin{aligned} P(X \leq 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= {}^5C_0 \times 0.25^0 \times 0.75^5 + {}^5C_1 \times 0.25^1 \times 0.75^4 + {}^5C_2 \times 0.25^2 \times 0.75^3 \\ &= 0.2373 + 0.3955 + 0.2637 = 0.8965 \end{aligned}$$

2. Tom's chance of passing an examination is $\frac{2}{3}$. If he sits for four examinations, calculate the probability he passes

- (i) Only two examinations

$$n = 4, p = \frac{2}{3}, q = \frac{1}{3}$$

$$P(X = 2) = {}^4C_2 \times \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^2 = 0.2963$$

- (ii) More than half of the examinations

$$P(X < 2) = P(X = 3) + P(X = 4) = {}^4C_3 \times \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^1 + {}^4C_4 \times \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^0 = 0.5926$$

3. A fair die is rolled 6 times, calculate the probability that

- (i) A 2 or 4 appears on the first throw

$$P(X = 2 \text{ or } 4) = P(X = 2) + P(X = 4) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

- (ii) Four 5's will appear in the six throws

$$n = 6, p = \frac{1}{6}, q = \frac{5}{6}$$

$$P(X = 4) = {}^6C_4 \times \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^2 = 0.0080$$

4. Usain Bolt makes 5 practice ran in the 100m sprint. A run is successful if he runs it in less than 11 seconds. There are 8 chances out of 10 that he is successful. Find the probability that;

(i) He records at least no success at all

$$n = 5, p = 0.8, q = 0.2$$

$$P(X = 0) = {}^5C_0 \times (0.8)^0 (0.2)^5 = 0.0003$$

(ii) Exactly 5 games

$$P(X = 5) = {}^5C_5 \times (0.8)^5 (0.2)^0 = 0.3277$$

5. The probability that Alex wins a chess game is $\frac{2}{3}$. He plays 8 games, what is the probability that he wins

(i) At least 7 games

$$n = 8, p = \frac{2}{3}, q = \frac{1}{3}$$

$$P(X \leq 7) = P(X = 7) + P(X = 8) = {}^8C_7 \times \left(\frac{2}{3}\right)^7 \left(\frac{1}{3}\right)^1 + {}^8C_8 \times \left(\frac{2}{3}\right)^8 \left(\frac{1}{3}\right)^0 \\ = 0.1561 + 0.390 = 0.1951$$

(ii) Exactly 5 games

$$P(X = 5) = {}^8C_5 \times \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^3 = 0.2731$$

6. Usain Bolt makes 5 practice ran in the 100m sprint. A run is successful if he runs it in less than 11 seconds. There are 8 chances out of 10 that he is successful. Find the probability that;

(i) He records at least no success at all

$$n = 5, p = 0.8, q = 0.2$$

$$P(X = 0) = {}^5C_0 \times (0.8)^0 (0.2)^5 = 0.0003$$

(ii) He record at least 2 success

$$P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X=0) + P(X = 1)]$$

$$= 1 - [{}^5C_0 \times (0.8)^0 (0.2)^5 + {}^5C_1 \times (0.8)^1 (0.2)^4] = 1 - (0.0003 + 0.0064) = 0.9933$$

(iii) If he is successful in 5 practice runs, he makes two additional runs. The probability of success in either of the additional runs is 0.7. Determine the probability that Bolt will make 7 successful runs

$$\text{Probability} = {}^5C_5 \times 0.8^5 \times 0.2^0 \text{ and } {}^2C^2 \times 0.7^2 (0.3)^0 \\ = 0.32768 \times 0.49 = 0.1606$$

7. In a test there are 10 objective questions each with a choice of five possible alternatives out of which only one is correct. If a student guesses each of the answers, find the probability that he gets at least two answers correct.

Solution

$$n = 10, p = 0.2, q = 0.8$$

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - [P(X = 0) + P(X = 1)]$$

$$= 1 - [{}^{10}C_0 \times 0.2^0 \times 0.8^{10} + {}^{10}C_1 \times 0.2^1 \times 0.8^9]$$

$$= 1 - (0.1074 + 0.2684) = (1 - 0.3758) = 0.6242$$

8. 30% of the students in the school are day scholars. From a sample of 10 students chosen at random, find the probability that

(i) Only 3 are day scholars
 $n = 10, p = 0.3, q = 0.7$
 $P(X = 3) = {}^{10}C_3(0.3)^3(0.7)^7 = 0.267$

(ii) Less than half are day scholars
 $P(X < 5) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$
 $= {}^{10}C_0(0.3)^0(0.7)^{10} + {}^{10}C_1(0.3)^1(0.7)^9 + {}^{10}C_2(0.3)^2(0.7)^8 + {}^{10}C_3(0.3)^3(0.7)^7$
 $+ {}^{10}C_4(0.3)^4(0.7)^6$
 $= 0.850$

9. The probability that a shopper buys a cake is 0.25. Find the probability that in a random sample of 9 shoppers

(i) Exactly 3 buys a cake
 $n = 9, p = 0.25, q = 0.75$
 $P(X = 3) = {}^9C_3(0.25)^3(0.75)^6 = 0.2334$

(ii) More than 7 buy a cake
 $P(X > 7) = P(X = 8) + P(X = 9)$
 $= {}^9C_8(0.25)^8(0.75)^1 + {}^9C_9(0.25)^9(0.75)^0 = 0.0001$

10. A bag contains counter books of which 40% are blue and the rest are black. A counter book is taken from the bag, its colour is noted then replaced. This is performed 8 times in all. Calculate the probability that

(i) Exactly 3 will be blue
 $n = 8, p = 0.4, q = 0.6$
 $P(X = 3) = {}^8C_3(0.4)^3(0.6)^5 = 0.279$

(ii) At least one will be blue
 $P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0) = 1 - {}^8C_0(0.4)^0(0.6)^8 = 0.983$

(iii) More than half will be black
 $P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$
 $= {}^8C_0(0.4)^0(0.6)^8 + {}^8C_1(0.4)^1(0.6)^7 + {}^8C_2(0.4)^2(0.6)^6 + {}^8C_3(0.4)^3(0.6)^5$
 $= 0.594$

11. At a certain school, the records taken from admission's office the ratio of male to female s. 5 applicants is 4:6. Basing on this experience, what is the probability that will be more female applicants in a random collection of a dozen applicants?

Solution

$n = 12, p = 0.4, q = 0.6$
 $P(X \leq 5) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$
 $= {}^{12}C_0(0.4)^0(0.6)^{12} + {}^{12}C_1(0.4)^1(0.6)^{11} + {}^{12}C_2(0.4)^2(0.6)^{10} + {}^{12}C_3(0.4)^3(0.6)^9 +$
 ${}^{12}C_4(0.4)^4(0.6)^8 + {}^{12}C_5(0.4)^5(0.6)^7$
 $= 0.6652$

12. The random variable X is B(6, 0.42). find

(i) $P(X = 6)$
 $n = 6, p = 0.42, q = 1 - 0.42 = 0.58$
 $P(X = 6) = {}^6C_6(0.42)^6(0.58)^0 = 0.00549$

(ii) $P(X = 4)$

$${}^6C_4(0.42)^4(0.58)^2 = 0.157$$

(iii) $P(X \leq 2)$

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= {}^6C_0(0.42)^0(0.58)^6 + {}^6C_1(0.42)^1(0.58)^5 + {}^6C_2(0.42)^2(0.58)^4 = 0.503$$

13. An unbiased die is thrown seven times. find the probability of throwing at least 5 sixes $n = 7$, $p =$

$$\frac{1}{6}, q = \frac{5}{6}$$

$$P(X \leq 5) = P(X = 5) + P(X = 6) + P(X = 7)$$

14.
$$= {}^7C_5\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^2 + {}^7C_6\left(\frac{1}{6}\right)^6(0.58)^1 + {}^7C_7\left(\frac{1}{6}\right)^7\left(\frac{5}{6}\right)^0 = 0.002$$

15. In a family a couple is likely to produce a girl or a boy. Find the probability that in a sample of 5 children there will be more boys than girls

$$n = 5, p = 0.5, q = 0.5$$

$$P(X \geq 3) = P(X = 3) + P(X = 4) + P(X = 5)$$

$$= {}^5C_3(0.5)^3(0.5)^2 + {}^5C_4(0.5)^4(0.5)^1 + {}^5C_5(0.5)^5(0.5)^0 = 0.5$$

16. The probability that it will rain on any given day during examination period is 0.3. calculate the probability that in a given week during examination period, it will rain on;

(i) Exactly 2 days

$$n = 7, p = 0.3, q = 0.7$$

$$P(X = 2) = {}^7C_2(0.3)^2(0.7)^5 = 0.318$$

(ii) At most two days = 0.671

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= {}^7C_0(0.3)^0(0.7)^7 + {}^7C_1(0.3)^1(0.7)^6 + {}^7C_2(0.3)^2(0.7)^5 = 0.647$$

17. A fair coin is tossed 6 times. find the probability of throwing at least four heads

$$n = 6, p = 0.5, q = 0.5$$

$$P(X \leq 4) = P(X = 4) + P(X = 5) + P(X = 6)$$

$$= {}^6C_4(0.5)^4(0.5)^2 + {}^6C_5(0.5)^5(0.5)^1 + {}^6C_6(0.5)^6(0.5)^0 = 0.344$$

18. The random variable x is $B(4, p)$ and $P(X = 4) = 0.0256$. find $P(X = 2)$

$$n = 4, p = p, q = (1-p)$$

$$P(X = 4) = {}^4C_4(p)^4(q)^0 = 0.0256$$

$$(p)^4 = 0.0256; p = 0.4$$

$$P(X = 2) = {}^4C_2(0.4)^2(0.6)^2 = 0.3456$$

19. In agriculture lab, Silvia plants bean seeds and the probability that they germinate successfully is $\frac{1}{3}$.

(a) She takes 9 seeds. Find the probability that

(i) More than five seeds germinate

$$P(X < 5) = P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9)$$

$$= {}^9C_6\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^3 + {}^9C_7\left(\frac{1}{3}\right)^7\left(\frac{2}{3}\right)^2 + {}^9C_8\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)^1 + {}^9C_9\left(\frac{1}{3}\right)^9\left(\frac{2}{3}\right)^0 = 0.0424$$

(ii) At least three seeds germinate

$$P(X \geq 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - [{}^9C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^9 + {}^9C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^8 + {}^9C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^7] = 0.623$$

(b) Find the number of seeds that she needs to take in order to 99% certain at least one germinate

$$P(X \leq 1) = 1 - P(X = 0) = 1 - {}^n C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^n = 0.99$$

$$\left(\frac{2}{3}\right)^n = 0.01$$

$$n = 11.36$$

therefore the number = 12

20. In a shooting competition, the probability of hitting the target with a single shot is 0.6, if 7 shots are taken; find the probability that the target is hit more than twice.

$$n = 7, p = 0.6, q = 0.4$$

$$\begin{aligned} P(X > 2) &= 1 - P(X \leq 2) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)] \\ &= 1 - [{}^7 C_0 (0.6)^0 (0.4)^7 + {}^7 C_1 (0.6)^1 (0.4)^6 + {}^7 C_2 (0.6)^2 (0.4)^5] \\ &= 0.9037 \end{aligned}$$

21. In mass production of shirts, it is found that 5% are defective. Shirts are selected at random of put into packets of 10.

(a) A packet is selected at random. Find the probability that it contains

(i) Three defective shirt

$$P(X = 3) = {}^{10} C_3 (0.05)^3 (0.95)^7 = 0.0105$$

(ii) Less than three defective shirts

$$\begin{aligned} P(X < 3) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= {}^{10} C_0 (0.05)^0 (0.95)^{10} + {}^{10} C_1 (0.05)^1 (0.95)^9 + {}^{10} C_2 (0.05)^2 (0.95)^8 = 0.9885 \end{aligned}$$

(b) Two packets are selected at random. Find the probability that there are no defective shirts in either packet = $({}^{10} C_0 (0.05)^0 (0.95)^{10})^2 = 0.358$

22. A biased coin is such that it is twice as likely to show a head as a tail. If the coin is tossed 5 times. find the probability that

(i) Exactly three heads are obtained

$$n = 5, p = \frac{2}{3}, q = \frac{1}{3}$$

$$P(X = 3) = {}^5 C_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^2 = 0.3292$$

(ii) More than three heads are obtained = 0.3333

$$\begin{aligned} P(X > 3) &= P(X = 4) + P(X = 5) \\ &= {}^5 C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^1 + {}^5 C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^0 = \end{aligned}$$

23. The probability that a target is hit 0.3. Find the probability the least number of times shots should be fired if the probability that the target is hit is at least once is greater than 0.95. =9

$$n = ? \quad p = 0.3, q = 0.7$$

$$P(X \geq 1) = 1 - P(X = 0) = 1 - {}^n C_0 (0.3)^0 (0.7)^n > 0.95$$

$$(0.7)^n > 0.05$$

$$n = \frac{\log 0.05}{\log 0.7} = 8.399$$

therefore n = 9

24. 1% of the light bulbs in a box are faulty. Find the largest sample size which can be taken if it required that the probability that there is no faulty bulb in the sample is greater than 0.5

$$(0.99)^n > 0.5$$

$$n = 68$$

25. In a test there are 10 multiple choice questions. Each question has got four possible alternatives out of which only one is correct. If a student guesses each of the answers, find the probability that he gets

- (i) More than 7 correct answers = 0.0004
 - (ii) More than half correct answers = 0.0197
26. $X \sim B(n, 0.3)$. find the value of n such that $P(X \geq 1) = 0.8 : 0.7^n = 0.2. = 5$
27. The random variable X is $B(n, 0.6)$. find the value of n such that $P(X < 1) = 0.0256$
 $0.4^n = 0.0256, n = 4$

Expectation, variance and standard deviation of binomial distribution

If $X \sim B(n, p)$

$E(X) = np$

$Var(X) = npq$ where $q = 1 - p$

$s.d = \sqrt{npq}$

Example 9

The random variable x is $B(4, 0.8)$. the mean, variance and standard deviation

Mean = $np = 4 \times 0.8 = 3.2$

Variance = $npq = 4 \times 0.8 \times 0.2 = 0.64$

$s.d = \sqrt{npq} = \sqrt{0.64} = 0.8$

Example 11

The probability that, it will be a sunny day is 0.4. Find the expected number if sunny days in a week and also find the standard deviation

Solution

$E(X) = np = 7 \times 0.4 = 2.8$

$s.d = \sqrt{npq} = \sqrt{(7 \times 0.4 \times 0.6)} = 1.296$

Example 12

X is $B(n, p)$ with mean 5 and standard deviation 2. Find the value of n and p.

$E(X) = np$

$np = 5$ (i)

$s.d = \sqrt{npq}$

$\sqrt{npq} = 2$

$npq = 4$ (ii)

Eqn. (ii) \div eqn. (i)

$$\frac{npq}{np} = \frac{4}{5} = 0.8$$

$$q = 0.8$$

$$p = 0.2$$

$$n = \frac{5}{0.2} = 25$$

Mode of the binomial distribution

The mode is the value of X that is most likely to occur. The value of X with the highest probability and its close to the mean gives the mode.

Example 13

The probability that a student is awarded a distinction in mathematics examination is 0.15. In a randomly selected group of 15 students, what is the most likely number of students awarded a distinction

Solution

$$E(X) = np = 15 \times 0.15 = 2.25$$

$$P(X = 2) = {}^{15}C_2(0.15)^2(0.85)^{13} = 0.286$$

$$P(X = 3) = {}^{15}C_3(0.15)^3(0.85)^{12} = 0.216$$

The most likely number of students awarded a distinction = 2

Example 14

In a school 80% of the students find difficulties in Physics. If a sample of 12 students is chosen

- (i) What is the most likely number of students who find it difficult in physics.

$$E(X) = np = 0.8 \times 12 = 9.6$$

$$P(X = 9) = {}^{12}C_9(0.8)^9(0.2)^3 = 0.236$$

$$P(X = 10) = {}^{12}C_{10}(0.8)^{10}(0.2)^2 = 0.294$$

The most likely number is 10

- (ii) Find the probability that fewer than half find difficulty in Physics.

$$P(X < 6) = 0.0004$$

Revision exercise 2

- 10% of drugs at a certain Pharmacy are expired. A sample of 25 drugs is taken. Find the expected number and standard deviation of expired drugs (25,1.5)
- The probability that a student scores above 60% in mathematics test is 0.5. In a random sample of 15 students, what is the most likely number of students who score above 60%. (7 and 8)
- The probability that an apple picked at random from a sack is bad is 0.15.
 - Find the standard deviation of the number of bad apples in a sample of 15 apples.=1.38
 - What is the most likely number of bad apples in a sample of 30 apples = 4

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4. The random variable X is $B(n, 0.3)$ and $E(X) = 2.4$. Find n and standard deviation (8, 1.30)
5. In a group of people, the expected number who wear glasses is 2 and the variance is 1.6, find the probability that;
 - (i) A person chosen at random from the group wear glasses = 0.2
 - (ii) 6 people in the group wear glasses. = 0.00551
6. New vision publishes a governance article in its newspaper each day of the week Sunday. A man is able to read 8 out of ten articles
 - (i) Find the expected value and the standard deviation of the number of read articles in a given week = 4.8
 - (ii) What is the probability that he will read at least 5 articles in a given week? = 0.98
7. A die is biased and probability, p of throwing a six is known to be less than $\frac{1}{6}$. An experiment consists of recording the number of sixes n 25 throws of the die. The standard deviation of the number of sixes is 1.5. calculate the
 - (i) value of p . = 0.1
 - (ii) the probability that exactly three sixes are recorded during a particular experiment = 0.23
8. The random variable X is $B(10, p)$ where $p < 0.5$. The variance of X is 1.875. find
 - (i) Value of p . = 0.25
 - (ii) $E(X) = 2.5$
 - (iii) $P(X = 2) = 0.282$
9. In a bag there are 6 red pens, 8 blue pens and 6 black pens. an experiment consists of taking a pen at random from the bag, noting its colour and then replacing it in the bag. This procedure is repeated 10 times in all. find
 - (i) Expected number of red pens drawn = 3
 - (ii) Most likely number of black pens drawn = 3
 - (iii) Probability that not more than four blue pens are drawn = 0.633
10. The random variable X is distributed binomially with mean 2 and variance 1.6, Find
 - (i) the probability that x is less than 6 = 0.994
 - (ii) the most likely value of $X = 2$
11. Each day a bakery delivers the same number of loaves to a certain shop which sells on average 98% of them. Assuming that the number of loaves sold per day has a binomial distribution with standard deviation 7. Find the expected number of loaves the shop would expect to sell per day = 2500
12. On average 20% of the bolts produced by a machine are faulty. Samples of 10 bolts are to be selected at random each day. Each bolt will be selected and replaced in the set of bolts which have been produced on that day.
 - (i) Find the probability that in any one sample, two bolts or less will be faulty = 0.68
 - (ii) Calculate the expected value and variance of the number of bolts in a sample which will not be faulty. = 8, 1.6
13. An experiment consists of taking 12 shots at a target and counting the number of hits. The expected number of hit was found to be 3. Calculate
 - (i) The probability of hitting the target with a single shot. = 0.25
 - (ii) Standard deviation of the number of hits in an experiment = 1.5
14. In a certain family, the probability that they will have a baby boy is 0.6. If there are 5 children in a family determine

- (a) Expected number of girls = 2
 (b) The probability that there are at least three girls = 0.317
 (c) The probability they are all boys. = 0.0778
15. The probability of winning a game is 0.8. Ten games are played. What is the;
 (a) Mean number of success and variance. = 8, 1.6
 (b) The probability of at least 8 success in the ten = 0.6778
16. In a test there are 10 multiple choices questions. Each question has four possible alternatives out of which one is correct. If a student guesses each of the answers, find the
 (i) The probability that at least four answers are correct = 0.2241
 (ii) Most likely number of correct answers = 2
17. A biased coin is such that a head is twice as likely to occur as a tail. the coin is tossed 15 times. Find the
 (i) The expected number of heads = 10
 (ii) Probability that at most two tails occur = 0.0793

Solutions to revision exercise 2

14. In a certain family, the probability that they will have a baby boy is 0.6. If there are 5 children in a family determine
 (a) Expected number of girls
 $n = 5, p = 0.4, q = 0.6$
 $E(X) = np = 5 \times 0.4 = 2$
 (b) The probability that there are at least three girls
 $P(X \leq 3) = P(X = 3) + P(X = 4) + P(X = 5)$
 $= {}^5C_3(0.4)^3(0.6)^2 + {}^5C_4(0.4)^4(0.6)^1 + {}^5C_5(0.4)^5(0.6)^0 = 0.317$
 (c) The probability they are all boys.
 $P(X = 0) = {}^5C_0(0.4)^0(0.6)^5 = 0.0778$
15. The probability of winning a game is 0.8. Ten games are played. What is the;
 (a) Mean number of success and variance.
 Mean = $np = 10 \times 0.8 = 8$
 Variance = $npq = 10 \times 0.8 \times 0.2 = 1.6$
 (b) The probability of at least 8 success in the ten
 $P(X \leq 8) = P(X = 8) + P(X = 9) + P(X = 10)$
 $= {}^{10}C_8(0.8)^8(0.2)^2 + {}^{10}C_9(0.8)^9(0.2)^1 + {}^{10}C_{10}(0.8)^{10}(0.2)^0 = 0.6778$
16. In a test there are 10 multiple choices questions. Each question has four possible alternatives out of which one is correct. If a student guesses each of the answers, find the
 (i) The probability that at least four answers are correct
 $n = 10, p = 0.25, q = 0.75$
 $P(X \geq 4) = 1 - P(X \leq 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)]$
 $= 1 - [{}^{10}C_0(0.25)^0(0.75)^{10} + {}^{10}C_1(0.25)^1(0.75)^9 + {}^{10}C_2(0.25)^2(0.75)^8 + {}^{10}C_3(0.25)^3(0.75)^7]$
 $= 0.2241$
 (ii) Most likely number of correct answers
 $E(X) = np = 0.25 \times 10 = 2.5$

$$P(X = 2) = {}^{10}C_2(0.25)^2(0.75)^8 = 0.2816$$

$$P(X = 3) = {}^{10}C_3(0.25)^3(0.75)^7 = 0.2503$$

\therefore the most likely number of correct answers = 2

17. A biased coin is such that a head is twice as likely to occur as a tail. The coin is tossed 15 times. Find the

(i) The expected number of heads

$$n = 15, p = \frac{2}{3}, q = \frac{1}{3}$$

$$E(X) = np = \frac{2}{3} \times 15 = 10$$

(ii) Probability that at most two tails occur

$$n = 15, p = \frac{1}{3}, q = \frac{2}{3}$$

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= {}^{15}C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{15} + {}^{15}C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{14} + {}^{15}C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{13} = 0.0793$$

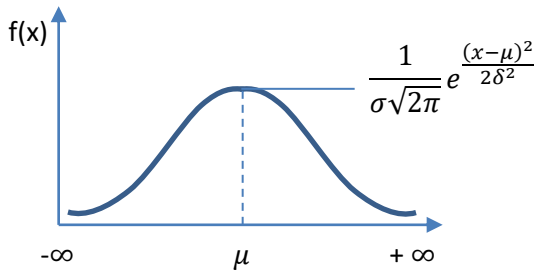
15.2. Normal distribution

A continuous random variable X follows a normal distribution with mean, μ and variance, σ^2 if

$X \sim N(\mu, \sigma^2)$ root

Its p.d.f is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$

A sketch of $f(x)$ gives a normal curve



Properties of the curve

- It is bell shaped
- It is symmetrical about μ
- It extends from $-\infty < x < \infty$

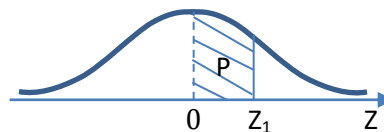
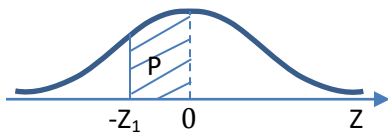
The maximum value of $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

- The total area under the curve = 1

How to read the cumulative normal distribution table

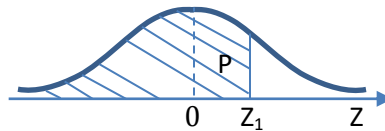
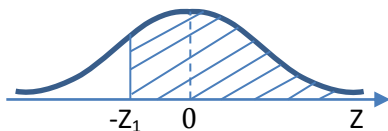
(i) Between 0 and any z value

- (a) $P(0 \leq Z \leq Z_1) = \phi(Z_1) = \text{region P}$ (b) $P(-Z_1 \leq Z \leq 0) = P(0 \leq Z \leq Z_1) = \phi(Z_1) = \text{region P}$
By symmetrical



(ii) Less than any positive z value

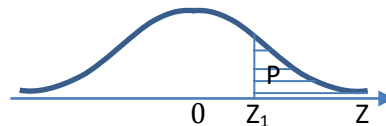
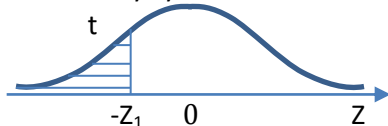
- (a) $P(Z < Z_1) = 0.5 + P(0 \leq Z \leq Z_1) = \phi(Z_1) = \text{region P}$
(b) $P(Z > -Z_1) = P(Z < Z_1) = 0.5 + P(0 \leq Z \leq Z_1) = 0.5 + \phi(Z_1) = \text{region P}$
By symmetrical



(iii) Greater than any positive z value

- $P(Z > Z_1) = 0.5 - P(0 \leq Z \leq Z_1) = 0.5 - \phi(Z_1) = \text{region P}$
 $P(Z < -Z_1) = P(Z > Z_1) = 0.5 - P(0 \leq Z \leq Z_1) = 0.5 - \phi(Z_1) = \text{region P}$

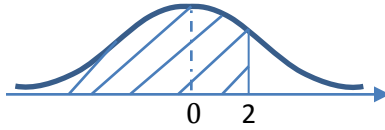
By symmetrical



Example 15

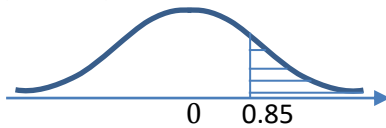
Find

- (i) $P(Z < 2)$
By symmetry



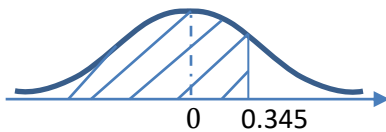
$$\begin{aligned} P(Z < 2) &= 0.5 + \phi(2) \\ &= 0.5 + 0.4772 \\ &= 0.9772 \end{aligned}$$

- (ii) $P(Z > 0.85)$



$$\begin{aligned} P(Z > 0.85) &= 0.5 - \phi(0.85) \\ &= 0.5 - 0.3023 \\ &= 0.1977 \end{aligned}$$

- (iii) $P(X < 0.345)$

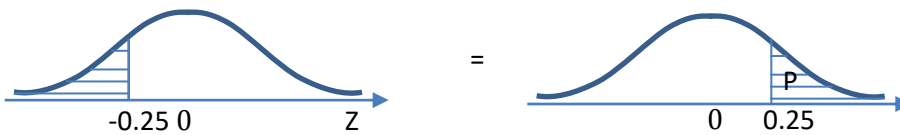


$$\begin{aligned} P(X < 0.345) &= 0.5 + \phi(0.345) \\ &= 0.5 + 0.1331 + 0.0019 \\ &= 0.6350 \end{aligned}$$

Example 16

Find

- (i) $P(Z < -0.25)$
By symmetry



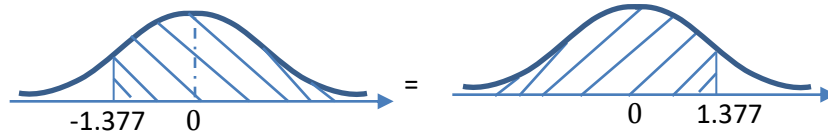
$$P(Z < -0.25) = P(Z > 0.25) = 0.5 - \phi(0.25)$$

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$$= 0.5 - 0.0987$$

$$= 0.4013$$

- (ii) $P(Z > -1.377)$
By symmetry



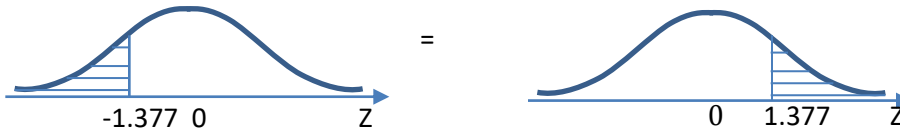
$$P(Z > -1.377) = P(Z < 1.377)$$

$$= 0.5 + \phi(1.377)$$

$$= 0.5 + 0.4147 + 0.0011$$

$$= 0.9158$$

- (iii) $P(Z < -1.377)$
By symmetry



$$P(Z < -1.377) = P(Z > 1.377)$$

$$= 0.5 - \phi(1.377)$$

$$= 0.5 - (0.4147 + 0.0011)$$

$$= 0.0842$$

Standardizing a random variable X

If a random variable X follows a normal distribution with mean, μ and variance, σ^2 , then $X \sim N(\mu, \sigma^2)$ and can be standardized using the equation below and read from a cumulative normal table

$$Z = \frac{X - \mu}{\sigma}$$

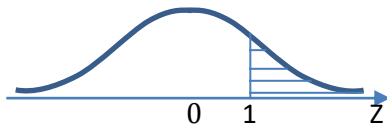
Example 17

Given that the random variable X is $X \sim N(300, 25)$. Find

- (i) $P(X > 305)$

$$P(X > 305) = P\left(Z < \frac{305 - 300}{5}\right)$$

$$= P(Z < 1)$$

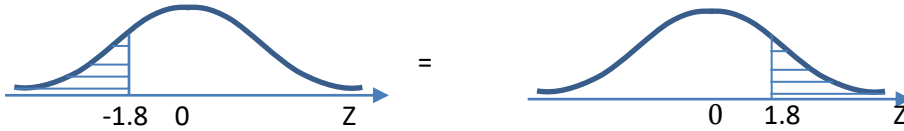


$$= 0.5 - \phi(1)$$

$$= 0.5 - 0.3413 = 0.1587$$

(ii) $P(X < 291)$
 $P(X < 291) = P\left(Z < \frac{291-300}{5}\right) = P(Z < -1.8)$

By symmetry



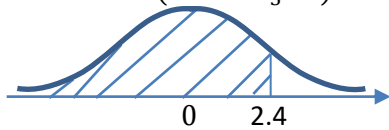
$$P(Z < -1.8) = P(Z > 1.8)$$

$$= 0.5 - \phi(1.8)$$

$$= 0.5 - 0.4641$$

$$= 0.0359$$

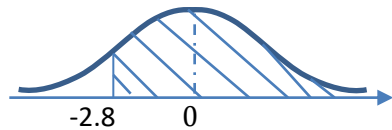
(iii) $P(X < 312)$
 $P(X < 312) = P\left(Z < \frac{312-300}{5}\right) = P(Z < 2.4)$



$$= P(Z < 2.4) = 0.5 + \phi(2.4)$$

$$= 0.5 + 0.4918 = 0.9918$$

(iv) $P(X > 286)$
 $P(X > 286) = P\left(Z < \frac{286-300}{5}\right) = P(Z < -2.8)$



$$= P(Z < -2.8)$$

$$= 0.5 + \phi(2.8)$$

$$= 0.5 + 0.4974 = 0.9974$$

Applications

Example 18

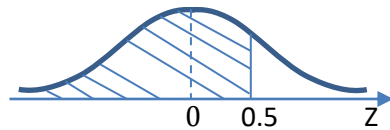
A bakery produces loaves of bread whose weight is normally distributed with mean 1,000g and standard deviation 40g.

- (a) Find the probability that a randomly selected loaf has a weight of utmost 1,020g.

$$\mu = 1000, \sigma = 40$$

Let x be the weight of bread

$$P(x \leq 1020) = P\left(z \leq \frac{1020 - 1000}{40}\right) = P(z \leq 0.5)$$



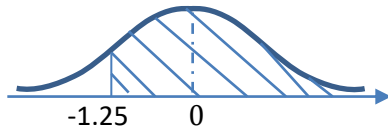
$$P(z \leq 0.5) = 0.5 + P(0 \leq x \leq 0.5)$$

$$= 0.5 + 0.1915$$

$$= 0.6915$$

- (b) Assuming that the bakery makes 10,500 loaves, find the approximate number of loaves with weight greater than 950g.

$$P(x \leq 950) = P\left(z \geq \frac{950 - 1000}{40}\right) = P(z \geq -1.25)$$



$$P(z \geq -1.25) = P(-1.25 \leq z \leq 0) + 0.5$$

$$= (0 \leq z \leq 1.25) + 0.5$$

$$= 0.5 + 0.3944$$

$$= 0.8944$$

$$\text{Number of loaves} = 0.8944 \times 10,500 = 9391$$

Example 19

A factory sells animals food in bags. The weights of the bags are normally distributed with mean weight 50kg and standard deviation 2.8kg.

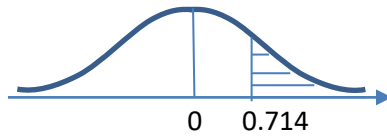
- (a) Find the probability that the weight of any bag selected at random;

Let X = random variable for weight of bags

Given $\mu = 50$ and $\sigma = 2.8$ kg

(i) is more than 53kg (04marks)

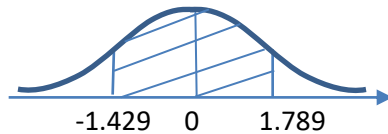
$$P(X > 52) = P\left(Z > \frac{52-50}{2.8}\right) = P(Z > 0.714)$$



$$\begin{aligned} P(Z > 0.714) &= 0.5 - P(0 < Z < 0.714) \\ &= 0.5 - 0.2623 \\ &= 0.2377 \end{aligned}$$

(ii) lies between 46 and 55kg (05marks)

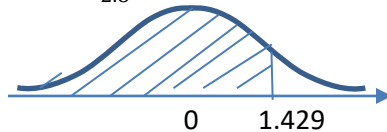
$$\begin{aligned} P(46 < X < 55) &= P\left(\frac{46-50}{2.8} < Z < \frac{55-50}{2.8}\right) \\ &= P(-1.429 < Z < 1.786) \end{aligned}$$



$$\begin{aligned} P(-1.429 < Z < 1.786) &= P(0 < Z < 1.429) + P(0 < Z < 1.789) \\ &= 0.4235 + 0.4630 \\ &= 0.8865 \end{aligned}$$

(b) Determine the percentage of bags whose weights are less than 54kg. (06marks)

$$P(X < 54) = P\left(Z < \frac{54-50}{2.8}\right) = P(Z < 1.429)$$



$$\begin{aligned} P(Z < 1.429) &= 0.5 + P(0 < Z < 1.429) \\ &= 0.5 + 0.4235 \\ &= 0.9235 \end{aligned}$$

Example 20

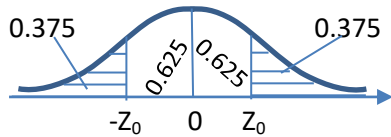
The marks scored by candidates in an examination are normally distributed with a mean score of μ and standard deviation of δ . Given that 37.5% of the candidates scored below 40 and that 12.5% scored above 60, find the;

(i) values of μ and δ . (09marks)

Let X be marks scored

$$P(X < 40) = \frac{37.5}{100} = 0.375$$

$$P(X < 40) = P(Z < Z_0) = 0.375$$



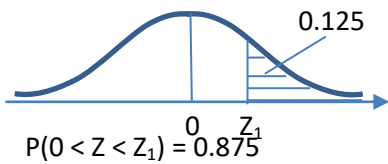
By symmetry, $P(Z < Z_0) = P(Z > Z_0)$

$$\text{But } Z = \frac{X - \mu}{\sigma}$$

$$-0.319 = \frac{40 - \mu}{\sigma}$$

$$\mu - 0.319\sigma = 40 \dots\dots\dots(i)$$

$$P(X > 60) = P(Z > Z_1) = \frac{12.5}{100} = 0.125$$



$$Z_1 = 1.15$$

$$\Rightarrow 1.15 = \frac{60 - \mu}{\sigma}$$

$$\mu - 0.319\sigma = 60 \dots\dots\dots(ii)$$

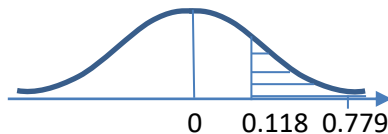
from equation (i) and (ii)

$$\sigma = 13.6 \text{ and } \mu = 44.4$$

(ii) probability that a candidate score between 46 and 55. (06 marks)

$$P(46 < X < 55) = \left[\frac{46 - 44.4}{13.6} < Z < \frac{55 - 44.4}{13.6} \right]$$

$$= P[0.118 < Z < 0.779]$$



$$\begin{aligned}
&= P(0 < Z < 0.779) - P(0 < Z < 0.118) \\
&= 0.2822 - 0.0470 \\
&= 0.2352
\end{aligned}$$

Example 21

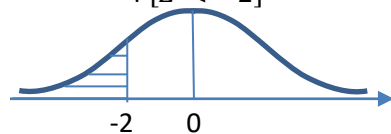
The time taken for a bus to make a journey is normally distributed with mean $3\frac{1}{2}$ hours and standard deviation $\frac{3}{4}$ hours.

(a) Determine the probability that the bus makes a journey:

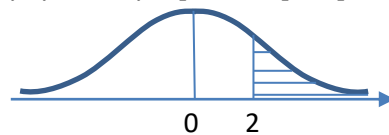
(i) in less than 2 hours (05marks)

Solution

$$\begin{aligned}
P(X < 2) &= P\left[Z < \frac{2-3.5}{0.75}\right] \\
&= P[Z < -2]
\end{aligned}$$



By symmetry, $P[Z < -2] = P[Z > 2]$



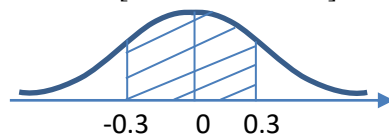
$$\begin{aligned}
P[Z > 2] &= 0.5 - (0 < Z < 2) \\
&= 0.5 - 0.4772 \\
&= 0.0228
\end{aligned}$$

Hence the probability = 0.0228

(ii) between $3\frac{1}{4}$ and $3\frac{3}{4}$ hours (07 marks)

Solution

$$\begin{aligned}
P(3.25 < X < 3.75) &= P\left[\frac{3.25-3.5}{0.75} < Z < \frac{3.75-3.5}{0.75}\right] \\
&= P[-0.3 < Z < 0.3]
\end{aligned}$$



$$\begin{aligned}
P[-0.3 < Z < 0.3] &= 2P(0 < Z < 0.3) \\
&= 2 \times 0.1179 \\
&= 0.2358
\end{aligned}$$

(b) If the bus made two hundred journeys, how many of these journeys did it take less than 2 hours? (03 marks)

$$\text{Number of journeys that take less than 2 hours} = 0.0228 \times 200 = 4$$

Example 22

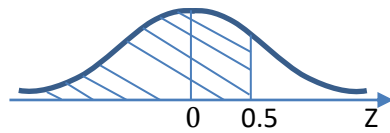
A bakery produces loaves of bread whose weight is normally distributed with mean 1,000g and standard deviation 40g.

- (a) Find the probability that a randomly selected loaf has a weight of at most 1,020g.

$$\mu = 1000, \sigma = 40$$

Let x be the weight of bread

$$P(x \leq 1020) = P(z \leq \frac{1020-1000}{40}) = P(z \leq 0.5)$$



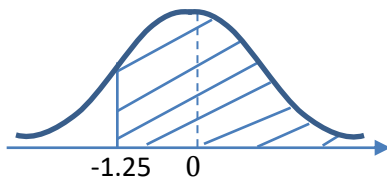
$$P(z \leq 0.5) = 0.5 + P(0 \leq z \leq 0.5)$$

$$= 0.5 + 0.1915$$

$$= 0.6915$$

- (b) Assuming that the bakery makes 10,500 loaves, find the approximate number of loaves with weight greater than 950g.

$$P(x \leq 950) = P(z \geq \frac{950-1000}{40}) = P(z \geq -1.25)$$



$$P(z \geq -1.25) = P(-1.25 \leq z \leq 0) + 0.5$$

$$= (0 \leq z \leq 1.25) + 0.5$$

$$= 0.5 + 0.3944$$

$$= 0.8944$$

$$\text{Number of loaves} = 0.8944 \times 10,500 = 9391$$

Example 24

Given that the random variable X is $X \sim N(10, 4)$. Find

Find (i) $P(X < 7)$ (ii) $P(X > 12)$ (iii) $P(7 < X < 12)$ (iv) $P(9 < X < 11)$

Solution

$$\begin{aligned} \text{(i)} \quad P(X < 7) &= P\left(Z < \frac{7-10}{2}\right) = P(Z < -0.15) = P(Z > 1.5) \\ &= 0.5 - \phi(1.5) = 0.5 - 0.4332 = 0.0668 \\ \text{(ii)} \quad P(X > 12) &= P\left(Z > \frac{12-10}{2}\right) = P(Z > 1) = 0.5 - \phi(1) = 0.5 - 0.3413 = 0.1587 \\ \text{(iii)} \quad P(7 < X < 12) &= P\left(\frac{7-10}{2} < Z < \frac{12-10}{2}\right) \\ &= P(-1.5 < Z < 1) = \phi(1.5) + \phi(1) = 0.4332 + 0.3413 = 0.7745 \\ \text{(iv)} \quad P(9 < X < 11) &= P\left(\frac{9-10}{2} < Z < \frac{11-10}{2}\right) \\ &= P(-0.5 < Z < 0.5) = \phi(0.5) + \phi(0.5) = 2 \times 0.1915 = 0.3830 \end{aligned}$$

Example 25

Given that the random variable X is $X \sim N(50, 8)$. Find

(i) $P(48 < X < 54)$ (ii) $P(52 < X < 55)$ (iii) $P(46 < X < 49)$ (iv) $P(|X - 50| < \sqrt{8})$

Solution

$$\begin{aligned} \text{(i)} \quad P(48 < X < 54) &= P\left(\frac{48-50}{\sqrt{8}} < Z < \frac{54-50}{\sqrt{8}}\right) = P(-0.707 < Z < 1.414) \\ &= \phi(1.414) + \phi(0.707) = 0.4213 + 0.2601 = 0.6814 \\ \text{(ii)} \quad P(52 < X < 55) &= P\left(\frac{52-50}{\sqrt{8}} < Z < \frac{55-50}{\sqrt{8}}\right) = P(0.707 < Z < 1.768) \\ &= \phi(1.768) - \phi(0.707) = 0.4615 - 0.2601 = 0.2014 \\ \text{(iii)} \quad P(46 < X < 49) &= P\left(\frac{46-50}{\sqrt{8}} < Z < \frac{49-50}{\sqrt{8}}\right) = P(-1.414 < Z < -0.354) \\ &= \phi(1.414) - \phi(0.354) = 0.4213 - 0.1383 = 0.283 \\ \text{(iv)} \quad P(|X - 50| < \sqrt{8}) &= P\left(\frac{-\sqrt{8}+50-50}{\sqrt{8}} < Z < \frac{\sqrt{8}+50-50}{\sqrt{8}}\right) = P(-1 < Z < 1) = 2 \times \phi(1) = 2 \times 0.3413 = 0.6826 \end{aligned}$$

Example 26

A random variable X is normally distributed with mean 65 and variance 100, find the probability that X assumes a value between 50 and 90.

$$P(50 < X < 90) = P\left(\frac{50-65}{10} < Z < \frac{90-65}{10}\right) = P(-1.5 < Z < 2.5) = \phi(1.5) + \phi(2.5) = 0.4332 + 0.4938 = 0.927$$

Example 27

Lengths of metal strips produced by a machine are normally distributed with mean length of 150cm and standard deviation of 10cm. find the probability that the length of a randomly selected strip is

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(i) shorter than 165 (ii) within 5cm of the mean

Solution

$$P(X < 165) = P\left(Z < \frac{165-150}{10}\right) = P(Z < 1.5) = 0.5 + \phi(1.5) = 0.5 + 0.4332 = 0.9332$$

$$P(150 - 5 < X < 150 + 5) = P\left(\frac{-5}{10} < Z < \frac{5}{10}\right) = P(-0.5 < Z < 0.5) = 2 \times \phi(0.5) = 2 \times 0.1915 = 0.383$$

Example 28

In end of year exams, the marks are normally distributed with a mean mark of 50 and standard deviation 5. If a mark 45 is required to pass the exam, what percentage of the students failed the exam.

$$P(X < 45) = P\left(Z < \frac{45-50}{5}\right) = P(Z < -1) = P(Z > 1) = 0.5 - \phi(1) = 0.5 - 0.3413 = 0.1587$$

Example 29

A bakery supplies bread to a shop every day. The time to deliver bread to the shop is normally distributed with mean 12 minutes and standard deviation of 2 minutes. Estimate the number of days the year when he takes

(i) longer than 17 minutes (ii) less than 10 minutes (iii) between 9 and 13 minutes

Solution

$$(i) P(X > 17) = P\left(Z > \frac{17-12}{2}\right) = P(Z > 2.5) = 0.5 - \phi(2.5) = 0.5 - 0.4938 = 0.0062$$

The number of days = $0.0062 \times 365 = 2$ days

$$(ii) P(X < 10) = P\left(Z > \frac{10-12}{2}\right) = P(Z < -1) = P(Z > 1) = 0.5 - \phi(1) + 0.5 - 0.3413 = 0.1587$$

The number of days = $0.1587 \times 365 = 58$ days.

$$(iii) P(9 < X < 13) = P\left(\frac{9-12}{2} < Z < \frac{13-12}{2}\right) = P(-1.5 < Z < 0.5) = \phi(1.5) + \phi(0.5) = 0.4332 + 0.1915 = 0.6247$$

Number of days = $0.6247 \times 365 = 228$ days.

Example 30

(a) In a certain athletics competition, points are awarded according to level of performance. The average grade was 82 points with standard deviation of 5 points. All competitors whose grades ranged between 88 to 94 points received certificates. If the grades are normally distributed and 8 competitors received certificates. How many participants took part in the competition?

$$P(88 < X < 94) = P\left(\frac{88-82}{5} < Z < \frac{94-82}{5}\right) = P(1.2 < Z < 2.4) = \frac{8}{n}$$

$$\phi(2.4) - \phi(1.2) = 0.4918 - 0.3849 = 0.1069 = \frac{8}{n}; n = 74.84$$

hence 75 participants took part.

(b) If certificates were to be awarded to only those having between 90 and 94 points. What proportion of the participants would acquire certificates.

$$\begin{aligned}
P(90 < X < 94) &= P\left(\frac{90-82}{5} < Z < \frac{94-82}{5}\right) = P(1.6 < Z < 2.4) = \phi(2.4) - \phi(1.6) \\
&= 0.4918 - 0.4452 = 0.0466 \\
&= 0.0466 \times 100\% = 4.66\%
\end{aligned}$$

Revision exercise 3

- The amount of meat sold by a butcher is normally distributed with mean 43kg and standard deviation 4kg. Determine the probability that the amount of meat sold is between 40kg and 50kg. (0.7333)
- Given that a random variable X is $X \sim N(2, 2.89)$. Find $P(X < 0)$ (0.1198)
- In a school of 800 students their average weight is 54.5kg and standard deviation 6.8kg. given that the weight of students are normally distributed, find
 - Probability that the weight of any student randomly selected is 52.8 kg or less = 0.4014
 - Number of students who weigh over 75kg = 1
 - Weight of the middle 56% of the students ($49.251 < X < 59.750$)
- A sugar factory sells sugar in bags of mean weight 50kg and standard deviation 2.5kg. given that the weight of the bags is normally distributed, find the
 - Probability that the weight of any bag of sugar randomly selected lies between 51.5kg and 53kg = 0.1592
 - Percentage of bags whose weight exceeds 54kg = 5.48%
 - Number of bags that will be rejected out of 1000 bags purchased for weighing below 45.0kg = 23
- A certain maize firm sells maize in bags of mean weight 40kg and standard deviation 2kg. given that the weight of the bags are normally distributed, find
 - Probability that the weight of any bag of maize randomly selected lies between 41.0 and 42.5kg = 0.2029
 - Percentage of bags whose weight exceeds 43kg = 6.68%
 - Number of bags that will be rejected out of 500 bags purchased for weighing below 38.5kg = 113
- Given that the random variable X is $X \sim N(300, 25)$ Find
 - $P(X > 308) = 0.0548$
 - $P(X > 311.5) = 0.0107$
 - $P(X < 294) = 0.8849$
 - $P(X < 290.5) = 0.9713$
 - $P(X > 302) = 0.6554$
 - $P(X > 312) = 0.9918$
- If $X \sim N(50, 20)$. Find
 - $P(X > 60.3) = 0.0106$
 - $P(X < 47.3) = 0.273$
 - $P(X > 48.9) = 0.5972$
 - $P(X > 53.5) = 0.2831$
 - $P(X < 59.8) = 0.9857$
 - $P(X < 62.3) = 0.9970$
- If $X \sim N(-8, 12)$. Find
 - $P(X < -9.8) = 0.1587$
 - $P(X > 0) = 0.8413$
 - $P(X < -3.4) = 0.9079$
 - $P(X > -5.7) = 0.2533$
 - $P(X < 10.8) = 0.2097$
 - $P(X > -1.6) = 0.0323$
- If $X \sim N(\alpha, \alpha^2)$. Find
 - $P(X < 0) = 0.1587$
 - $P(X > 0) = 0.8413$
 - $P(X < 0.5\alpha) = 0.6915$
 - $P(X > 0.5\alpha) = 0.3085$
- If $X \sim N(100, 80)$. Find
 - $P(85 < X < 112) = 0.8634$
 - $P(105 < X < 115) = 0.2413$
 - $P(85 < X < 92) = 0.1388$
 - $P(|X| < \sqrt{80}) = 0.6826$

11. If $X \sim N(84, 12)$. Find
 - (i) $P(80 < X < 89) = 0.8014$
 - (ii) $P(X < 79 \text{ or } X > 92) = 0.085$
 - (iii) $P(76 < X < 82) = 0.2714$
 - (iv) $P(|X - 84| > 2.9) = 0.4028$
 - (v) $P(87 < X < 93) = 0.1886$
12. The masses of packages from a particular machine are normally distributed with a mean of 200g and standard deviation of 2g, find the probability that a randomly selected package from the machine weighs
 - (i) less than 197g = 0.0668
 - (ii) more than 200.5g = 0.4013
 - (iii) between 198.5g and 199.5g = 0.1747
13. The heights of boys at a certain school follow a normal distribution with mean = 150.3cm and variance 25cm, find the probability that a boy picked at random from the group has a height;
 - (i) less than 153cm = 0.7054
 - (ii) more than 158cm = 0.018
 - (iii) between 150 cm and 158 cm = 0.4621
 - (iv) more than 10cm difference from the mean height = 0.0046
14. The masses of a certain type of cabbages are normally distributed with mean of 1000g and standard deviation of 0.15kg. In a batch of 800 cabbages, estimate how many have a mass between 750g and 1290g = 740
15. Cartons of milk from quality super market are advertised as containing 1 litre, but in fact the volume of the content is normally distributed with a mean of 1012ml and standard deviation of 15ml.
 - (i) Find the probability that a randomly chosen carton contains more than 1010ml = 0.6554
 - (ii) In a batch of 1000 cartons, estimate the number of cartons containing less than the advertised volume of milk = 8
16. A random variable X is such that $X \sim N(-5, 9)$. Find the probability that;
 - (i) A randomly chosen item from the population will have positive value = 0.0478
 - (ii) Out of 10 items chosen randomly, exactly 4 will have a positive value = 0.00082
17. The life of a laptop is normally distributed with a mean of 2000 hours and standard deviation of 120 hours. Estimate the probability that the life of such a laptop will be
 - (i) greater than 2150 hours = 0.1056
 - (ii) greater than 1910hours = 0.7734
 - (iii) within a range 1850 hours to 2090 hours = 0.6678

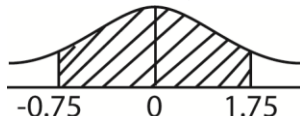
Solutions to revision questions 3

1. The amount of meat sold by a butcher is normally distributed with mean 43kg and standard deviation 4kg. Determine the probability that the amount of meat sold is between 40kg and 50kg.

$$X \sim N(43, 4)$$

$$P(40 < x < 50) = P\left(\frac{40-43}{4} < Z < \frac{50-43}{4}\right)$$

$$= P(-0.75 < Z < 1.75)$$



$$P(40 < x < 50) = P(-0.75 < Z < 0) + P(0 < Z < 1.75)$$

By property of symmetry

$$\begin{aligned} P(40 < x < 50) &= P(-0.75 < Z < 0) + P(0 < Z < 1.75) \\ &= 0.2735 + 0.4599 \\ &= 0.733 \end{aligned}$$

2. Given that a random variable X is $X \sim N(2, 2.89)$. Find $P(X < 0)$

$$\mu = 2, \sigma = \sqrt{2.89} = 1.7$$

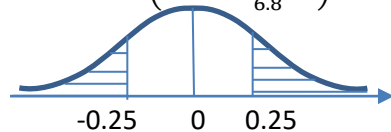
$$\begin{aligned} P(X < 0) &= P\left(Z < \frac{0-2}{1.7}\right) = P(Z < -1.176) = P(X > 1.176) \\ &= 0.5 - P(0 < Z < 1.176) \\ &= 0.5 - 0.3802 = 0.1198 \end{aligned}$$

3. In a school of 800 students their average weight is 54.5kg and standard deviation 6.8kg. given that the weight of students are normally distributed, find

- (i) Probability that the weight of any student randomly selected is 52.8 kg or less

Let x be the weight of the student

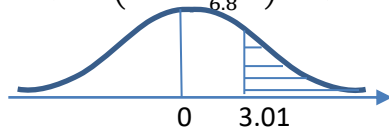
$$P(x \leq 52.8) = P\left(Z < \frac{52.8-54.5}{6.8}\right) = P(Z < -0.25)$$



$$= P(Z > 0.25) = 0.5 - P(0 < Z < 0.25) = 0.5 - 0.0987 = 0.4013$$

- (ii) Number of students who weigh over 75kg = 1

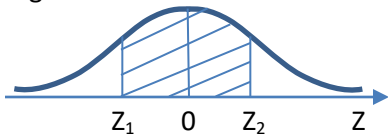
$$P(Z > 75) = P\left(Z > \frac{75-54.5}{6.8}\right) = P(Z > 3.01)$$



$$P(Z > 3.01) = 0.5 - P(0 < Z < 3.01) = 0.5 - 0.4990 = 0.001$$

Number of students who weigh more than 75g = $800 \times 0.001 = 1$

- (iii) Weight of the middle 56% of the students



$$P(X_1 < X < X_2) = P(Z_1 < Z < Z_2) = 0.56$$

But $P(0 < Z < Z_2) = 2.8$; $Z_2 = 0.772$ and $Z_1 = -0.772$

$$\begin{aligned} Z_1 &= \frac{x_1 - 54.5}{6.8} \\ -0.772 &= \frac{x_1 - 54.5}{6.8}; x_1 = 49.251 \\ Z_2 &= \frac{x_2 - 54.5}{6.8} \\ 0.772 &= \frac{x_2 - 54.5}{6.8}; x_2 = 59.750 \end{aligned}$$

Hence the weight range of the middle 56% of students of the school is $49.251 < X < 59.750$

4. A sugar factory sells sugar in bags of mean weight 50kg and standard deviation 2.5kg. given that the weight of the bags is normally distributed, find the

- (i) Probability that the weight of any bag of sugar randomly selected lies between 51.5kg and 53kg

$$P(51.5 < X < 53) = \frac{51.5 - 50}{2.5} < Z < \frac{53 - 50}{2.5} = P(0.6 < Z < 1.2)$$

$$= \phi(1.2) - \phi(0.6) = 0.3849 - 0.2257 = 0.1592$$

- (ii) Percentage of bags whose weight exceeds 54kg

$$P(X > 54) = P\left(Z > \frac{54-50}{2.5}\right) = P(Z > 1.6) = 0.5 - \phi(1.6) = 0.5 - 0.4452 = 0.0548$$

$$= 0.0548 \times 100 = 5.48\%$$

- (iii) Number of bags that will be rejected out of 1000 bags purchased for weighing below 45.0kg

$$P(X < 45) = P\left(Z < \frac{45-50}{2.5}\right) = P(Z < -2) = P(Z < 2) = 0.5 - \phi(2) = 0.5 - 0.4772 = 0.0228$$

$$\text{Number of bags rejected} = 0.0228 \times 1000 = 22.8 \approx 23$$

5. A certain maize firm sells maize in bags of mean weight 40kg and standard deviation 2kg, given that the weight of the bags are normally distributed, find

- (i) Probability that the weight of any bag of maize randomly selected lies between 41.0 and 42.5kg

$$P(41.0 < X < 42.5) = P\left(\frac{41-40}{2} < Z < \frac{42.5-40}{2}\right) = P(0.5 < Z < 1.25)$$

$$= \phi(1.25) - \phi(0.5) = 0.3944 - 0.1915 = 0.2029$$

- (ii) Percentage of bags whose weight exceeds 43kg

$$P(X > 43) = P\left(Z > \frac{43-40}{2}\right) = P(Z > 1.5) = 0.5 - \phi(1.5) = 0.5 - 0.4332 = 0.0668$$

$$= 0.0668 \times 100 = 6.68\%$$

- (iii) Number of bags that will be rejected out of 500 bags purchased for weighing below 38.5kg

$$P(X < 38.5) = P\left(Z < \frac{38.5-40}{2}\right) = P(Z < -0.77) = P(Z < 0.75) = 0.5 - \phi(0.75)$$

$$= 0.5 - 0.2734 = 0.2266$$

$$\text{Number of bags rejected} = 0.2266 \times 500 = 113$$

7. If $X \sim N(50, 20)$. Find

- (i) $P(X > 60.3)$

$$P(X > 60.3) = P\left(Z > \frac{60.3-50}{\sqrt{20}}\right) = P(Z > 2.303) = 0.5 - \phi(2.303)$$

$$= 0.5 - (0.4893 + 0.0001) = 0.0106$$

- (ii) $P(X < 47.3)$

$$P(X < 47.3) = P\left(Z < \frac{47.3-50}{\sqrt{20}}\right) = P(Z < -0.6037) = P(Z > 0.6037) = 0.5 - \phi(0.6037)$$

$$= 0.5 - (0.2257 + 0.0013) = 0.273$$

- (iii) $P(X > 48.9)$

$$P(X > 48.9) = P\left(Z > \frac{48.9-50}{\sqrt{20}}\right) = P(Z > -0.246) = P(Z < 0.246) = 0.5 + \phi(0.246)$$

$$= 0.5 + 0.0948 + 0.0022 = 0.597$$

- (iv) $P(X > 53.5)$

$$P(X > 53.5) = P\left(Z > \frac{53.5-50}{\sqrt{20}}\right) = P(Z > 0.783) = 0.2823 + 0.0008 = 0.2831$$

- (v) $P(X < 59.8)$

$$P(X < 59.8) = P\left(Z < \frac{59.8-50}{\sqrt{20}}\right) = P(Z < 2.191) = 0.5 + \phi(2.191)$$

$$= 0.5 + 0.4826 + 0.0001 = 0.9857$$

- (vi) $P(X < 62.3)$

$$P(X < 62.3) = P\left(Z < \frac{62.3-50}{\sqrt{20}}\right) = P(Z < 2.750) = 0.5 + \phi(2.730) \\ = 0.5 + 0.4970 = 0.9970$$

How to obtain Z-values from a given probabilities

If you are interested in finding the Z-values whose probabilities are given, it is important to note that the Z-value may be positive or negative.

Sign	Probability	Z-value
<	< 0.5	-
>	>0.5	-
<	>0.5	+
>	<0.5	+

Note: for the above table the probability given in the question always correspond to Q in the critical table

Example 31

$$P(Z < Z_1) = 0.5, \text{ find } Z_1$$

$$P(Z < Z_1) = 0.5, \text{ find } Z_1$$

Solution

$$P(Z < Z_1) = 0.5(Q)$$

$$P(Z < Z_1) = 0.5, \text{ find } Z_1$$

$Z_1 = -0.674$ (negative since $0.25 < 0.5$ read directly from a critical table)

Example 32

$$P(Z < Z_1) = 0.0968, \text{ find } Z_1$$

Since 0.0968(Q) is not on critical table

$$P(Z < Z_1) = 0.5 - 0.0968 = 0.403(P)$$

$Z_1 = -1.3$ (negative since $0.0968 < 0.5$ read directly from a critical table)

Example 33

$$P(Z < Z_1) = 0.5, \text{ find } Z_1$$

Solution

$$P(Z < Z_1) = 0.05(Q)$$

$Z_1 = -1.645$ (negative since $0.05 < 0.5$ read directly from a critical table)

Example 34

$$P(Z < a) = 0.787, \text{ find } a$$

Since 0.787 is not on critical table

$$P(Z < a) = 0.787 - 0.5 = 0.287$$

and 0.2881. Since the extra information to the right hand side is **add**, we consider the smallest value i.e. 0.2852 but 2852 corresponds to 0.79

to get the next

$$0.2870 - 0.285 = 0.0018$$

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From the table 0.287 lies between 0.2852

So we look for 0.0018 on the add column which gives 6

$$\therefore a = 0.79 + 0.006 = 0.796$$

Example 35

$P(Z > b) = 0.01$, find b

$P(Z > b) = 0.01(Q)$

$b = 2.326$ read directly from critical table

Inverse process (De-standardizing Z)

It involves converting the Z-value to raw data (X) form

Example 36

If $X \sim (100, 36)$ and $P(X > \alpha) = 0.8907$, find the value of α

Since 0.8907 is not critical on a critical table

$$P\left(Z < \frac{\alpha - 100}{6}\right) = 0.8907 - 0.5 = 0.3907(P)$$

From the table $Z = 1.23$

$$1.23 = \frac{\alpha - 100}{6}$$

$$\alpha = 100 + 1.23 \times 6 = 107.38$$

Example 37

If $X \sim (24, 9)$ and $P(X > b) = 0.974$, find the value of b

$$P\left(Z < \frac{b - 24}{3}\right) = 0.974 - 0.5 = 0.474(P)$$

From the table $Z = -1.943$

$$-1.943 = \frac{b - 24}{3}$$

$$\alpha = 24 - 1.943 \times 3 = 18.171$$

Example 38

The height of flowers in a farm is normally distributed with the mean 169 cm and standard deviation 9cm. if X stands for the height of flowers in cm, find X values for

(a) $P(X < a) = 0.8$

Solution

$$P(X < a) = 0.8(Q)$$

$$P\left(Z < \frac{a-169}{9}\right) = 0.8 - 0.5 = 0.3(P)$$

From the table $Z = 0.842$

$$a = 0.842 \times 9 + 169 = 176.38$$

(b) $P(X > b) = 0.6$

Solution

$$P(X > b) = 0.6 (Q)$$

$$P\left(Z < \frac{a-169}{9}\right) = 0.6 - 0.5 = 0.1(P)$$

From the table $Z = -0.253$

$$b = -0.253 \times 9 + 169 = 166.72$$

Example 39

The period of a certain machine approximately follows a normal distribution with mean of five years and standard deviation of 1 year. Given that the manufacturer of this machine replaces the machine that fails under guarantee, determine the

- (i) Length of the guarantee required so that not more than 2% of the machine that fail are replaced.

$$P(X < X_0) = 0.02(Q)$$

$$P\left(Z < \frac{X_0-5}{1}\right) = 0.02(Q)$$

From the table $Z = -2.054$

$$X_0 = -2.054 \times 1 + 5 = 2.946$$

∴ the guarantee period is 2.946 years

- (ii) The proportion of the machines that would be replaced if the guarantee period was four years

$$P(X < 4) = P\left(Z < \frac{4-5}{1}\right) = P(Z < -1) = P(Z > 1) = 5 - \phi(1) = 5 - 0.3413 = 0.1587$$

$$P(Z < 4) = 0.1587 \times 100 = 15.87\%$$

Example 40

The marks of 500 students in a mock examination for which the pass mark was 50%. Their marks are normally distributed with mean 45 marks and standard deviation 20 marks.

- (a) Given that the pass mark is 41, estimate the number of candidates who passed the examination.

$$P(X \geq 41) = P\left(Z < \frac{41-45}{20}\right) = P(Z \geq -0.2) = P(Z \leq 0.2) = 0.5 + \phi(0.2) = 0.5 + 0.0793 = 0.5793$$

$$\text{Number of candidates who passed} = 0.5793 \times 500 = 290$$

- (b) If 5% of the candidates obtain a distinction by scoring X marks or more, estimate the value of X.

$$P(X > X_0) = 0.05(Q)$$

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$$P\left(Z < \frac{X_0 - 45}{20}\right) = 0.5 - 0.05 = 0.45(P); \text{ from the table } Z = 1.645$$

$$X_0 = 1.645 \times 20 + 45 = 78$$

∴ the distinction starts at 78%

(c) Estimate the interquartile range of the distribution

$$\text{Interquartile range} = q_3 - q_1$$

$$P\left(0 < Z < \frac{q_3 - 45}{20}\right) = 0.25(P); \text{ from the table } Z = 0.674$$

$$q_3 = 0.674 \times 20 + 45 = 58.48$$

$$P\left(\frac{q_1 - 45}{20} < Z < 0\right) = 0.25(P); \text{ from the table } Z = -0.674$$

$$q_1 = -0.674 \times 20 + 45 = 31.52$$

$$\therefore \text{interquartile range} = 58.48 - 31.52 = 26.96$$

Example 41

If $X \sim N(70, 25)$ and $P(|X - 70| < a) = 0.8$, find the value of a and hence the limits within which the central 80% of the distribution lies.

$$P(|X - 70| < a) = P(-a < X - 70 < a) = P(-a + 70 < X < a + 70) = 0.8$$

$$P\left(\frac{-a + 70 - 70}{5} < X < \frac{a + 70 - 70}{5}\right) = P\left(\frac{-a}{5} < X < \frac{a}{5}\right) = 0.8$$

$$2 \times P\left(0 < X < \frac{a}{5}\right) = 0.8; P\left(0 < X < \frac{a}{5}\right) = 0.4(P)$$

From table $Z = 1.282$

$$1.282 = \frac{a}{5}; a = 6.41$$

$$\text{But } P(-a + 70 < X < a + 70) = 0.8$$

$$P(63.59 < X < 76.41) = 0.8$$

∴ Central 80% of the distribution lies between 63.59 and 76.41

Revision Exercise 4

1. Find the value of the following

$$(i) P(Z < a) = 0.506 [a = 0.015] \quad (ii) P(Z < a) = 0.787 [a = 0.796] \quad (iii) P(Z < a) = 0.0296 [a = -0.1887]$$

$$(iv) P(Z > a) = 0.713 [a = -0.562] \quad (v) P(Z < a) = 0.325 [a = -0.454] \quad (vi) P(|Z| > a) = 0.5 [a = 0.674]$$

$$(vii) P(|Z| > a) = 0.6 [a = 0.842] \quad (viii) P(Z < a) = 0.9738 [a = 1.94] \quad (ix) P(Z < a) = 0.2435 [a = -0.695]$$

$$(x) P(Z > a) = 0.82 [a = -0.915] \quad (xi) P(Z > a) = 0.2351 [a = 0.628] \quad (xii) P(|Z| < a) = 0.6372 [a = 0.91]$$

$$(xiii) P(Z > a) = 0.097 [a = 1.66] \quad (xiv) P(|Z| > a) = 0.0404 [a = 2.05]$$

2. Find the value of a if

$$(i) P(Z < a) = 0.9693 [a = 1.87] \quad (ii) P(Z > a) = 0.3802 [a = 0.305] \quad (iii) P(Z > a) = 0.7367 [a = -0.633]$$

$$(iv) P(Z < a) = 0.0793 [a = -1.41] \quad (v) P(|Z| < a) = 0.9 [a = 1.645]$$

3. If $X \sim N(60, 25)$ find a if

$$(i) P(X > a) = 0.2324 [a = 63.66] \quad (ii) P(X > a) = 0.0702 [a = 67.37]$$

$$(iii) P(X > a) = 0.837 [a = 55.09] \quad (iv) P(X > a) = 0.7461 [a = 56.69]$$

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4. If $X \sim N(45, 16)$ find a if
- (i) $P(X < a) = 0.0317$ [$a = 37.57$] (ii) $P(X < a) = 0.895$ [$a = 50.01$]
 (iii) $P(X < a) = 0.0456$ [$a = 38.24$] (iv) $P(X < a) = 0.996$ [$a = 55.6$]
5. If $X \sim N(400, 64)$ find a if
- (i) $P(|X-400| < a) = 0.75$ [9.2] (ii) $P(|X-400| < a) = 0.98$ [18.61]
 (iii) $P(|X-400| < a) = 0.95$ [15.68] (iv) $P(|X-400| < a) = 0.975$ [17.92]
 (v) The limits within which the central 95% of distribution lies. [384.32 < X < 415.68]
 (vi) Interquartile range of distribution [394.61, 405.39]
6. Bags of flour paced by a particular machine have masses which are normally distributed with mean 500g and standard deviation 20g. 2% of the bags are rejected for being overweight. Between what ranges of values should the mass of a bag of flour lie if it is to be accepted. [0458.92, 546.52]
7. The masses of mangoes sold at a market are normally distributed with mean mass 600g and standard deviation 20g.
- (i) If a mango is chosen at random, find the probability that its mass lies between 570g and 610g [0.6247]
 (ii) Find the mass exceeded by 7% of mangoes [629.52]
 (iii) In one day 1000 mangoes are sold. Estimate how many weigh less than 545g [3]
8. The length of metal strips are normally distributed with mean of 120cm and standard deviation of 10cm.
- (a) Find the probability that a strip selected at random has a length
 (i) greater than 105cm [0.9332] (ii) within 5cm of the mean =[0.383]
 (b) Strips that are shorter than L cm are rejected. Estimate the value of L, if 9% or all the strips are rejected. [106.6cm]
 (c) In a sample of 500 strips, estimate the number having a length over 126cm. [137]
9. The number of shirts sold in a week by a boutique are normally distributed with a mean 2080 and standard deviation of 50. Estimate
- (i) The probability that in a given week fewer than 2000 shirts are sold [0.0548]
 (ii) The number of weeks in a year that between 2060 and 2130 shirts are sold [26]
 (iii) The least number n of shirts such that the probability that more than n are sold in a given week is less than 0.02 [2183]
10. Batteries for a transistor radio have a mean life under normal usage of 160 hours, with standard deviation of 30 hours. Assuming the battery life follow normal distribution
- (i) Find the percentage of batteries which have a life between 150 hours and 180 hours. [37.8%]
 (ii) Calculate the range, symmetrical about the mean, within which 75% of the batteries lives lie. [125.5, 194.5]
 (iii) If the radio takes four of these batteries and require all of them to be working, find the probability that the radio will run for at least 135 hours. [0.405]
11. The length of type A rod is normally distributed with mean of 15cm and a standard deviation of 0.1cm. the length of another type B is also normally distributed with mean of 20cm and standard deviation 0.16cm. For type A rod to be acceptable, its length must be between 14.8cm and 15.2 cm and type B rod, the length must be between 19.8cm and 20.2cm.
- (i) What is the proportion of type a rod is of acceptable length? [95.44%]
 (ii) What is the probability that one of them is of acceptable length [0.7528, 0.2375]

12. The marks of 1000 students in an examination were normally distributed with mean 55 marks and standard deviation 8 marks.
- (i) If a mark of 71 or more is required for A-pass, estimate the number of a-passes awarded. [23]
 - (ii) If 15% of the candidates failed, estimate the minimum mark required to pass. [47]
 - (iii) Calculate the probability that two candidates chosen at random both passes examination [0.7225]
13. The burning life of a bulb approximately follows a normal distribution with mean of 1300hours and standard deviation of 125 hours
- (i) What is the probability that the bulb selected at random will burn for more than 1500 hours. [0.0548]
 - (ii) Given that the manufacturer guarantees to replace any bulb that burns for less than 1050hours, what percentage of the bulbs will have to be replaced. [2.28%]
 - (iii) If two bulbs are installed at the same time, what is the probability that bot will burn less than 1400 hours but more than 1200 hours [0.3320]
14. The marks in an examination were found to be normally distributed with mean 53.9 and standard deviation 16.5. 20% od the candidates who sat this examination failed. Find the pass mark [40.007]

Finding the value of mean, μ or standard deviation, σ or both for normal distribution

Hint: $X = Z\sigma + \mu$

Example 42

If $X \sim N(100, \sigma^2)$ and $P(X < 106) = 0.8849$, find the value of standard deviation, σ .

Solution

$$\begin{array}{l}
 P(X < 106) = 0.8849 \text{ (Q-value)} \\
 P\left(Z < \frac{106-100}{\sigma}\right) = 0.8849 - 0.5 = 0.3849 \text{ (P-value)} \\
 \text{From table } Z = 1.2
 \end{array}
 \quad \left| \begin{array}{l}
 \frac{106-100}{\sigma} = 1.2 \\
 \sigma = 5
 \end{array} \right.$$

Example 43

The length of a certain item follows a normal distribution with mean, μ cm and standard deviation of 6cm. it is known that 4.78% of the items have length greater than 82cm, find the mean, μ .

Solution

$$\begin{array}{l}
 P(X > 82) = 0.0478 \text{ (Q-value)} \\
 P\left(Z > \frac{82-\mu}{6}\right) = 0.5 - 0.0478 = 0.4522 \text{ (P-value)} \\
 \text{From table } Z = 1.667
 \end{array}
 \quad \left| \begin{array}{l}
 \frac{82-\mu}{6} = 1.667 \\
 \mu = 72 \text{ cm}
 \end{array} \right.$$

Example 44

The masses of boxes of oranges are normally distributed such that 30% of them are greater than 4.00kg and 20% are greater than 4.53kg. Estimate the mean and standard deviation of the masses

Solution

$$\begin{array}{l}
 P(X > 4) = 0.3 \text{ (Q-value)} \\
 P\left(Z > \frac{4-\mu}{\sigma}\right) = 0.3 \text{ (Q-value)} \\
 \frac{4-\mu}{\sigma} = 0.524 \\
 4 = \mu + 0.524\sigma \text{(i)} \\
 P\left(Z > \frac{4.53-\mu}{\sigma}\right) = 0.2 \text{ (Q-value)}
 \end{array}
 \quad \left| \begin{array}{l}
 \frac{4.53-\mu}{\sigma} = 0.842 \\
 4.53 = \mu + 0.842\sigma \text{(ii)} \\
 \text{Eqn. (ii) - eqn. (i): } 0.53 = 0.318\sigma; \sigma = 1.67\text{kg} \\
 \text{From eqn. (i)} \\
 \mu = 4 - 0.524 \times 1.67 = 3.13\text{kg}
 \end{array} \right.$$

Example 45

The speed of cars passing certain Entebbe high way can be taken to be normally distributed. 95% of the cars are travelling at less than 85m/s and 10% are travelling at less than 55m/s.

- (i) Find the average speed of the cars passing through the highway
- (ii) Find the proportion of the cars that travel at more than 70m/s

Solution

$$P(X < 85) = 0.95 \text{ (Q - value)}$$

$$P\left(Z < \frac{85-\mu}{\sigma}\right) = 0.95 \text{ (Q - value)}$$

$$\frac{85-\mu}{\sigma} = 1.645$$

$$85 = \mu + 1.645\sigma \text{(i)}$$

$$P(X < 55) = 0.1 \text{ (Q - value)}$$

$$P\left(Z < \frac{55-\mu}{\sigma}\right) = 0.1 \text{ (Q - value)}$$

$$\frac{55-\mu}{\sigma} = -1.282$$

$$55 = \mu - 1.282\sigma \text{ (ii)}$$

$$\text{Eqn. (i) - eqn. (ii): } 30 = 2.927\sigma; \sigma = 10.25\text{m/s}$$

From eqn. (i)

$$\mu = 85 - 1.645 \times 10.25 = 68.14\text{m/s}$$

$$\begin{aligned} \text{(ii) } P(X > 70) &= P\left(Z < \frac{70-68.14}{10.25}\right) = P(Z > 0.182) \\ &= 0.5 - 0.0722 \\ &= 0.4278 \end{aligned}$$

Example 46

The masses of articles produced in a particular shop are normally distributed with mean μ and standard deviation σ . 5% of the articles have greater than 85g and 10% have masses less than 25g.

- (i) Find the values of μ and σ
- (ii) Find the symmetrical limits, about the mean, within which 75% of the masses lie.

Solution

$$P(X > 85) = 0.05 \text{ (Q - value)}$$

$$P\left(Z > \frac{85-\mu}{\sigma}\right) = 0.05 \text{ (Q - value)}$$

$$\frac{85-\mu}{\sigma} = 1.645$$

$$85 = \mu + 1.645\sigma \text{(i)}$$

$$P(X < 25) = 0.1 \text{ (Q - value)}$$

$$P\left(Z < \frac{25-\mu}{\sigma}\right) = 0.1 \text{ (Q - value)}$$

$$\frac{25-\mu}{\sigma} = -1.282$$

$$25 = \mu - 1.282\sigma \text{ (ii)}$$

$$\text{Eqn. (i) - eqn. (ii): } 60 = 2.927\sigma; \sigma = 20.5\text{m/s}$$

From eqn. (i)

$$\mu = 85 - 1.645 \times 20.5 = 51.3\text{g}$$

$$\text{(ii) } P(|X-51.3| < a) = 0.75$$

$$P(-a + 51.3 < X < a + 51.3) = 0.75$$

$$P\left(\frac{-a + 51.3 - 51.3}{20.5} < Z < \frac{a + 51.3 - 51.3}{20.5}\right) = 0.75$$

$$P\left(\frac{-a}{20.5} < Z < \frac{a}{20.5}\right) = 0.75$$

$$2 \times P\left(0 < Z < \frac{a}{20.5}\right) = 0.375 \text{ (P - value)}$$

$$\frac{a}{20.5} = 1.15; a = 23.575$$

$$\text{Lower limit} = -23.575 + 51.3 = 27.73$$

$$\text{Upper limit} = 23.575 + 51.3 = 74.88$$

Example 47

A total population of 700 students sat a mock examination for which the pass mark was 50%. Their marks were normally distributed. 28 students scored below 40% while 35 students scored above 60%.

- (a) Find the mean mark and standard deviation of the students' marks.
- (b) What is the probability that a student chosen at random passed the exam?
- (c) Suppose the pass mark is lowered by 2%, how many more students will pass.

Solution

$$P(X < 40) = \frac{28}{700} = 0.04 \text{ (Q - value)}$$

$$P\left(Z > \frac{40 - \mu}{\sigma}\right) = 0.04 \text{ (Q - value)}$$

$$\frac{40 - \mu}{\sigma} = -1.751$$

$$40 = \mu - 1.751\sigma \text{(i)}$$

$$P(X > 60) = \frac{35}{700} = 0.05 \text{ (Q - value)}$$

$$P\left(Z > \frac{60 - \mu}{\sigma}\right) = 0.05 \text{ (Q - value)}$$

$$\frac{60 - \mu}{\sigma} = 1.645$$

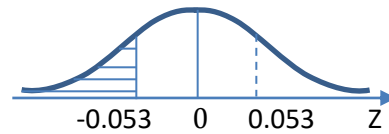
$$60 = \mu + 1.645\sigma \text{ (ii)}$$

$$\text{Eqn. (ii) - eqn. (i): } 20 = 3.396\sigma; \sigma = 5.889$$

From eqn. (i)

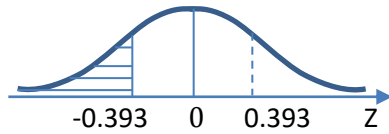
$$\mu = 40 + 1.751 \times 5.889 = 50.312$$

$$(ii) P(X \geq 50) = P\left(Z \geq \frac{50 - 50.312}{5.889}\right) = P(Z \geq -0.053)$$



$$P(Z \geq -0.053) = 0.5 + P(0 < Z < 0.053) \\ = 0.5 + 0.0211 = 0.5211$$

$$(iii) P(X \geq 48) = P\left(Z \geq \frac{48 - 50.312}{5.889}\right) = P(Z \geq -0.393)$$



$$P(Z \geq -0.392) = 0.5 + P(0 < Z < 0.392) \\ = 0.5 + 0.1528 = 0.6528$$

$$\text{More proportion} = 0.6528 - 0.5211 = 0.1317$$

$$\text{More students} = 0.1317 \times 700 = 92$$

Example 48

A random variable X has a normal distribution with $P(X > 55) = 0.2$ and $P(35 < X < 55) = 0.5$. Find

- (a) The value of the mean, μ and standard deviation, σ .
- (b) The percentage of those with $P(X > 45)$

Solution

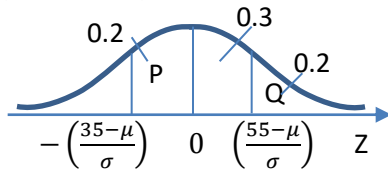
$$P(X > 55) = \frac{28}{700} = 0.2 \text{ (Q - value)}$$

$$P\left(Z > \frac{55 - \mu}{\sigma}\right) = 0.2 \text{ (Q - value)}$$

$$\frac{40 - \mu}{\sigma} = 0.842$$

$$55 = \mu + 0.842\sigma \text{(i)}$$

$$P(35 < X < 55) = 0.5$$



$$P\left(\frac{35 - \mu}{\sigma} < Z < 0\right) = 0.2 \text{ (P - value)}$$

$$\frac{35 - \mu}{\sigma} = -0.842$$

$$35 = \mu - 0.842\sigma \text{(ii)}$$

$$\mu = 70.8797, \sigma = 42.5872$$

$$(b) P(X > 45) = P\left(Z > \frac{45 - 70.8797}{42.5872}\right) = P(Z > -0.608)$$

$$= 0.5 + P(0 < Z < 0.608)$$

$$= 0.5 + 0.2283 = 0.7283$$

$$\text{Percentage} = 0.7283 \times 100 = 72.83$$

Revision exercise 4

- $X \sim N(45, \sigma^2)$ and $P(X > 51) = 0.288$. find σ . [$\sigma = 10.7$]
- $X \sim N(21, \sigma^2)$ and $P(X < 27) = 0.9332$. find σ . [$\sigma = 4$]
- $X \sim N(\mu, 25)$ and $P(X < 27.5) = 0.3085$. find μ . [$\mu = 30$]
- $X \sim N(\mu, 12)$ and $P(X > 32) = 0.8438$. find μ . [$\mu = 35.5$]
- $X \sim N(\mu, \sigma^2)$ and $P(X > 80) = 0.0113$ and $P(X > 30) = 0.9713$. find σ and μ . [$\mu = 52.73$ and $\sigma = 11.96$]
- $X \sim N(\mu, \sigma^2)$ and $P(X > 102) = 0.42$ and $P(X < 97) = 0.25$. find σ and μ . [$\mu = 100.8$ and $\sigma = 5.71$]
- $X \sim N(\mu, \sigma^2)$ and $P(X < 57.84) = 0.90$ and $P(X < 50) = 0.5$. find σ and μ . [$\mu = 50$ and $\sigma = 6.12$]
- $X \sim N(\mu, \sigma^2)$ and $P(X < 35) = 0.20$ and $P(35 < X < 45) = 0.65$. find σ and μ . [$\mu = 39.5$ and $\sigma = 5.32$]
- The length of rods produced in a workshop follow a normal distribution with mean μ and variance 4. 10% of the rods are less than 17.4cm long. Find the probability that a rod chosen at random will be between 18cm and 23 cm. [0.7725]
- The length of a stick follow a normal distribution. 10% are of length 250cm or more while 55% have a length over 240cm. Find the probability that a stick chosen at random is less than 235cm long. [0.203]
- A certain make of car tyres can be safely used for 25000km on average before replaced. The makers guarantee to pay compensation to anyone whose tyre does not last for 22000km. they expect 7.5% of all the tyres sold to qualify for compensation. If the distance X travelled before a tyre is replaced has normal distribution.
 - Find the standard deviation [2080]
 - Estimate the number of tyres per 1000 which will not have been replaced when they have covered 26500km. [236]
- The continuous random variable X is normally distributed with mean μ and standard deviation σ . If $P(X < 53) = 0.04$ and $P(X < 65) = 0.97$, find the interquartile range [4.46]
- Tea sold in packages marked 750g. The masses are normally distributed with mean 760g and standard deviation σ . What is the maximum value of σ , if less than 1% of the packages are underweight? [4.299]
- In an examination 30% of the candidates fail and 10% achieve distinction. Last year the pass mark (out of 200) was 84 and the minimum mark required for a distinction was 154. Assuming that the marks of

candidates are normally distributed, estimate the mean mark and standard deviation. [$\mu = 104.31$, $\sigma = 38.76$]

15. At St Noa junior, the heights of students are normally distributed. 10% are over 1.8m and 20% are below 1.6m.

(i) Find the mean height μ and standard deviation σ . [$\mu = 1.68$, $\sigma = 0.09$]

(ii) Find the interquartile range [0.13]

16. Observation of a very large number of cars at certain point on a motorway established that the speeds are normally distributed. 90% of the cars have speed less than 77.7km/h and only 5% of cars have speed less than 63.1km/h. Find the mean speed μ and standard deviation σ . [$\mu = 71.305$, $\sigma = 4.988$]

17. A sample of 100 apples is taken from a load. The apples have the following distribution of size.

Diameter (cm)	6	7	8	9	10
Frequency	11	21	38	17	13

Assuming that the distribution is approximately normal with mean μ and standard deviation σ .

(i) Determine μ and σ [$\mu = 8$, $\sigma = 1.16$]

(ii) Find the range of sizes of apples for packing, if 5% are to be rejected as too small and 5% are to be rejected as too large [6.10, 9.90]

18. The volumes of soda in bottles are normally distributed with mean of 333ml. Given that 20% of the bottles contain more than 340ml, find

(i) Standard deviation of the volume of bottle. [8.31]

(ii) Percentage of bottles that contain less than 330ml. [35.9%]

19. The heights of 500 students are normally distributed with a standard deviation of 0.080cm. If the heights of 129 of the students are greater than the mean height but less than 1.806m find the mean height. [1.75]

20. The masses of boxes of apples are normally distributed such that 20% of them are greater than 5.08kg and 15% are greater than 5.62kg; find the mean and standard deviation. [$\mu = 2.74$, $\sigma = 2.78$]

21. The masses of sugar are normally distributed. If 5% of the packets have mass greater than 510g and 2% have masses greater than 515g. Find the mean and standard deviation. [$\mu = 490$, $\sigma = 12.2$ g]

22. Sugar packed in 500g packets is observed to be approximately normally distributed with standard deviation of 4. If only 2% of the packets contained less than 500g of sugar. Find the mean weight of sugar in the packets. [508.216g]

23. Sixty students sat for a mathematics contest whose pass mark was 40marks. Their scores in the contest were approximately normally distributed. 9 students scored less than 20 marks while 3 scored more than 70 marks. Find the

(i) Mean score and the standard deviation of the contest. [$\mu = 39.32$, $\sigma = 18.65$]

(ii) Find the probability that a student chosen at random passed the contest. [0.4856]

24. The number of cows owned by residents in a village is assumed to be normally distributed. 15% of the residents have less than 60 cows while 5% of residents have over 90 cows.

(a) Determine the values of the mean and standard deviations of cows [$\mu = 71.5926$, $\sigma = 11.1899$]

(b) If there are 200 residents, find how many have more than 80 cows. [45]

25. A random variable X has a normal distribution when $P(X > 9) = 0.9192$ and $P(X < 11) = 0.7580$. Find

(a) the value of the mean and standard deviation. [$\mu = 10.3333$, $\sigma = 0.9524$]

(b) $P(X > 10)$ [0.6386]

26. The marks in an examination were normally distributed with mean μ and standard deviation σ . 20% of the candidates scored less than 40marks and 10% more than 75 marks. Find the
- values of μ and σ . (08marks)
 - percentage of the candidate who scored more than 50 marks.(04marks)

Solutions to revision exercise 4

20. The masses of boxes of apples are normally distributed such that 20% of them are greater than 5.08kg and 15% are greater than 5.62kg; find the mean and standard deviation.

Solution

$$P(X > 5.08) = 0.2 \text{ (Q - value)}$$

$$P\left(Z > \frac{5.08 - \mu}{\sigma}\right) = 0.2 \text{ (Q - value)}$$

$$\frac{5.08 - \mu}{\sigma} = 0.842$$

$$5.08 = \mu + 0.842\sigma \text{(i)}$$

$$P(X > 5.62) = 0.15 \text{ (Q - value)}$$

$$P\left(Z > \frac{5.62 - \mu}{\sigma}\right) = 0.15 \text{ (Q - value)}$$

$$\frac{5.62 - \mu}{\sigma} = 1.036$$

$$5.62 = \mu + 1.036\sigma \text{(ii)}$$

Eqn. (ii) – eqn. (i)

$$0.54 = 0.194\sigma; \sigma = 2.7835\text{kg}$$

From eqn. (i)

$$\mu = 5.08 - 0.842 \times 2.7835 = 2.7363\text{kg}$$

21. The masses of sugar are normally distributed. If 5% of the packets have mass greater than 510g and 2% have masses greater than 515g. Find the mean and standard deviation.

Solution

$$P(X > 510) = 0.05 \text{ (Q - value)}$$

$$P\left(Z > \frac{510 - \mu}{\sigma}\right) = 0.05 \text{ (Q - value)}$$

$$\frac{510 - \mu}{\sigma} = 1.645$$

$$510 = \mu + 1.645\sigma \text{(i)}$$

$$P(X > 515) = 0.02 \text{ (Q - value)}$$

$$P\left(Z > \frac{515 - \mu}{\sigma}\right) = 0.02 \text{ (Q - value)}$$

$$\frac{515 - \mu}{\sigma} = 2.054$$

$$515 = \mu + 2.054\sigma \text{(ii)}$$

Eqn. (ii) – eqn. (i)

$$5 = 0.409\sigma; \sigma = 12.225\text{kg}$$

From eqn. (i)

$$\mu = 510 - 1.645 \times 12.225 = 489.89\text{kg}$$

22. Sugar packed in 500g packets is observed to be approximately normally distributed with standard deviation of 4. If only 2% of the packets contained less than 500g of sugar. Find the mean weight of sugar in the packets.

$$P(X < 500) = 0.02 \text{ (Q - value)}$$

$$P\left(Z < \frac{500 - \mu}{\sigma}\right) = 0.02 \text{ (Q - value)}$$

$$\frac{500 - \mu}{\sigma} = -2.054$$

$$\text{Mean weight, } \mu = 500 + 2.054 \times 4 = 508.216\text{g}$$

23. Sixty students sat for a mathematics contest whose pass mark was 40marks. Their scores in the contest were approximately normally distributed. 9 students scored less than 20 marks while 3 scored more than 70 marks. Find the

(i) Mean scored and the standard deviation of the contest.

Solution

$$P(X < 20) = \frac{9}{60} = 0.15 \text{ (Q - value)}$$

$$P\left(Z > \frac{20 - \mu}{\sigma}\right) = 0.15 \text{ (Q - value)}$$

$$\frac{20 - \mu}{\sigma} = -1.036$$

$$20 = \mu - 1.036\sigma \text{(i)}$$

$$P(X > 70) = \frac{3}{60} = 0.05 \text{ (Q - value)}$$

$$P\left(Z > \frac{70 - \mu}{\sigma}\right) = 0.05 \text{ (Q - value)}$$

$$\frac{70 - \mu}{\sigma} = 1.645$$

$$70 = \mu + 1.645\sigma \text{(ii)}$$

Eqn. (ii) – eqn. (i)

$$50 = 2.681; \sigma = 18.65$$

From eqn. (i)

$$\mu = 20 + 1.036 \times 18.65 = 39.3214$$

(ii) Find the probability that a student chosen at random passed the contest.

$$P(X > 40) = P\left(Z > \frac{40 - 39.3214}{18.65}\right) = P(Z > 0.0364) = 0.5 - \phi(0.036) = 0.5 - 0.0144 = 0.4856$$

24. The number of cows owned by residents in a village is assumed to be normally distributed. 15% of the residents have less than 60 cows while 5% of residents have over 90 cows.

(a) Determine the values of the mean and standard deviations of cows

(b) If there are 200 residents, find how many have more than 80 cows. [45]

Solution

(i) $P(X < 60) = 0.15$ (Q – value)

$$P\left(Z < \frac{60 - \mu}{\sigma}\right) = 0.15 \text{ (Q – value)}$$

$$\frac{20 - \mu}{\sigma} = -1.036$$

$$60 = \mu - 1.036\sigma \text{(i)}$$

$P(X > 90) = 0.05$ (Q – value)

$$P\left(Z > \frac{70 - \mu}{\sigma}\right) = 0.05 \text{ (Q – value)}$$

(ii) $P(X > 80) = P\left(Z > \frac{80 - 71.5927}{11.1899}\right) = P(Z > 0.751)$

$$\frac{70 - \mu}{\sigma} = 1.645$$

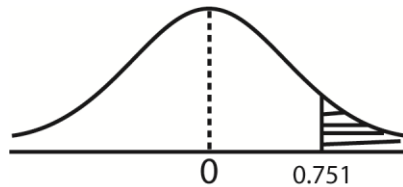
$$90 = \mu + 1.645\sigma \text{(ii)}$$

Eqn. (ii) – eqn. (i)

$$30 = 2.681; \sigma = 11.1899$$

From eqn. (i)

$$\mu = 60 + 1.036 \times 11.1899 = 71.5927$$



$$\begin{aligned} P(Z > 0.751) &= 0.5 - (0 < Z < 0.751) \\ &= 0.5 - 0.2737 \\ &= 0.2263 \end{aligned}$$

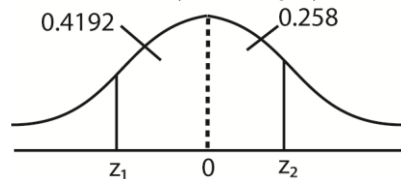
Number of residents = $200 \times 0.2263 = 45$

25. A random variable X has a normal distribution when $P(X > 9) = 0.9192$ and $P(X < 11) = 0.7580$. find

(a) The values of the mean and standard deviation (08marks)

$$P(X, >9) = P\left(z_1 > \frac{9 - \mu}{\delta}\right) = 0.9192$$

$$P(X, < 11) = P\left(z_2 > \frac{11 - \mu}{\delta}\right) = 0.0.7580$$



$$z_1 = -\phi(0.4192) = -1.4$$

$$z_2 = \phi(0.258) = 0.7$$

$$\Rightarrow \frac{9 - \mu}{\delta} = -1.4$$

$$9 - \mu = -1.4\delta \text{ (i)}$$

$$\Rightarrow \frac{11 - \mu}{\delta} = 0.7$$

$$11 - \mu = 0.7\delta \text{ (ii)}$$

Eqn (i) – Eqn (ii)

$$-2 = -2.1\delta$$

$$\delta = \frac{-2}{-2.1} = 0.9524$$

From (i)

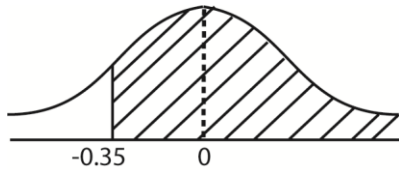
$$9 - \mu = -1.4 \times 0.9524$$

$$\mu = 10.333$$

(b) $P(X > 10)$ (04marks)

$$P(X > 10) = P\left(z > \frac{10 - 10.333}{0.9524}\right)$$

$$= P(z > -0.35)$$



$$P(X > 10) = P(0.5 + P(0 < z < 0.35))$$

$$= 0.5 + 0.1368$$

$$= 0.6368$$

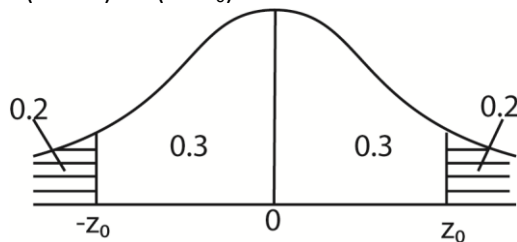
26. The marks in an examination were normally distributed with mean μ and standard deviation σ . 20% of the candidates scored less than 40 marks and 10% more than 75 marks. Find the

(c) values of μ and σ . (08marks)

Let x = marks scored

$$P(x < z < z_0) = 20\% = 0.2$$

$$P(x < 40) = P(z < z_0) = 0.2$$



$$P(0 < x < z_0) = 0.3$$

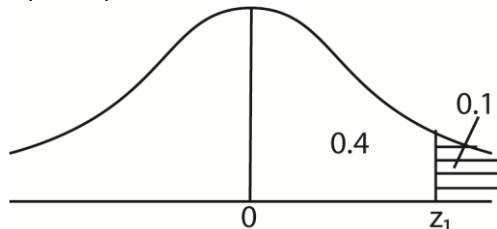
$$z_0 = -0.842$$

$$\text{but } z = \frac{40 - \mu}{\sigma}$$

$$-0.842 = \frac{40 - \mu}{\sigma}$$

$$-0.842\sigma = 40 - \mu \dots\dots\dots (i)$$

$$P(x > 75) = 10\% = 0.1$$



$$P(0 < z < z_1) = 0.4$$

$$z_1 = 1.282$$

$$1.282 = \frac{75 - \mu}{\sigma}$$

$$1.282\sigma = 75 - \mu \dots\dots\dots (ii)$$

Eqn. (ii) – eqn. (i)

$$2.124\sigma = 35$$

$$\sigma = 16.478 \text{ (3D)}$$

substituting σ into eqn. (ii)

$$1.282 \times 16.478 = 75 - \mu$$

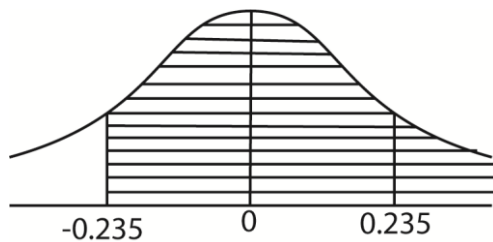
$$\mu = 53.875$$

Hence $\mu = 53.875$ and $\sigma = 16.478$

(d) percentage of the candidate who scored more than 50 marks.(04marks)

$$P(x > 50) = P\left(z - \frac{50 - 53.875}{16.478}\right)$$

$$= P(z > -0.235)$$



$$= 0.5 + P(0 < z < 0.235)$$

$$= 0.5 + 0.0929$$

$$= 0.5929$$

$$= 59.29\%$$

15.3 Binomial approximation to a normal distribution

Under the following conditions, the normal distribution is used to approximate binomial distribution

Conditions

- (i) the number of trials of the binomial experiment should be large, $n > 20$.
- (ii) The probability of success not to small or too large i.e. p constant and very close to 0.5
 $X \sim N(np, npq)$

The z-value is obtained from $Z = \frac{X \pm 0.5 - np}{\sqrt{npq}}$

Where ± 0.5 is used to make the binomial distribution continuous.

Note; 0.5 must be subtracted from the minimum value and added to the maximum value

$$(i) \quad P(X \geq x_1) = P\left(Z \geq \frac{(x_1 - 0.5) - np}{\sqrt{npq}}\right)$$

$$(ii) \quad P(X \leq x_1) = P\left(Z \leq \frac{(x_1+0.5)-np}{\sqrt{npq}}\right)$$

$$(iii) \quad P(x_1 \leq X \leq x_2) = P\left(\frac{(x_1-0.5)-np}{\sqrt{npq}} \leq Z \leq \frac{(x_2+0.5)-np}{\sqrt{npq}}\right)$$

Example 49

In a box containing different pens, the probability that a pen is red is 0.35. Find the probability that in a random sample of 400 pens from the box

- (i) Less than 120 are red pens
- (ii) More than 160 are red pens
- (iii) Between 120 and 150 inclusive are red pens.

Solution

$$N = 400, p = 0.35, q = 0.65$$

$$\text{Mean, } \mu = np = 400 \times 0.35 = 140$$

$$\sigma = \sqrt{npq} = \sqrt{400 \times 0.35 \times 0.65} = \sqrt{91}$$

$$(i) \quad P(X < 120) = P\left(Z \leq \frac{119.5-140}{\sqrt{91}}\right) = P(Z \leq -2.149)$$

$$= P(Z \geq 2.149) = 0.5 - \phi(2.149)$$

$$= 0.5 - 0.4842 = 0.0158$$

$$(ii) \quad P(X > 160) = P(X \geq 161)$$

$$= P\left(Z \leq \frac{116.5-140}{\sqrt{91}}\right) = P(Z \geq 2.149)$$

$$= 0.5 - \phi(2.149) = 0.5 - 0.4821 = 0.0158$$

$$(iii) \quad P(120 \leq X \leq 150)$$

$$= P\left(\frac{119.5-140}{\sqrt{91}} \leq Z \leq \frac{150.5-140}{\sqrt{91}}\right)$$

$$= P(-2.149 \leq Z \leq 1.101)$$

$$= 0.4842 + 0.3645 = 0.8487$$

Example 50

In unbiased coin is tossed 100 times, what is the probability that

- (i) There will be more than 60 heads
- (ii) there will be less than 43 head
- (iii) there will be between 45 heads and 55 head

Solution

$$N = 100, p = 0.5, q = 0.5$$

$$\text{Mean, } \mu = np = 100 \times 0.5 = 50$$

$$\sigma = \sqrt{npq} = \sqrt{100 \times 0.5 \times 0.5} = 5$$

$$(i) \quad P(X > 60) = P(X \geq 61)$$

$$= P\left(Z \geq \frac{60.5-50}{5}\right) = P(Z \geq 2.1)$$

$$= 0.5 - \phi(2.149) = 0.5 - 0.4821 = 0.0179$$

$$(ii) \quad P(X < 43) = P(X \leq 42)$$

$$= P\left(Z \leq \frac{42.5-50}{5}\right) = P(Z \leq -1.5)$$

$$= P(Z \geq 1.5) = 0.5 - \phi(1.5)$$

$$= 0.5 - 0.4332 = 0.0668$$

$$(iii) \quad P(45 \leq X \leq 55)$$

$$= P\left(\frac{44.5-50}{5} \leq Z \leq \frac{55.5-50}{5}\right)$$

$$= P(-1.1 \leq Z \leq 1.1) = 2 \times \phi(1.1)$$

$$= 2 \times 0.3643 = 0.7286$$

Example 51

It is known that 72% of NTV viewers watch news at 9 pm. What is the probability that a sample of 500 viewers chosen at random

- (i) More than 350 watch news (ii) fewer than 340 watch news (iii) exactly 350 watch news

Solution

$$\begin{aligned}
 \text{(i) } P(X > 350) &= P(X \geq 351) \\
 &= P\left(Z \geq \frac{350.5 - 500 \times 0.72}{\sqrt{500 \times 0.72 \times 0.28}}\right) \\
 &= P(Z \geq -0.946) = P(Z \leq 0.946) \\
 &= 0.5 + \phi(0.946) = 0.5 + 0.328 = 0.8280 \\
 \text{(ii) } P(X < 340) &= P(X \leq 339) \\
 &= P\left(Z \leq \frac{338.5 - 500 \times 0.72}{\sqrt{500 \times 0.72 \times 0.28}}\right) \\
 &= P(Z \leq -2.042) = P(Z \geq 2.042) \\
 &= 0.5 - \phi(2.042) = 0.5 - 0.4794 = 0.0206
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii) } P(X = 350) &= P(349.5 \leq X \leq 350.5) \\
 &= P\left(\frac{349.5 - 500 \times 0.72}{\sqrt{500 \times 0.72 \times 0.28}} \leq Z \leq \frac{350.5 - 500 \times 0.72}{\sqrt{500 \times 0.72 \times 0.28}}\right) \\
 &= \phi(-1.046) - \phi(0.946) \\
 &= 0.3522 - 0.3280 = 0.0242 \\
 \therefore P(X = 350) &= 0.0242
 \end{aligned}$$

Example 52

A pair of balanced dice, each numbered 1 to 6 is tossed 150 times. Determine the probability that a sum of seven appears at least 26 times

$$\begin{aligned}
 P(\text{sum of 7}) &= p = \frac{6}{36} = \frac{1}{6} \\
 P(X \geq 26) &= P\left(Z \geq \frac{(26-0.5) - 150 \times \frac{1}{6}}{\sqrt{150 \times \frac{1}{6} \times \frac{5}{6}}}\right) \\
 &= P\left(Z \geq \frac{25.5 - 25}{4.56}\right) = P(Z \geq 0.11) \\
 &= 0.5 - \phi(0.11) = 0.5 - 0.0438 \\
 &= 0.4562
 \end{aligned}$$

Example 53

Two players play a game in which each of them tosses a balanced coin. The game ends in a draw if both get the same result. Determine the probability that in 100 trials, the game ends in a draw.

- (i) At least 53 times (ii) at most 53 times

$$\begin{aligned}
 P(\text{sum of 7}) &= p = \frac{2}{4} = \frac{1}{2} \\
 P(X \geq 53) &= P\left(Z \geq \frac{(53-0.5) - 100 \times \frac{1}{2}}{\sqrt{100 \times \frac{1}{2} \times \frac{1}{2}}}\right) \\
 &= P\left(Z \geq \frac{52.5 - 50}{5}\right) = P(Z \geq 0.5) \\
 &= 0.5 - \phi(0.5) = 0.5 - 0.1915 = 0.3085 \\
 P(X \leq 53) &= P\left(Z \leq \frac{(53+0.5) - 100 \times \frac{1}{2}}{\sqrt{100 \times \frac{1}{2} \times \frac{1}{2}}}\right) \\
 &= P\left(Z \leq \frac{53.5 - 50}{5}\right) = P(Z \leq 0.7) \\
 &= 0.5 + \phi(0.7) = 0.5 + 0.2580 = 0.7580 \\
 \therefore P(X \leq 53) &= 0.7580
 \end{aligned}$$

Example 54

In a certain book of words per page follow normal distribution with mean 800 words and standard deviation 40 words. Three pages are chosen at random, what is the probability that

- (i) None of them has between 830 and 845 words.
- (ii) At least two pages have between 830 and 845 words

Solution

$$(i) \quad P(830 \leq X \leq 845) = P\left(\frac{830-800}{40} \leq Z \leq \frac{845-800}{40}\right) = P(0.75 \leq Z \leq 1.25) \\ = \phi(1.25) - \phi(0.75) = 0.3522 - 0.3280 = 0.0962$$

$$P(X = 0) = {}^3C_0(0.0962)^0(0.9038)^3 = 0.7383$$

$$(ii) \quad P(X \geq 2) = P(X = 2) + P(X = 3) \\ = {}^3C_2(0.0962)^2(0.9038)^1 + {}^3C_3(0.0962)^3(0.9038)^0 = 0.02509 + 0.00089 = 0.02598$$

Revision exercise 5

1. A random variable $X \sim B(200, 0.7)$. Find
 - (i) $P(X \geq 130)$ [0.9474]
 - (ii) $P(136 \leq X < 148)$ [0.6325]
 - (iii) $P(X < 142)$ [0.5914]
 - (iv) $P(X = 152)$ [0.0111]
2. An ordinary unbiased die is thrown 120 times. Find the probability of obtaining at least 24 sixes. [0.1958]
3. A pair of dice is tossed 144 times and the sum of the outcomes recorded. Find the probability that a sum of 7 occurs at least 26 times. [0.3688]
4. In a school 45% of the boys are circumcised. Find the probability that in a group of 200 boys 97 are circumcised. [0.1432]
5. 10% of phones imported to Uganda are I-phones, a random sample of 1000 phones is taken. Find the probability that
 - (i) Less than 80 are I-phones [0.0154]
 - (ii) Between 90 and 115 inclusive are I-phones [0.8145]
 - (iii) 120 or more are I-phones [0.02]
6. During Christmas, the probability that a message is sent on phone successfully is 0.85.
 - (i) When 8 messages are sent, find the probability that at least 7 are successfully sent [0.657]
 - (ii) When 50 messages are sent, find the probability that at least 45 are successfully sent [0.2142]
7. One-fifth of tourists have COVID 19. Find the probability that the number of tourists with COVID 19 is
 - (i) More than 20 in a sample of 100 people [0.4502]
 - (ii) Exactly 20 in a random sample of 100 people [0.0996]
 - (iii) More than 200 in a random sample of 1000 people [0.484]
8. If a fair die is thrown 300 times, what is the probability that
 - (i) There will be more than 60 sixes [0.0519]
 - (ii) There will be fewer than 45 sixes [0.1971]
9. A coin is biased such that head is twice as likely to occur as a tail. The coin is tossed 120 times. Find the probability that there will be
 - (i) Between 42 and 51 tails inclusive [0.3729]

- (ii) 48 tails or less [0.9501]
 - (iii) Less than 34 tails [0.1039]
 - (iv) At least 72 and at most 90 heads [0.9290]
10. A lorry of potatoes has average one rotten potato in six. A green grocer tests a random sample of 100 potatoes and decides to turn away the lorry if he finds more than 18 rotten potatoes in the sample. Find the probability that he accepts the consignment. [0.6886]
 11. On a certain farm, 20% of all the cows are infected by a tick disease. Find the probability that in a sample of 50 cows selected at random not more than 10% of the cows are infected. [0.0558]
 12. A pair of balanced dice, each numbered from 1 to 6 is tossed 180 times. determine the probability that a sum of seven appears;
 - (i) Exactly 40 times [0.0108]
 - (ii) Between 25 and 35 inclusive times [0.7286]
 13. On average 20% of all the eggs supplied by a farm have cracks. Find the probability that in a sample of 900 eggs supplied by a far will have more than 200 cracked eggs. [0.0439]
 14. On average 15% of all boiled eggs sold in a restaurant have cracks. Find the probability that in a sample of 300 boiled eggs will have more than 50 cracked eggs [0.215]
 15. Among spectators watching a football watch, 80% were the home supporters while 20% were the visiting team supporters. If 2500 of the spectators are selected randomly, what is the probability that there are at least 541 visitors in the sample? [0.0215]
 16. A die is tossed 40 times and the probability of getting at any one toss is 0.122, estimate the probability of getting between 6 to 10 sixes. [0.2048]
 17. In an examination which consists of 100 questions, a student has a probability of 0.6 of getting each answer correct. A student fails the examination if he obtains a mark less than 55, and obtains a distinction for a mark of 68 or more. Calculate
 - (i) The probability that he fails the examination [0.1308]
 - (ii) The probability that he obtains a distinction [0.0629]
 18. A research station supplies three varieties of seeds S_1 , S_2 and S_3 in the ratio 4:2:1. The probabilities of germination of S_1 , S_2 and S_3 are 50%, 60% and 80% respectively
 - (i) Find the probability that a selected seed will germinate [
 - (ii) Given that 150 seeds are selected at random, find the probability that less than 90 seed will germinate.
 19. A biased die with faces labelled 1, 2, 3, 4, 5, and 6 is tossed 45 times. calculate the probability that 2 appears
 - (i) More than 18 times [0.1342]
 - (ii) Exactly 11 times[0.0568]

Solutions to revision exercise 5

11. On a certain farm, 20% of all the cows are infected by a tick disease. Find the probability that in a sample of 50 cows selected at random not more than 10% of the cows are infected. [0.0558]

Since n is large, we use the normal distribution to approximate binomial distribution

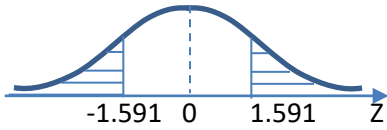
$$\text{Given: } n = 50, p = 0.2, q = 1 - 0.2 = 0.8, \mu = np = 50 \times 0.2 = 10,$$

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$$=\sqrt{npq} = \sqrt{50 \times 0.2 \times 0.8} = \sqrt{8}$$

$$10\% \text{ of } 50 \text{ cows} = 0.1 \times 50 = 5$$

$$P(x < 5) = P\left(Z < \frac{5.5}{\sqrt{8}}\right) = P(Z < -1.591)$$



$$\begin{aligned} P(Z < -1.591) &= P(Z > 1.591) \\ &= 0.5 - P(0 < Z < 1.591) \\ &= 0.5 - 0.4442 = 0.0558 \end{aligned}$$

12. A pair of balanced dice, each numbered from 1 to 6 is tossed 180 times. determine the probability that a sum of seven appears;
- Exactly 40 times [0.0108]
 - Between 25 and 35 inclusive times [0.7286]

Solution

Dice	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Let E = event that the sum 7 is picked when a pair of dice is tossed

$$n(E) = 6 \text{ and } P(E) = \frac{6}{36} = \frac{1}{6}$$

Since n is large, we use the normal distribution to approximate binomial distribution

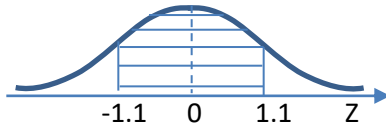
$$\text{Given: } n = 180, p = \frac{1}{6}, q = 1 - \frac{1}{6} = \frac{5}{6}, \mu = np = 180 \times \frac{1}{6} = 30,$$

$$=\sqrt{npq} = \sqrt{180 \times \frac{1}{6} \times \frac{5}{6}} = 5$$

Let x = number of times a sum of 7 appears

$$\begin{aligned} \text{(i)} \quad P(x = 40) &= P(39.5 < X < 40.5) = P\left(\frac{39.5-30}{5} < Z < \frac{40.5-30}{5}\right) = P(1.9 < Z < 2.1) \\ &= P(0 < Z < 2.1) - P(0 < Z < 1.9) = 0.4821 - 0.4713 = 0.0108 \end{aligned}$$

$$(ii) \quad P(25 \leq x \leq 35) = P\left(\frac{24.5-30}{5} < Z < \frac{35.5-30}{5}\right) = P(-1.1 < Z < 1.1)$$



$$P(-1.1 < Z < 1.1) = 2 \times P(0 < Z < 1.1) = 2 \times 0.3643 = 0.7286$$

$$\text{Hence } P(25 \leq x \leq 35) = 0.7286$$

13. On average 20% of all the eggs supplied by a farm have cracks. Find the probability that in a sample of 900 eggs supplied by a farm will have more than 200 cracked eggs.

Since n is large, we use the normal distribution to approximate binomial distribution

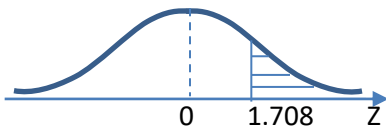
Given: $n = 900$, $p = 0.2$, $q = 1 - 0.2 = 0.8$, $\mu = np = 900 \times 0.2 = 180$,

$$= \sqrt{npq} = \sqrt{900 \times 0.2 \times 0.8} = 12$$

let x = number of eggs with cracks

$$P(x > 200) = P(Z > Z_1)$$

$$\text{Where } Z_1 = \frac{200.5 - 180}{12} = 1.708$$



$$P(Z > 1.708) = 0.5 - P(0 < Z < 1.708)$$

$$= 0.5 - 0.4561 = 0.0439$$

14. On average 15% of all boiled eggs sold in a restaurant have cracks. Find the probability that in a sample of 300 boiled eggs will have more than 50 cracked eggs

Since n is large, we use the normal distribution to approximate binomial distribution

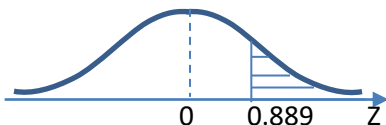
Given: $n = 300$, $p = 0.15$, $q = 1 - 0.15 = 0.85$, $\mu = np = 300 \times 0.15 = 45$,

$$= \sqrt{npq} = \sqrt{300 \times 0.15 \times 0.85} = 6.1847$$

let x = number of eggs with cracks

$$P(x > 50) = P(Z > Z_1)$$

$$\text{Where } Z_1 = \frac{50.5 - 45}{6.1847} = 0.889$$



$$P(Z > 0.889) = 0.5 - P(0 < Z < 0.889)$$

$$= 0.5 - 0.2850 = 0.215$$

15. Among spectators watching a football match, 80% were the home supporters while 20% were the visiting team supporters. If 2500 of the spectators are selected randomly, what is the probability that there are at least 541 visitors in the sample?

Solution

Since n is large, we use the normal distribution to approximate binomial distribution

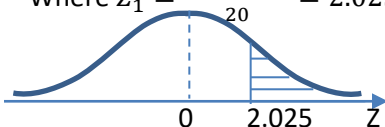
Given: $n = 2500$, $p = 0.2$, $q = 1 - 0.2 = 0.8$, $\mu = np = 2500 \times 0.2 = 500$,

$$\sigma = \sqrt{npq} = \sqrt{2500 \times 0.2 \times 0.8} = 20$$

let x = number of visitors to support their team

$$P(x > 540) = P(Z > Z_1)$$

$$\text{Where } Z_1 = \frac{540.5 - 500}{20} = 2.025$$



$$P(Z > 2.025) = 0.5 - P(0 < Z < 2.025)$$

$$= 0.5 - 0.4785 = 0.0215$$

16. A die is tossed 40 times and the probability of getting at any one toss is 0.122, estimate the probability of getting between 6 to 10 sixes.

Solution

Given: $n = 40$, $p = 0.122$, $q = 1 - 0.122 = 0.878$, $\mu = np = 40 \times 0.122 = 4.88$,

σ

$$= \sqrt{npq} = \sqrt{40 \times 0.122 \times 0.878} = 2.07$$

Let x be the number of sixes

$$P(6 < x < 10) = P(7 \leq x \leq 9)$$

Using normal approximation to binomial

$$Z = \frac{X \pm 0.5 - \mu}{\sigma}$$

$$P(7 \leq x \leq 9) = P\left(\frac{6.5 - 4.88}{2.07} \leq Z \leq \frac{9.5 - 4.88}{2.07}\right)$$

$$= P(0.78 < Z < 2.23)$$

$$= P(0 < Z < 2.23) - P(0 < Z < 0.78)$$

$$= 0.4871 - 0.2823$$

$$= 0.2048$$

17. In an examination which consists of 100 questions, a student has a probability of 0.6 of getting each answer correct. A student fails the examination if he obtains a mark less than 55, and obtains a distinction for a mark of 68 or more. Calculate

(i) The probability that he fails the examination [0.1308]

(ii) The probability that he obtains a distinction [0.0629]

Solution

Given: $n = 100$, $p = 0.6$, $q = 0.4$, $\mu = np = 100 \times 0.6 = 60$ and $\sigma = \sqrt{npq} = \sqrt{100 \times 0.6 \times 0.4} = 4.899$

Let x = mark scored

$$(i) P(x < 55) = P(x \leq 54) = P\left(Z \leq \frac{54.5 - 60}{4.899}\right)$$

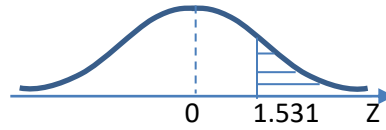
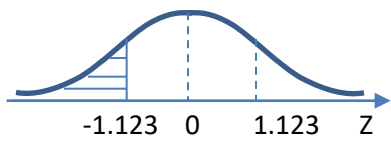
$$(ii) P(x \geq 68) = P\left(Z \leq \frac{67.5 - 60}{4.899}\right)$$

$$= P(Z \geq 1.531)$$

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$$= (Z \leq -1.123)$$



$$\begin{aligned} P(Z < -1.123) &= P(Z > 1.123) \\ &= 0.5 - P(0 \leq Z \leq 1.123) \\ &= 0.5 - 0.3692 \\ &= 0.1308 \end{aligned}$$

18. A research station supplies three varieties of seeds s_1 , s_2 and s_3 in the ratio 4:2:1. The probabilities of germination of s_1 , s_2 and s_3 are 50%, 60% and 80% respectively

- (i) Find the probability that a selected seed will germinate [
- (ii) Given that 150 seeds are selected at random, find the probability that less than 90 seed will germinate.

Solution

Given

$$4:2:1$$

$$4 + 2 + 1 = 7$$

$$P(S_1) = \frac{4}{7}; P(S_2) = \frac{2}{7}; P(S_3) = \frac{1}{7}$$

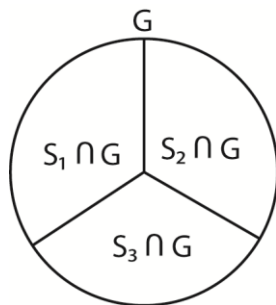
Let G = germination of seeds

$$P(G/S_1) = 50\% = 0.5$$

$$P(G/S_2) = 60\% = 0.6$$

$$P(G/S_3) = 80\% = 0.8$$

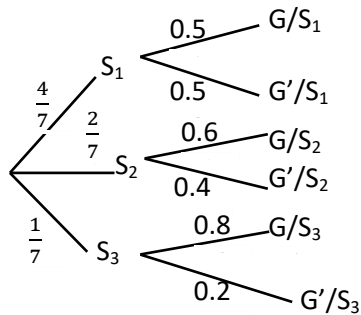
- (a) Find the probability that a seed selected at random will germinate.



$$\begin{aligned} P(G) &= P(s_1 \cap G) + P(s_2 \cap G) + P(s_3 \cap G) \\ &= P(S_1) \cdot P(G/S_1) + P(S_2) \cdot P(G/S_2) + P(S_3) \cdot P(G/S_3) \end{aligned}$$

$$\begin{aligned}
&= \frac{4}{7} \times 0.5 + \frac{2}{7} \times 0.6 + \frac{1}{7} \times 0.8 \\
&= \frac{2}{7} + \frac{1.2}{7} + \frac{0.8}{7} \\
&= \frac{4}{7}
\end{aligned}$$

Or Using factor tree diagram



$$\begin{aligned}
P(G) &= \frac{4}{7} \times 0.5 + \frac{2}{7} \times 0.6 + \frac{1}{7} \times 0.8 \\
&= \frac{2}{7} + \frac{1.2}{7} + \frac{0.8}{7} \\
&= \frac{4}{7}
\end{aligned}$$

- (b) Given that 150 seeds are selected at random, find the probability that less than 90 of the seeds will germinate. Give your answer to two decimal places.

$$n = 150; P = \frac{4}{7}; q = \frac{3}{7}$$

since n is large ($= 150$), we use the normal approximate this binomial

$$\mu = np = \frac{4}{7} \times 150 = \frac{600}{7}$$

$$\sigma = \sqrt{npq} = \sqrt{\frac{600}{7} \times \frac{3}{7}} = \frac{30\sqrt{2}}{7}$$

Let X = number of seeds that will germinate

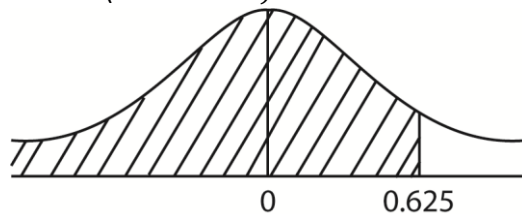
$$P(x < 90) = P(x \leq 89)$$

$$= P\left(z \leq \frac{89.5 - \frac{600}{7}}{\frac{30\sqrt{2}}{7}}\right)$$

$$= P\left(z \leq \frac{7(89.5 - \frac{600}{7})}{30\sqrt{2}}\right)$$

$$= P\left(z \leq \frac{628.5 - 600}{30\sqrt{2}}\right)$$

$$= P(z \leq 0.6250)$$



$$= 0.5 + P(0 \leq z \leq 0.625)$$

$$= 0.5 + 0.2340$$

$$= 0.7340$$

$$= 0.73 \text{ (2D)}$$

19. A biased die with faces labelled 1, 2, 3, 4, 5, and 6 is tossed 45 times. calculate the probability that 2 appears

(i) More than 18 times (07marks)

$$n=45, p = \frac{2}{6} = \frac{1}{3}, q = \frac{2}{3}$$

$$\mu = np = 45 \times \frac{1}{3} = 15$$

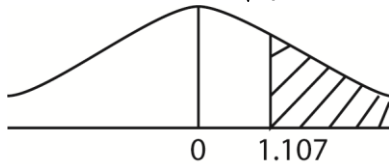
$$\sigma = \sqrt{npq} = \sqrt{45 \times \frac{1}{3} \times \frac{2}{3}} = \sqrt{10}$$

Changing binomial to normal distribution.

$$P(X > x) = P(X > 18 + 0.5) = P(X > 18.5)$$

Standardizing using $z = \frac{x - \mu}{\sigma}$

$$P(X > 18.5) = P\left(z > \frac{18.5 - 15}{\sqrt{10}}\right) = P(z > 1.107)$$



$$P(z > 1.107) = 0.5 - P(0 < z < 1.107)$$

$$= 0.5 - 0.3658$$

$$= 0.1342$$

$$\therefore P(X > 18) = 0.1342$$

(ii) Exactly 11 times (05marks)

$$P(X = 11) = P(11 - 0.5 < X < 11 + 0.5)$$

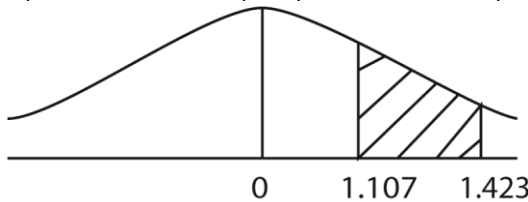
$$= P(10.5 < X < 11.5)$$

$$= P\left(\frac{10.5 - 15}{\sqrt{10}} < z < \frac{11.5 - 15}{\sqrt{10}}\right)$$

$$= P(-1.423 < z < 1.107)$$

By symmetry

$$P(-1.423 < z < 1.107) = P(1.107 < z < 1.423)$$



$$P(1.107 < z < 1.423) = P(0 < z < 1.423) - P(0 < z < 1.107)$$

$$= 0.4226 - 0.3658$$

$$= 0.0568$$

Distribution of sample mean of a normal distribution population

If a random variable X of a sample of size n from a normal distribution with mean μ and variance σ^2 , then distribution of the sample mean \bar{x} is also said to be normally distributed with mean μ and variance $\frac{\sigma^2}{n}$, such that $\bar{x} \approx \left(\mu, \frac{\sigma^2}{n} \right)$

$$\text{Then } Z = \frac{\bar{x} - \mu}{\frac{\sigma^2}{n}}$$

Example 55

At a certain school, the masses of students are normally distributed with mean 70kg and standard deviation 5kg. If 4 students are randomly selected, find the probability that their mean is less than 65.

$$P(\bar{X} < 65) = P\left(Z < \frac{65-70}{\frac{5}{\sqrt{4}}}\right) = P(Z < -2)$$

$$P(Z < -2) = P(Z > 2) = 0.5 - P(0 < Z < 2) = 0.5 - 0.4772 = 0.0228$$

Example 56

A random sample of size 15 is taken from a normal population with mean 60 and standard deviation 4. Find the probability that the mean of the sample is less than 58

$$P(\bar{X} < 58) = P\left(Z < \frac{58-60}{\frac{4}{\sqrt{15}}}\right) = P(Z < -1.936)$$

$$P(Z < -1.936) = P(Z > 1.936) = 0.5 - P(0 < Z < 1.936) = 0.5 - 0.4736 = 0.0264$$

Example 57

The height of students are normally distributed with mean 164cm and standard deviation 7.2cm. Calculate the probability that the mean height of a sample of 36 students will be between 162cm and 166cm.

$$P(162 < \bar{X} < 166) = P\left(\frac{162-164}{\frac{7.2}{\sqrt{36}}} < Z < \frac{166-164}{\frac{7.2}{\sqrt{36}}}\right) = P(-1.667 < Z < 1.667)$$

$$P(-1.667 < Z < 1.667) = 2 \times P(0 < Z < 1.667) = 2 \times 0.4522 = 0.9044$$

Example 58

The height of a certain plant follows a normal distribution with mean 21cm and standard deviation $\sqrt{90}$ cm. A random sample of 10 plants is taken and the mean height calculated. Find the probability that this sample mean lies between 18cm and 27 cm

$$P(18 < \bar{X} < 27) = P\left(\frac{18-21}{\frac{\sqrt{90}}{\sqrt{10}}} < Z < \frac{27-21}{\frac{\sqrt{90}}{\sqrt{10}}}\right) = P(-1 < Z < 2)$$

$$P(-1 < Z < 2) = P(0 < Z < 1) + P(0 < Z < 2) = 0.3413 + 0.4772 = 0.8185$$

Example 59

A large number of random sample of size n is taken from a distribution X where $X \sim N(74, 36)$ and the sample mean \bar{x} for each sample is noted. If $P(\bar{x} > 72) = 0.854$, find the value of n .

$$P(\bar{X} > 72) = P\left(Z > \frac{72-74}{\frac{6}{\sqrt{n}}}\right) = 0.854$$

$$P\left(Z > \frac{-\sqrt{n}}{3}\right) = 0.854$$

From table $Z = -1.054$

$$\frac{-\sqrt{n}}{3} = -1.054$$

$$n = 10$$

Example 60

The distribution of a random variable x is $X \sim N(25, 340)$ and the sample mean \bar{x} for each sample is calculated. If $P(\bar{x} > 28) = 0.005$, find the value of n .

$$P(\bar{X} > 28) = P\left(Z > \frac{28-25}{\frac{\sqrt{340}}{\sqrt{n}}}\right) = 0.005$$

$$P\left(Z > \frac{3\sqrt{n}}{\sqrt{340}}\right) = 0.005$$

From table $Z = 2.576$

$$\frac{3\sqrt{n}}{\sqrt{340}} = 2.576$$

$$n = 250$$

Revision exercise 6

1. If $X \sim N(200, 80)$ and a random sample of size 5 is taken from the distribution, find the probability that the sample mean
 - (i) is greater than 207 [0.0401]
 - (ii) lies between 201 and 209 [0.3891]
2. If $X \sim N(200, 10)$ and a random sample of size 10 is taken from the distribution, find the probability that the sample mean lies outside the range 198 and 205 [0.3206]
3. If $X \sim N(50, 12)$ and a random sample of size 12 is taken from the distribution, find the probability that the sample mean
 - (i) Is less than 48.5 [0.0668]
 - (ii) Is less than 52.3 [0.9893]
 - (iii) Lie between 50.7 and 51.7 [0.1974]
4. Biscuits are produced with weight (W g) where W is $N(10, 4)$ and are packed at random into boxes consisting of 25 biscuits. Find the probability that
 - (i) a biscuit chosen at random weigh between 9.25g and 10.7g [0.2924]

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- (ii) the content of a box weighs between 245g and 255g [0.0796]
 - (iii) the average weight of the biscuit in the box lies between 9.7g and 10.3g [0.5468]
5. A normal distribution has a mean of 40 and standard deviation of 4. If 25 items are drawn at random, find the probability that their mean
 - (i) 41.4 or more [0.0401]
 - (ii) Between 38.7 and 40.7 [0.7571]
 - (iii) Less than 39.5 [0.2660]
 6. A random sample of size 25 is taken from a normal population with mean 60 and standard deviation 4. Find the probability that the mean of the sample
 - (i) Less than 58 [0.0062]
 - (ii) Greater than 58 [0.9918]
 - (iii) Between 58 and 62 [0.9876]
 7. At St. Noa Junior, the marks of the pupils can be modelled by a normal distribution with mean 70% and standard deviation 5%. If four pupils are chosen at random, find the probability that the mean mark is
 - (i) Less than 65% [0.9772]
 - (ii) Greater than 65% [0.0228]
 - (iii) Greater than 75% [0.0228]
 - (iv) Between 72% and 75% [0.1891]
 8. The volume of soda in bottle are normally distributed with mean 758ml and standard deviation of 12ml. a random sample of 10 bottles is taken and mean volume is found. Find the probability that the sample mean is less than 750ml. [0.0176]
 9. The height of cassava plants are normally distributed with mean of 2m and standard deviation of 40cm. a random sample of 50 cassava plants is taken and the mean height found. Find the probability that the sample mean lies between 195cm and 205cm. [0.6234]
 10. In an examination, marks are normally distributed with mean 64.5 and variance 64. The mean mark in a random sample of 100 scripts is denoted by \bar{X} . find
 - (i) $P(\bar{X} > 65.5)$ [0.1056]
 - (ii) $P(63.8 < \bar{X} < 64.5)$ [0.3092]
 11. The marks of an examination were normally distributed. 20% of the students scored below 40 marks while 10% of the students scored above 75 marks
 - (i) Find the mean mark and standard deviation of the students [$\mu = 53.87, \sigma = 16.473$]
 - (ii) If 25 students were chosen at random from those who sat for the examination, what is the probability that their average mark exceeds 60. [0.0313]
 - (iii) If a sample of 8 students were chosen, find the probability that not more than 3 scored between 45 and 65 marks. [0.5419]
 12. The life time of batteries produced by a certain factory is normally distributed. Out of 10,000 batteries selected at random, 668 have life time less than 130 hours and 228 have life time more than 200 hours.
 - (i) Find the mean mark and standard deviation of the battery life time [$\mu = 160, \sigma = 20$]
 - (ii) Find the percentage of the batteries with life time between 150 and 180 hours.
 - (iii) If the sample of 25 batteries is selected at random, find the probability that the mean of the life time exceeds 165 hours [0.1056]
 13. A normal distribution has a mean of 30 and a variance of 5. Find the probability that

- (i) The average of 10 observation exceeds 30.5 [0.2399]
 - (ii) The average of 40 observation exceeds 30.5 [0.0787]
 - (iii) The average of 100 observation exceeds 30.5 [0.0127]
 - (iv) Find n such that the probability that the average of observations exceed 30.5 is less than 1%, [n > 108]
14. The random variable is such that $X \sim N(\mu, 4)$. A random sample size n is taken from the population. Find the least n such that $P(|\bar{X} - \mu| < 0.5) = 0.95$ [62]
15. Boxes made in a factory have weight which are normally distributed with a mean of 4.5kg and a standard deviation of 2.0kg. if a sample of 16 boxes is drawn at random, find the probability that their mean is
- (i) between 4.6 and 4.7 kg [0.0761]
 - (ii) between 4.3 and 4.7g [0.3108]
16. the masses of soap powder in a certain packet are normally distributed with mean 842g and variance 225g. find the probability that a random sample of 120 packets has sample mean mass
- (i) between 844g and 846g [0.0702]
 - (ii) less than 843g [0.7673]

Estimation of population parameters

Statistical estimation is used to describe the unknown characteristics of the population (population parameters) by using sample characteristics.

A sample is a representation of the population parameter such as population mean, μ and population variance, σ^2 .

Types of parameter estimation

- point estimation
- interval estimation

(a) point estimates

- (i) the unbiased estimate of the population mean, μ is

$$\bar{x} = \frac{\sum x}{n} \text{ or } \bar{x} = \frac{\sum fx}{\sum f}$$
 where \bar{x} is sample mean
- (ii) the unbiased estimate of the population variance, σ^2 is $\hat{\sigma}^2$ where $\hat{\sigma}^2 = \frac{n}{n-1} s^2$ where s^2 is sample variance

$$\text{OR } \hat{\sigma}^2 = \frac{n}{n-1} \left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n} \right)^2 \right] \text{ or } \hat{\sigma}^2 = \frac{n}{n-1} \left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f} \right)^2 \right]$$

Example 61

Find the best unbiased estimate of mean μ and variance σ^2 of the population from each of the following sample is drawn

- (i) 46, 48, 50, 45, 53, 50, 48, 51

Solution

x	f	fx	fx ²
45	1	45	2025
46	1	46	2116
48	2	96	4608
50	2	100	5000
51	1	51	2601
53	1	53	2809
	$\sum f = 8$	$\sum fx = 391$	$\sum fx^2 = 19159$

Unbiased estimate for the mean $\bar{x} = \frac{\sum fx}{\sum f} = \frac{391}{8} = 48.875$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1} \left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f} \right)^2 \right]$
 $= \frac{8}{8-1} \left[\frac{19159}{8} - \left(\frac{391}{8} \right)^2 \right] = 6.982$

(ii) $\sum x = 100, \sum x^2 = 1028, n = 10$

Unbiased estimate for the mean $\bar{x} = \frac{\sum fx}{\sum f} = \frac{100}{10} = 10$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1} \left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f} \right)^2 \right]$
 $= \frac{10}{10-1} \left[\frac{1028}{10} - \left(\frac{100}{10} \right)^2 \right] = 3.11$

(iii) $\sum x = 120, \sum x^2 = 2102, n = 8$

Unbiased estimate for the mean $\bar{x} = \frac{\sum fx}{\sum f} = \frac{120}{8} = 15$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1} \left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f} \right)^2 \right]$
 $= \frac{8}{8-1} \left[\frac{2102}{8} - \left(\frac{120}{8} \right)^2 \right] = 43.14$

(iv) $\sum x = 330, \sum x^2 = 23700, n = 34$

Unbiased estimate for the mean $\bar{x} = \frac{\sum fx}{\sum f} = \frac{330}{34} = 9.71$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1} \left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f} \right)^2 \right]$

$$= \frac{34}{34-1} \left[\frac{23700}{34} - \left(\frac{330}{34} \right)^2 \right] = 621.12$$

(v) $\sum x = 738, \sum x^2 = 16526, n = 50$

Unbiased estimate for the mean $\bar{x} = \frac{\sum fx}{\sum f} = \frac{738}{50} = 14.76$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1} \left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f} \right)^2 \right]$

$$= \frac{50}{50-1} \left[\frac{16526}{50} - \left(\frac{738}{50} \right)^2 \right] = 114.96$$

Example 62

The fuel consumption of a new car model was being tested. In one trials 8 cars chosen at random were driven under identical conditions and distance x km covered on one litre of petro was recorded. The following results were obtained. $\sum x = 152.98, \sum x^2 = 2927.1$. Calculate the unbiased estimate of the mean and variance of the distance covered by the car.

Solution

Unbiased estimate for the mean $\bar{x} = \frac{\sum fx}{\sum f} = \frac{152.98}{8} = 19.1225$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1} \left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f} \right)^2 \right]$

$$= \frac{8}{8-1} \left[\frac{2927.1}{8} - \left(\frac{152.98}{8} \right)^2 \right] = 0.25$$

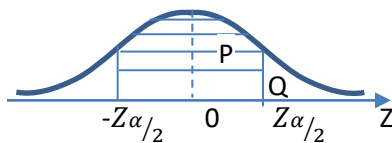
(b) Interval estimate

Here we are interested in obtaining the interval over which the true population mean lies (confidence interval)

The unbiased estimate of the population mean, μ is \bar{x}

$Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$ where n is the sample size

Z is the area under the normal curve leaving an area of $\frac{\alpha}{2}$ on either side of the curve



$$P \left(-Z_{\alpha/2} < Z < Z_{\alpha/2} \right) = \frac{\alpha}{2}$$

$$P \left(-Z_{\alpha/2} < \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} < Z_{\alpha/2} \right) = \frac{\alpha}{2}$$

$$P \left(\bar{x} - Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right) = \frac{\alpha}{2}$$

Confidence interval $\left[\bar{x} - Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right]$

Confidence Limits $\left[\bar{x} - Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right]$

Or $\mu < \bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

(i) Confidence interval for population mean μ

- of a normal or non-normal population
- with known population variance σ^2 or standard deviation σ
- using any sample size

The confidence interval is obtained from $\mu < \bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$ where \bar{x} is sample mean

Example 63

The length of a bar of a metal is normally distributed with mean of 115cm and standard deviation of 3cm. find the 95% confidence limits for the length of the bar

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$	$\mu < \bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$ $\mu < 115 \pm 1.96 \frac{3}{\sqrt{1}}$	Lower limit = 109.12
$Z_{\alpha/2} = 1.96$		Upper limit = 120.88

Example 64

The mass of vitamin in a capsule is normally distributed with standard deviation 0.042mg. a random sample of 5 capsules was taken and the mean mass of vitamin e found to be 5.12. calculate a symmetric confidence interval for the population mean mass.

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$	$\mu < \bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$ $\mu < 5.12 \pm 1.96 \frac{0.042}{\sqrt{5}}$	Lower limit = 5.08
$Z_{\alpha/2} = 1.96$		Upper limit = 5.16

Example 65

It is known that an examination paper is marked in such a way that the standard deviation of the marks is 15.1. in a certain school, 80 candidates took the examination and they had an average mark of 57.4. find

(i) 95% confidence limits for the mean mark in the exam.

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$	$\mu < \bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$ $\mu < 57.4 \pm 1.96 \frac{15.1}{\sqrt{80}}$	Lower limit = 54.091
$Z_{\alpha/2} = 1.96$		Upper limit = 60.709

(ii) 99% confidence limits for the mean mark in the exam.

$\frac{\alpha}{2} = \frac{0.99}{2} = 0.495$	$\mu < \bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$ $\mu < 57.4 \pm 2.575 \frac{15.1}{\sqrt{80}}$	Lower limit = 53.053
$Z_{\alpha/2} = 2.575$		Upper limit = 61.746

Example 66

After a particular rainy night, 12 worms were picked and their length in cm measure: 9.5, 9.5, 11.2, 10.6, 9.9, 11.1, 10.9, 9.8, 10.1, 10.2, 10.9, 11.0

Assuming that this sample came from a normal population with standard deviation 2, find the 95% confidence interval for the mean length of all the worms

$$\bar{x} = \frac{\sum x}{n} = \frac{9.5 + 9.5 + 11.2 + 10.6 + 9.9 + 11.1 + 10.9 + 9.8 + 10.1 + 10.2 + 10.9 + 11.0}{12}$$
$$= 10.39$$

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$	$\mu < \bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$	Lower limit = 9.81
$Z_{\alpha/2} = 1.96$		Upper limit = 10.97
	$\mu < 10.39 \pm 1.96 \frac{2}{\sqrt{12}}$	

The height of students are normally distributed with mean μ and standard deviation σ . On the basis of results obtained from a random sample of 100 students from school, the 95% confidence interval of the mean was calculated and found to be (177.22cm, 179.18cm). Calculate

- (i) the value of the sample mean

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$	$179.18 < \bar{x} + 1.96 \frac{\sigma}{\sqrt{100}} \dots (ii)$
$Z_{\alpha/2} = 1.96$	
$\mu < \bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$	Eqn. (i) + eqn. (ii)
$177.22 < \bar{x} - 1.96 \frac{\sigma}{\sqrt{100}} \dots (i)$	

- (ii) the value of standard deviation

$$177.22 < 178.2 - 1.96 \frac{\sigma}{\sqrt{100}};$$
$$\sigma = 5$$

- (iii) 90% confidence interval of the mean, μ

$$\frac{\alpha}{2} = \frac{0.90}{2} = 0.45; Z_{\alpha/2} = 1.645$$

Example 67

A plant produces steel sheets whose weight are normally distributed with standard deviation of 2.4kg. A random sample of 36 sheets had a mean weight of 31.4kg.

- (i) Find the 99% confidence limit for the population

$\frac{\alpha}{2} = \frac{0.99}{2} = 0.495$	$\mu < \bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$	Lower limit = 30.37kg
$Z_{\alpha/2} = 2.575$		Upper limit = 32.43kg
	$\mu < 31.4 \pm 2.595 \frac{2.4}{\sqrt{36}}$	

- (ii) Find the width of the 99% confidence limit

$$= 32.43\text{kg} - 30.37\text{kg} = 2.06\text{kg}$$

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Example 69

The marks scored by students are normally distributed with mean μ and standard deviation 1.3. it is required to have 95% confidence interval for μ with width less than 2. Find the least number of students that be sampled to achieve this.

$$\begin{array}{l} \frac{\alpha}{2} = \frac{0.95}{2} = 0.475 \\ Z_{\alpha/2} = 1.96 \end{array} \quad \left| \begin{array}{l} \text{width} = 2 \times Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} < 2 \\ 2 \times 1.96 \frac{\sigma}{\sqrt{n}} < 2 \end{array} \right. \quad \left| \begin{array}{l} 2 \times 1.96 \frac{\sigma}{2} < \sqrt{n}; n < 6.49 \\ n > 6.49 \\ \text{the least number} = 7 \end{array} \right.$$

(ii) Confidence interval for population mean μ

- of a normal or non-normal population
- with unknown population variance σ^2 or standard deviation σ
- using a large sample size ($n \geq 30$)

If the population variance σ^2 is not given or unknown, the confidence interval is obtained from $\mu < \bar{x} \pm Z_{\alpha/2} \frac{\hat{\sigma}}{\sqrt{n}}$ where \bar{x} is sample mean, $\hat{\sigma} = \frac{n}{n-1} s^2$ and s = sample variance

$$\hat{\sigma}^2 = \frac{n}{n-1} \left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n} \right)^2 \right]$$

Or

$$\hat{\sigma}^2 = \frac{n}{n-1} \left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f} \right)^2 \right]$$

Example 70

the fuel consumption of a new car was being tested. In one trials 50 cars chosen at random were driven under identical conditions and the distance x km covered on one litre of petrol was recorded. the following results were obtained. $\sum x = 525$, $\sum x^2 = 5625$. Calculate the 95% confidence interval for the mean petrol consumption, in km per litre of cars of this type..

$$\text{Unbiased estimate for the mean } \bar{x} = \frac{\sum fx}{\sum f} = \frac{525}{50} = 10.5$$

$$\begin{aligned} \text{The unbiased estimate of the population variance, } \hat{\sigma}^2 &= \frac{n}{n-1} \left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f} \right)^2 \right] \\ &= \frac{50}{50-1} \left[\frac{5625}{50} - \left(\frac{525}{50} \right)^2 \right] = 2.2952 \end{aligned}$$

$$\begin{array}{l} \frac{\alpha}{2} = \frac{0.95}{2} = 0.475 \\ Z_{\alpha/2} = 1.96 \end{array} \quad \left| \begin{array}{l} \mu < \bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \\ \mu < 10.5 \pm 1.96 \frac{\sqrt{2.2952}}{\sqrt{50}} \end{array} \right. \quad \left| \begin{array}{l} \text{Lower limit} = 10.08 \text{ km/litre} \\ \text{Upper limit} = 10.92 \text{ km/litre} \end{array} \right.$$

Example 71

The height x cm of each man in a random sample of 200 men living in Nairobi was measured. The following results were obtained $\sum x = 35050$, $\sum x^2 = 6163109$.

(a) calculate the unbiased estimate of the mean and variance of the heights of men living Nairobi

$$\text{Unbiased estimate for the mean } \bar{x} = \frac{\sum fx}{\sum f} = \frac{35050}{200} = 175.25$$

$$\begin{aligned} \text{The unbiased estimate of the population variance, } \hat{\sigma}^2 &= \frac{n}{n-1} \left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f} \right)^2 \right] \\ &= \frac{200}{200-1} \left[\frac{6163109}{200} - \left(\frac{35050}{200} \right)^2 \right] = 103.5 \end{aligned}$$

(b) Determine the 90% confidence interval for the mean height of mean living in Nairobi.

$$\begin{array}{l|l|l} \frac{\alpha}{2} = \frac{0.90}{2} = 0.45 & \left| \begin{array}{l} \mu < \bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \\ \mu < 175.25 \pm 1.645 \frac{\sqrt{103.5}}{\sqrt{200}} \end{array} \right. & \begin{array}{l} \text{Lower limit} = 174.07\text{cm} \\ \text{Upper limit} = 176.43\text{cm} \end{array} \end{array}$$

Example 72

A random sample of 100 observations from a normal population with mean μ gave the following results $\sum x = 8200$, $\sum x^2 = 686000$.

(a) calculate the unbiased estimate of the mean and variance of the heights of men living Nairobi

$$\text{Unbiased estimate for the mean } \bar{x} = \frac{\sum fx}{\sum f} = \frac{8200}{100} = 82$$

$$\begin{aligned} \text{The unbiased estimate of the population variance, } \hat{\sigma}^2 &= \frac{n}{n-1} \left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f} \right)^2 \right] \\ &= \frac{200}{200-1} \left[\frac{686000}{100} - \left(\frac{8200}{100} \right)^2 \right] = 11.72 \end{aligned}$$

(b) Determine the 98% confidence interval for the mean

$$\begin{array}{l|l|l} \frac{\alpha}{2} = \frac{0.98}{2} = 0.49 & \left| \begin{array}{l} \mu < \bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \\ \mu < 82 \pm 2.326 \frac{11.72}{\sqrt{100}} \end{array} \right. & \begin{array}{l} \text{Lower limit} = 79.274 \\ \text{Upper limit} = 84.726 \end{array} \end{array}$$

(c) determine the 99% confidence interval for the mean

$$\begin{array}{l|l|l} \frac{\alpha}{2} = \frac{0.99}{2} = 0.495 & \left| \begin{array}{l} \mu < \bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \\ \mu < 82 \pm 2.595 \frac{11.72}{\sqrt{100}} \end{array} \right. & \begin{array}{l} \text{Lower limit} = 78.981 \\ \text{Upper limit} = 85.726 \end{array} \end{array}$$

Example 73

The mean and standard deviation of a random sample of size 100 is 900 and 60 respectively. Given that the population is normally distributed, find 96% confidence interval of the population mean.

$$\hat{\sigma} = \sqrt{\frac{n}{n-1}} s^2 = \sqrt{\frac{100}{100-1}} \times 60^2 = 60.302$$

$$\frac{\alpha}{2} = \frac{0.96}{2} = 0.48$$

$$Z_{\alpha/2} = 2.054$$

$$\mu < \bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$$\mu < 900 \pm 2.054 \frac{60.302}{\sqrt{100}}$$

$$\text{Lower limit} = 887.614$$

$$\text{Upper limit} = 912.386$$

Revision exercise 7

- the concentration in mg per litre of a trace element in 7 randomly chosen samples of water from a spring were 240.8, 237.3, 236.7, 236.6, 234.2, 233.9, 232.5.
Determine the unbiased mean and variance of the concentration of the trace element per litre from spring [236, 7.58]
- Cartons of oranges are filled by a machine. A sample of 10 cartons selected at random from the population contained the following quantities in ml) 201.2, 205.0, 209.1, 202.3, 204.6, 206.4, 210.1, 201.9, 203.7, 207.3. Determine the unbiased mean and variance of the population from which the sample was taken. [203.16, 9.223]
- A factory produces cans of meat whose masses are normally distributed with standard deviation 18g. A random sample of 25 cans is found to have the mean of 458g. find the 99% confidence interval for the population mean of a can of meat produced at the factory. [448.7, 467.3g]
- The height of bounce of a tennis ball is normally distributed with standard deviation 2cm. A sample of 60 tennis balls is tested and the mean height of bounce is 140cm. Find
(i) 95% [139.5, 140.51] (ii) 98% [139.4, 140.6] confidence interval for the mean height of bounce of the tennis ball
- A random sample of 100 is taken from a population. The sample is found to have a mean of 76.0 and standard deviation of 120. Find
(i) 90% [747.51, 748.49] (ii) 95% [747.42, 748.58] (iii) 98% [747.31, 748.69] confidence interval for the mean of the population
- 150 bags of flour of a particular brand are weighed and the mean mass is found to be 748g with standard deviation 3.6g. Find
(i) 90% [74.02, 77.98] (ii) 97% [73.38, 78.62] (iii) 99% [72.89, 79.11] confidence interval for the mean mass of bags of flour of this brand.
- A random sample of 100 readings taken from a normal population gave the following data: $\bar{x} = 82$, $\sum x^2 = 686800$. Find
(i) 98% [79.19, 84.81] (ii) 99% [78.89, 85.11] confidence interval of the population mean
- 80 people were asked to measure their pulse rates when they woke up in the morning. The mean was 69 beats and standard deviation 4 beats. find
(i) 95% [68.12, 69.88] (ii) 99% [67.84, 70.16] (iii) 97% [68.0, 70.0] confidence interval of the population mean

9. The 95% confidence interval for the mean length of a particular brand of light bulbs is [1023.3h, 1101.7h]. This interval is based on results from a sample of 36 light bulbs. Find the 99% confidence interval for the mean length of life of this brand of light bulbs assuming that the length of life is normally distributed. [1011, 1114]
10. A random sample of 6 items taken from a normal population with variance 4.5cm^2 gave the following data: 12.9cm, 13.2cm, 14.6cm, 12.6cm, 11.3cm, and 10.1 cm.
- Find the 95% confidence interval for the population mean. [10.75, 14.15].
 - What is the width of this confidence interval [3.4]
11. A random sample of 60 loaves is taken from a population whose mean masses are normally distributed with mean μ and standard deviation 10g.
- calculate the width of 95% confidence interval for μ based on the sample [5.06]
 - Find the confidence level having the same width as in (i) but based on a random sample of 40 loaves. [89%]
12. The distribution of measurements of masses of a random sample of bags packed in a factory is shown below

Mass (kg)	72.5	77.5	82.5	87.5	92.5	97.5	102.5	107.5
frequency	6	18	32	57	102	51	25	9

- Find the mean and standard deviation of the masses [$\mu = 91.317$, $\sigma = 7.41$]
- find the 95% confidence limits [90.5, 92.2]

Thank you

Dr. Bbosa Science