SENIOR SIX TERM 2

TOPIC 1/5: Dynamics 2

Competency: The learner analyses motion patterns of objects as observed from different reference points to predict their behaviour and solving kinematic issues.

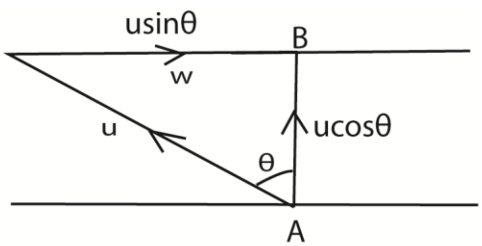
Contents

Crossing a river

There are two cases to consider when crossing a river

Case I: Shortest route

If the water is not still and boatman wishes to cross **directly opposite** to the standing point. In order to cross from point A to point B directly opposite A (perpendicularly), then the course set by the boat must be upstream of the river.



u = speed of the boat in still water

w= speed of running water

At point B: w = usinθ

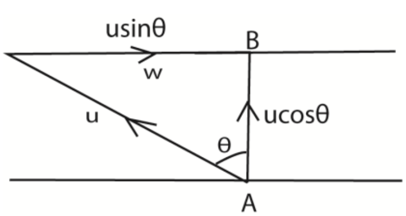
θ = $\sin^{-1}\left(\frac{w}{u}\right)$

θ = is the direction to the vertical but the direction to the bank is $(90 - \theta)^0$

Time taken = $\frac{AB}{u\cos\theta}$

Example 1

A man who can swim at 6km/h in still water would like to swim between two directly opposite points on the river banks of the river 300m flowing at 3km/h. Find the time taken to do this.



AB = 0.3km

θ = $\sin^{-1}\left(\frac{w}{u}\right) = \sin^{-1}\left(\frac{3}{6}\right) = 30^0$

Time taken = $\frac{AB}{u\cos\theta} = \frac{0.3}{6\cos 30}$

time = 0.058hrs = 3.46minute

He must swim at 300m to AB in order to cross directly and it takes him 3.46minutes.

Alternatively

Using Pythagoras theorem

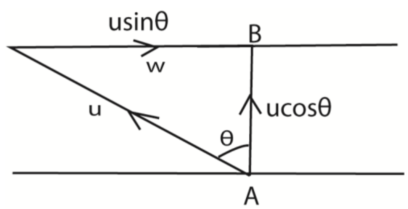
$6^2 = 3^2 + V_{AB}^2$

$V_{AB} = \sqrt{36 - 9} = 5.1962$km/h

Time = $\frac{AB}{V_{AB}} = \frac{0.3}{5.1963} = 0.058$hrs

Example 2

Two points A and B are on opposite banks of a river flowing at $\frac{5}{6}ms^{-1}$. A man who can swim at $\frac{25}{18}ms^{-1}$ in still water would like to swim directly from A to B. Find the width of the river if he takes 2minutes to cross the river.



$\theta = \sin^{-1}\left(\frac{w}{u}\right) = \sin^{-1}\left(\frac{5/6}{25/18}\right)$ =36.87^0

Time taken = $\frac{AB}{u\cos\theta} = \frac{0.3}{\frac{25}{18}\cos 36.87}$

AB = 133.333m

Alternatively: using Pythagoras theorem

$\left(\frac{25}{18}\right)^2 = \left(\frac{5}{6}\right)^2 + V_{AB}^2$

V_{AB}=1.1111m

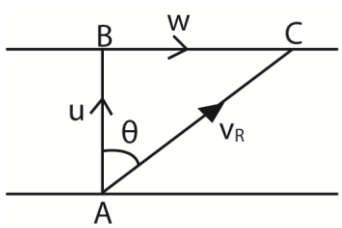
Time taken =$\frac{AB}{V_{AB}}$

2 x 60 = $\frac{AB}{1.1111}$

AB = 133.333m

Case II: the shortest time taken/ as quickly as possible

If the boatman wishes to cross the river as quickly as possible, then he should steer his boat directly from A to B as shown. The river pushes the boat down stream



Time taken to cross the river, t = $\frac{AB}{u}$

Distance covered downstream = wt

Where w is the velocity at which the river flows

u is the velocity of the boat

Or

Distance downstream = $w\frac{AB}{u}$

tanθ = $\frac{w}{u}$ or θ = $\tan^{-1}\frac{w}{u}$

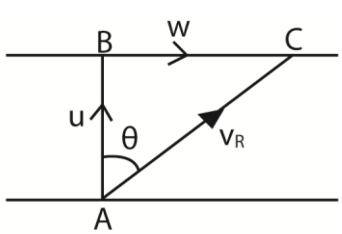
The resultant velocity downstream V_R

$V_R = \sqrt{w^2 + u^2}$

Example 3

A man who can swim at 2ms^{-1} in still water wishes to swim across a river 120m wide as quickly as possible. If the river flows at 0.5ms^{-1}, find the time the man takes to cross far downstream he travels.

Solution



u = 2ms^{-1}, w = 0.5ms^{-1}, AB = 120m

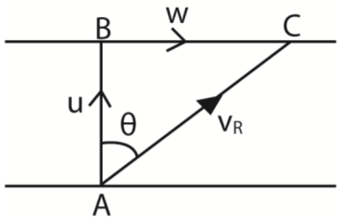
t = $\frac{AB}{u} = \frac{120}{2} = 60s$

Distance = wt = 0.5 x 60 = 30m

Example 4

A boat can travel at $3.5ms^{-1}$ in still water. A river is 80m wide and the current flows at $2ms^{-1}$, calculate

(a) the shortest time to cross the river and the distance downstream the oat is carried.

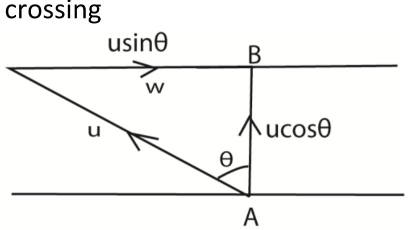


$u = 3.5ms^{-1}$, $w = 2ms^{-1}$, AB = 80m

$t = \frac{AB}{u} = \frac{80}{3.5} = 22.95s$

Distance, BC = wt = 2 x 22.95 = 45.8m

(b) the course that must be set to point exactly opposite the starting point and time taken for crossing



course, $\theta = \sin^{-1}\frac{w}{u} = \sin^{-1}\frac{2}{3.5} = 34.8^0$

Time for crossing = $\frac{80}{3.5cos34.8}$= 27.8s

$u = 3.5ms^{-1}$, $w = 2ms^{-1}$, AB = 80m

Revision exercise 1

1. A man who can row at $0.9ms^{-1}$ in still water wishes to cross a river of width 1000m as quickly as possible. If the current flows at a rate of $0.3ms^{-1}$. Find the time taken for journey. Determine the direction in which he should point the boat and position of the boat where he lands. [111.11s, 71.57^0 to the bank, 333.33 downstream]
2. A man swims at $5kmh^{-1}$ in still water. Find the time it takes the man to swim across the river250m wide, flowing at $3kmh^{-1}$, if he swims so as to cross the river
   (a) the shortest route [225s]
   (b) in the quickest time[180s]
3. A boy can swim in still water at $1ms^{-1}$, he swims across the river flowing at $0.6ms^{-1}$ which is 300m wide, find the time he takes
   (a) if he travels the shortest possible distance [375s]
   (b) if he travels as quickly as possible and the distance downstream, [300s, 180m]
4. A boy wishes to swim across a river 100m wide as quickly as possible. The river flows at $3kmh^{-1}$ and the boy can swim at $4kmh^{-1}$ in still water. Find the time that the boy takes to cross the river and how far downstream he travels. [90s, 75m]

Relative motion

It is composed of

(a) relative velocity
(b) Relative path

Relative velocity

This is the velocity a body would have as seen by an observer on another body. Suppose A and B are two moving bodies, the velocity of A relative to B is the velocity of A as it appears to observer on B

It is denoted by $V_{AB} = V_A - V_B$

Note that $V_{AB} \neq V_{BA}$ since $V_{BA} = V_B - V_A$.

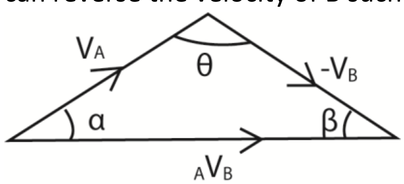
Numerical calculations

There are two methods used in calculations

- Geometric method and
- Vector method

(i) Geometric method.

If V_A and V_B are not given in vector form and the velocity of A relative to B is required, then we can reverse the velocity of B such that $V_{AB} = V_A + (-V_B)$ and the vector triangle is drawn as below.



$V_{AB}^2 = V_A^2 + V_B^2 - 2V_A \times V_B \cos\theta$

and

$\frac{V_{AB}}{sin\theta} = \frac{V_B}{sin\alpha} = \frac{V_A}{sin\beta}$

(ii) Vector method

Find components of velocity for each separately

$\therefore V_{AB} = V_A - V_B$

Example 5

Particle A is moving due north at $30ms^{-1}$ and particle B is moving due south at $20ms^{-1}$. Find the velocity of A relative to B.

Solution

$\uparrow V_A = 30ms^{-1}$ and $\downarrow V_B = 20ms^{-1}$

$V_{AB} = V_A - V_B$

$V_{AB} = \binom{0}{30} - \binom{0}{-20} = \binom{0}{50}$

$|V_{AB}| = \sqrt{0^2 + 50^2} = 50ms^{-1}$ due north

Example 6

A particle A has a velocity $(4i + 6j - 5k)ms^{-1}$ while B has a velocity of $(-10i - 2j + 6k)ms^{-1}$. Find the velocity of A relative to B.

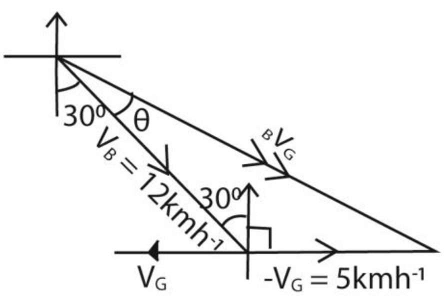
$V_{AB} = V_A - V_B$

$$V_{AB} = \begin{pmatrix} 4 \\ 6 \\ -5 \end{pmatrix} - \begin{pmatrix} -10 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 14 \\ 8 \\ -11 \end{pmatrix} \text{ms}^{-1}$$

Example 7

A girl walks at $5kmh^{-1}$ due west and a boy runs $12kmh^{-1}$ at a bearing of 150^0. Find the velocity of the boy relative to the girl

Method I (geometrical)



$V_{BG}^2 = V_B^2 + V_G^2 - 2V_B \times V_G \cos 120^0$

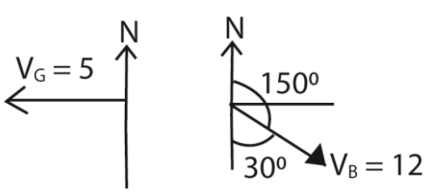
$V_{BG} = \sqrt{5^2 + 12^2 - 2 \times 5 \times 12 \cos 120^0}$ = $15.13ms^{-1}$

$\frac{5}{\sin\theta} = \frac{15.13}{\sin 120}$

$\theta = 16.63^0$

The relative velocity is $15.13ms^{-1}$ at $S46.63^0E$

Method II (Vector)

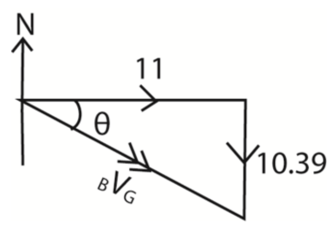


$V_{BG} = V_B - V_G$

$= \begin{pmatrix} 12\sin 30 \\ -12\cos 30 \end{pmatrix} - \begin{pmatrix} -5 \\ 0 \end{pmatrix} = \begin{pmatrix} 11 \\ -10.39 \end{pmatrix}$

$|V_{BG}| = \sqrt{11^2 + (-10.39)^2}$

$= 15.13ms^{-1}$

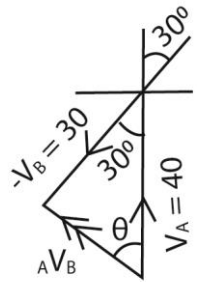


$\theta = \tan^{-1}\frac{10.39}{11} = 43.40$

The relative velocity is $15.13ms^{-1}$ at $S46.63^0E$

Example 8

Plane A is flying due north at $40kmh^{-1}$ while plane B is flying in the direction $N30^0E$ at $30kmh^{-1}$. Find the velocity of A relative B



$V_{AB}^2 = V_A^2 + V_B^2 - 2V_A \times V_B \cos 30^0$

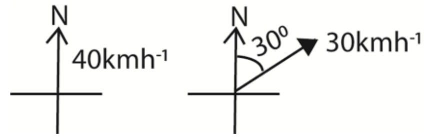
$V_{BG} = \sqrt{40^2 + 30^2 - 2 \,x\, 40 \,x\, 30 \cos 30^0}$

$= 20.53kmh^{-1}$

$\frac{30}{sin\theta} = \frac{20.53}{\sin 30}$; $\theta = 46.94^0$

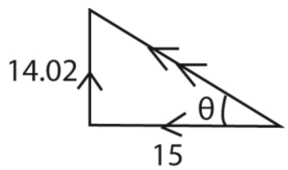
The relative velocity is $20.53kmh^{-1}$ at $N46.9^0W$

Method II (vectors)



$V_{AB} = V_A - V_B = \begin{pmatrix} 0 \\ 40 \end{pmatrix} - \begin{pmatrix} 30sin30 \\ -30cos30 \end{pmatrix} = \begin{pmatrix} -15 \\ 14.02 \end{pmatrix}$

$|V_{BG}| = \sqrt{(-15)^2 + (14.02)^2} = 20.53kmh^{-1}$



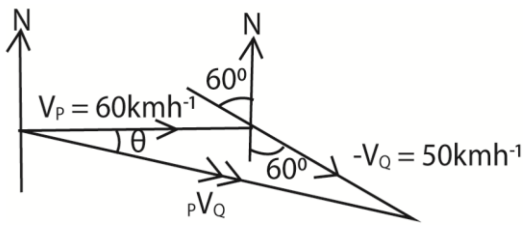
$\theta = \tan^{-1}\frac{14.02}{15} = 43.1^0$

The relative velocity is $20.53kmh^{-1}$ at $N46.9^0W$

Example 9

Ship P is steering 60kmh^{-1} due east while ship Q is steering in the direction N600W at 50kmh^{-1}. Find the velocity of P relative to Q.

Method I (Geometrical)



$V_{PQ}^2 = V_P^2 + V_Q^2 - 2V_P \times V_Q \cos 150^0$

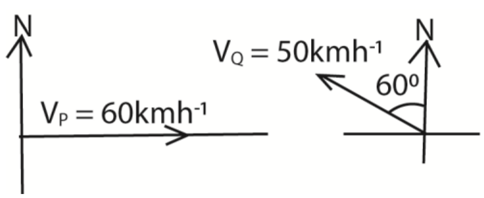
$V_{BG} = \sqrt{60^2 + 50^2 - 2 \times 60 \times 50 \cos 150^0}$

$= 106.28\text{kmh}^{-1}$

$\frac{50}{\sin\theta} = \frac{106.28}{\sin 30}$; θ = 13.6^0

The relative velocity is 106.28kmh^{-1} at S76.4^0E

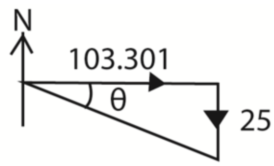
Method II (Vector)



$V_{PQ} = V_P - V_Q$

$= \begin{pmatrix}60\\0\end{pmatrix} - \begin{pmatrix}-50\sin 60\\50\cos 60\end{pmatrix} = \begin{pmatrix}103.301\\-25\end{pmatrix}$

$|V_{BG}| = \sqrt{(103.301)^2 + (-25)^2}$ =106.3kmh^{-1}



θ =$\tan^{-1}\frac{25}{103.301}$ = 13.6^0

Direction S($90^0 - 13.6^0$)E

The relative velocity is 106.28kmh^{-1} at S76.4^0E

Finding true velocity

Example 10

To a cyclist riding due north at 40kmh^{-1}, a steady wind appears to blow from N60^0E at 30kmh^{-1}. Find the true velocity of the wind

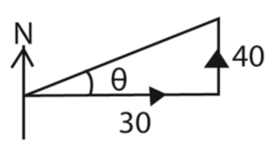
Solution

$V_{WC} = V_W - V_C$

$\begin{pmatrix}30\\0\end{pmatrix} = V_W - \begin{pmatrix}0\\40\end{pmatrix}$

$V_W = \begin{pmatrix}30\\40\end{pmatrix}$

$|V_W| = \sqrt{(30)^2 + (40)^2}$ = 50kmh^{-1}



θ =$\tan^{-1}\frac{40}{30}$ = 53.13^0

Direction N $(90 - 15.13)^0$E = N36.87^0E

Example 11

To a motorist travelling due north at $40kmh^{-1}$, a steady wind appears to blow from $N60^0E$ at $50kmh^{-1}$.

(a) find the true velocity of the wind

$V_{WM} = V_W - V_M$

$\begin{pmatrix}-50sin60\\-50cos60\end{pmatrix} = V_W - \begin{pmatrix}0\\40\end{pmatrix}$

$V_W = \begin{pmatrix}-43.3\\15\end{pmatrix}$

$|V_W| = \sqrt{(-43.3)^2 + (15)^2}$ = $46kmh^{-1}$

$\theta = \tan^{-1}\frac{15}{43.3}$ = 19^0

Direction: $N71^0W$

(b) If the wind velocity and direction remain constant but the speed of the motorist is increases, find his speed when the wind appears to be blowing from the direction $N45^0E$.

$V_{WM} = V_W - V_M$

$\begin{pmatrix}-bsin45\\-bcos45\end{pmatrix} = \begin{pmatrix}46sin71\\-46cos71\end{pmatrix} - \begin{pmatrix}0\\a\end{pmatrix}$

i components : -bsin45 = 46sin71

b = 61.5096

j components: -bcos45 = -46cos71 – a

a = $58.47kmh^{-1}$

Example 12

To a man travelling due north at $10kmh^{-1}$, a steady wind appears to blow from East. When he travels in the direction $N60^0W$ at $8kmh^{-1}$, it appears to come from south. Find the velocity of the wind.

$V_{WM} = V_W - V_M$

$\begin{pmatrix}-a\\0\end{pmatrix} = V_W - \begin{pmatrix}0\\10\end{pmatrix}$

$V_W = \begin{pmatrix}-a\\10\end{pmatrix}$ ……………………………(i)

Also

$V_{WM} = V_W - V_M$

$\begin{pmatrix}0\\b\end{pmatrix} = V_W - \begin{pmatrix}-8sin60\\b+8cos60\end{pmatrix}$ ….. (ii)

(i) and (ii)

$\begin{pmatrix}-a\\10\end{pmatrix} = \begin{pmatrix}-8sin60\\b+8cos60\end{pmatrix}$

a = 8sin60 = $4\sqrt{3}$

10 = b + 8cos60

b = 6

$V_W = \begin{pmatrix}-a\\10\end{pmatrix} = \begin{pmatrix}-4\sqrt{3}\\10\end{pmatrix}$

$|V_W| = \sqrt{\left(-4\sqrt{3}\right)^2 + 10^2}$ = $12.17kmh^{-1}$

$\theta = \tan^{-1}\left(\frac{10}{4\sqrt{3}}\right)$= 55.3^0

Direction: $N(90 - 53.3)^0W$= $N34.7^0W$

Example 13

To a cyclist riding due north at $40kmh^{-1}$, a steady wind appears to blow eastwards. On reducing his speed to 30kmh¯1 but moving in the same direction, the wind appears to come from southwest. Find the velocity of the wind

$V_{WC} = V_W – V_C$

$\begin{pmatrix} a \\ 0 \end{pmatrix} = V_W - \begin{pmatrix} 0 \\ 40 \end{pmatrix}$

$V_W = \begin{pmatrix} a \\ 10 \end{pmatrix}$ ……………………………(i)

Also

$V_{WC} = V_W – V_C$

$\begin{pmatrix} bsin45 \\ bcos45 \end{pmatrix} = V_W - \begin{pmatrix} 0 \\ 30 \end{pmatrix}$ ….. (ii)

(i) and (ii)

$\begin{pmatrix} a \\ 40 \end{pmatrix} = \begin{pmatrix} bsin45 \\ 30 + bcos45 \end{pmatrix}$

40 = 30 + bcos45

b = $10\sqrt{2}$

a = bsin45 = $10\sqrt{2}$sin45 = 10

$V_W = \begin{pmatrix} a \\ 40 \end{pmatrix} = \begin{pmatrix} 10 \\ 40 \end{pmatrix}$

$|V_W| = \sqrt{10^2 + 40^2}$ = 41.23kmh^{-1}

$\theta = \tan^{-1}\left(\frac{40}{10}\right)$= 75.96^0

Direction: N$(90 -76)^0$E= N14^0E

Exercise 2

1. Car A moving Eastward at 20ms^{-1} and car B is moving Northward at 10ms^{-1}. Find the
   (i) velocity of A relative to B [$10\sqrt{5}ms^{-1}$
   (ii) velocity of B relative to A [$10\sqrt{5}ms^{-1}$
2. A yatch and a trawler leave a harbour at 8am. The yatch travels due west at 10kmh-1 and trawler due east at 20kmh^{-1}
   (i) what is the velocity of the trawler relative to yatch [30kmh^{-1} east]
   (ii) how far apart are the boats at 9.30am [45km]
3. At 10.30am a car travelling at 25ms^{-1} due east overtakes a motor bike travelling at 10ms^{-1} due east. What is the velocity of the car relative to the motor bike and how far apart are the vehicle at 10.30am. [15ms^{-1} east, 900m]
4. Bird A has a velocity of (7i + 3j + 10k)ms^{-1} while bird B has a velocity (6i – 17k)ms^{-1}. Find the velocity of B relative to A [(-i -3j – 27k0ms^{-1}]
5. Joe rides his horse with a velocity $\begin{pmatrix} 5 \\ 24 \end{pmatrix}$kmh^{-1} while Jill is riding her horse with velocity $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$kmh^{-1}
   (i) Find Joe's velocity as seen by Jill $\left[\begin{pmatrix} 5 \\ 24 \end{pmatrix} \text{kmh}^{-1}\right]$
   (ii) What is Jill's velocity as seen by Joe. $\left[\begin{pmatrix} 0 \\ -124 \end{pmatrix} \text{kmh}^{-1}\right]$
6. In EPL football match, a ball is moving at 5ms^{-1}in the direction of N450E and the player is running due north at 8ms^{-1}. Find the velocity of the ball relative to the player. [5.69ms^{-1} at S38.38^0E]
7. An aircraft is flying at 250kmh^{-1} in direction N60^0E and a second aircraft is flying at 200kmh^{-1} in the direction N20^0W. Find the velocity of the first aircraft as seen by the pilot of the second aircraft. [292kmh^{-1} at S77.9^0E]

8. A ship is sailing southeast at 20kmh^{-1} and a second ship is sailing due west at 25kmh^{-1}. Find the magnitude and direction of the velocity of the first ship relative to the second. [41.62kmh^{-1} at S70.13^{0}E]
9. What is the velocity of a cruiser moving at 20kmh^{-1} due to north as seen by an observer on a liner moving at 15kmh^{-1} in the direction N30^{0}W [10.3kmh^{-1} at N46.9^{0}E]
10. A car is being driven at 20ms-1 on a bearing of 0400. Wind is blowing from 3000 with speed of 10ms-1. Find the velocity of the wind as experienced by the driver of the car. [48.13ms^{-1} at S18.13^{0}W]
11. An aircraft is moving at 250kmh^{-1} in direction N60^{0}E. The second aircraft is moving at 200kmh^{-1} in a direction N20^{0}W. Find the velocity of the first aircraft as seen by the pilot of the second aircraft. [292kmh^{-1} at S77.9^{0}E]
12. To a pigeon flying with velocity of (-2i + 3j + k)ms^{-1}, a hawk appears to have a velocity of (i – 5j -10k)ms^{-1}.Find the true velocity of the hawk[(-i -2j – 9k)ms^{-1}]
13. To a cyclist riding at 3ms^{-1} due east, the wind appears to come from the south with the speed $3\sqrt{3}ms^{-1}$. Find the true speed and direction of the wind. [6ms^{-1} from S30^{0}W]
14. To the pilot of an aircraft A travelling at300kmh^{-1} due south, it appears that an aircraft B is travelling at 600kmh^{-1} in a direction N60^{0}W. Find the true speed and direction of the aircraft B. [520kmh^{-1} west]
15. Jane is riding her horse at 5kmh^{-1} due north and sees Suzan riding her horse apparently with velocity 4kmh^{-1}, N60^{0}E. Find Suzan's true velocity. [7.81kmh^{-1} N26.3^{0}E]
16. A eagle flying at 8ms^{-1} on a bearing of 240^0 sees a chick apparently running at 5ms^{-1} on bearing300^0. Find true velocity of the chick. [11.4ms^{-1} at 262.4^0]
17. A train is travelling at 80kmh^{-1} in direction N15^{0}E. A passenger on the train observes a plane apparently moving at 125kmh^{-1} in the direction N50^{0}E. Find the true velocity of the plane. [196kmh^{-1} N36.5^{0}E]
18. To an athlete jogging at 12kmh^{-1} on a bearing of N10^{0}E, the wind seems to come from a direction N20^{0}W at 15kmh^{-1}. Find the true velocity of the wind. [7.57kmh^{-1} N72.5^{0}W]
19. To a passenger on a boat which is travelling at 20kmh^{-1} on a bearing 230^0, the wind seems to be blowing from 250^0 as 12kmh^{-1}. Find the true velocity of the wind [9.64kmh^{-1} N24.8^{0}E]
20. On a particular day wind is blowing N30^{0}E at a velocity of 4ms^{-1} and a motorist is driving at 40ms^{-1} in the direction of S60^{0}E.
    (a) Find the velocity of the wind relative to the motorist. [40.2ms^{-1} at N54.28^{0}W]
    (b) If the motorist changes the direction maintaining his speed and the wind appears to blow due east. What is the new direction of the motorist [N85.03^{0}W]
21. A, B and C are three aircrafts. A has velocity (200i + 170j)ms^{-1}. To the pilot of A it appears that B has velocity (50i – 270j)ms^{-1}. To the pilot of B it appears that C has a velocity (50i + 170j)ms^{-1}. Find the velocities of B and C [(250i -100j)ms^{-1}, (300i + 70j)ms^{-1}]
22. To a bird flying due east at 10ms^{-1}, the wind seems to come from south. When the bird alters its direction of flight to N30^{0}E without altering its speed, the wind seems to come from the north-west. Find the true velocity of wind. [10.6ms^{-1} from S69.9^{0}W]
23. To an observer on a trawler moving at 12kmh^{-1} in the direction S30^{0}W, the wind appears to come from N600W. To an observer on a ferry moving at 15kmh^{-1} in a direction S80^{0}E, the wind appears to come from the north. Find the true velocity of the wind. [26.8kmh^{-1} N33.4^{0}W]

Relative position

Consider two bodies A and B moving with V_A and V_B from points with position vectors OA and OB respectively.

Position of A at time t is $R_{A(t=t)} = R_{A(t=0)} + t \times V_A$

Position of A at time t is $R_{Bt=t)} = R_{B(t=0)} + t \times V_B$

Position of A relative to B at time t is $R_{ABt=t)} = R_{A(t=0)} - R_{B(t=0)}$

$R_{ABt=t)} = (R_{A(t=0)} + t \times V_A) - (R_{B(t=0)} + t \times V_B)$

$R_{ABt=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

Example 14

The velocities of ships P and Q are (i +6j)kmh^{-1} and (-i + 3j)kmh^{-1}. At a certain instant, the displacement between the two ship is (7i + 4j)km. Find the

(a) Relative velocity of ship P to Q

$V_{PQ} = V_P - V_Q$

$V_{PQ} = \begin{pmatrix}1\\6\end{pmatrix} - \begin{pmatrix}-1\\3\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix} kmh^{-1}$

(b) Magnitude of displacement between ships P and Q after 2hours.

$R_{PQt=t)} = (R_{P(t=0)} - R_{Q(t=0)}) + (V_{PQ})t$

$R_{PQt=t)} = \begin{pmatrix}7\\4\end{pmatrix} + t\begin{pmatrix}2\\3\end{pmatrix}$

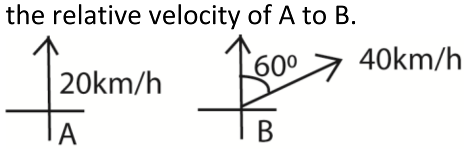
$R_{PQt=2)} = \begin{pmatrix}7\\4\end{pmatrix} + 2\,x\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}11\\10\end{pmatrix} km$

$|R_{PQ}| = \sqrt{11^2 + 10^2} = 14.87km$

Example 15

Two ship A and B move simultaneously with velocities 20kmh^{-1} and 40kmh^{-1}. Ship A moves in the northern direction while ship B moves in N60^{0}E. Initially ship B is 10km due west of A. Determine

(a) the relative velocity of A to B.

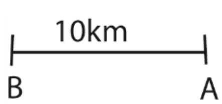


$V_{AB} = V_A - V_B$

$V_{AB} = \begin{pmatrix}0\\20\end{pmatrix} - \begin{pmatrix}40sin60\\40cos60\end{pmatrix} = \begin{pmatrix}-34.641\\0\end{pmatrix}$

$|V_{AB}| = \sqrt{(-34.641)^2 + 0^2} = 34.641$

(b) the position of A relative to B at any time t



$R_{AB(t=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix}10\\0\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix}\right] + t\begin{pmatrix}-34.641\\0\end{pmatrix}$

$R_{AB(t=t)} = \begin{pmatrix}10 - 34.641t\\0\end{pmatrix} km$

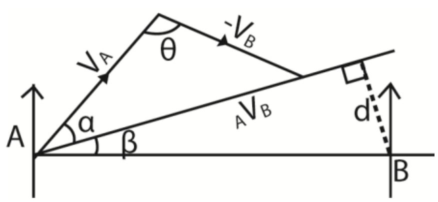
Shortest Distance and time of closest approach

(Shortest distance and time of shortest distance)

When two particles are moving simultaneously with specific velocities, time will come when they are closest to each other **without** colliding.

There are three methods used for the distance and time of closest approach, i.e. Geometrical, vector and differential method.

1. vector method
   Consider particle A and B moving with velocities VA and VB from point with position vector OA and OB respectively.
   For minimum distance to be attained then $V_{AB}.R_{AB(t=t)} = 0$. This gives the time.
   Then **shortest distance, d** = $|R_{AB(t=t)}|$
2. Differential method
   The minimum distance is reached when $\frac{d}{dt}|R_{AB(t=t)}|^2 = 0$. This gives time
   Then **shortest distance, d** = $|R_{AB(t=t)}|$
3. Geometrical method
   If V_A and V_B are not given in vector form, then the velocity of B is reversed such that $V_{AB} = V_A + (-V_B)$ and the vector triangle is drawn as below.
   The shortest distance, d will be perpendicular to $_{VAB}$



$${V_{AB}}^2 = {V_A}^2 + {V_B}^2 - 2V_AV_Bcos\theta$$

$$\frac{V_{AB}}{sin\theta} = \frac{V_B}{sin\alpha}$$

Shortest distance, d =ABsinβ

Time to the shortest distance, t = $\frac{ABcos\beta}{V_{AB}}$

Example 16

A particle P starts from rest from a point with position vector (2j + 2k)m with a velocity (j + k)ms^{-1}. A second particle Q starts at the same time from a point whose position vector is (–11i – 2j – 7k)m with a velocity of (2i + j + 2k)ms^{-1}. Find

(i) the shortest distance between the particles

(ii)

(iii) how far each has travelled by this time.

$$V_{PQ} = V_P - V_Q = \begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}-2\\0\\-1\end{pmatrix}$$

$$R_{PQ(t=t)} = \left(R_{P(t=0)} - R_{Q(t=0)}\right) + (V_{PQ})t$$

$$R_{PQ(t=t)} = \left[\begin{pmatrix}0\\2\\2\end{pmatrix} - \begin{pmatrix}-11\\-2\\-7\end{pmatrix}\right] + \begin{pmatrix}-2\\0\\-1\end{pmatrix}t$$

$$R_{PQ(t=t)} = \begin{pmatrix}11-2t\\4\\9-t\end{pmatrix}$$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

$$\begin{pmatrix}-2\\0\\-1\end{pmatrix}.\begin{pmatrix}11-2t\\4\\9-t\end{pmatrix} = 0$$

-22 + 4t + 0 -9 + t = 0

t = $\frac{31}{5} = 6.2s$

(i) Then **shortest distance, d** = $\left|R_{PQ(t=t)}\right|$

$$R_{PQ(t=6.2)} = \begin{pmatrix}11-2t\\4\\9-t\end{pmatrix}$$

$$R_{PQ(t=6.2)} = \begin{pmatrix}11-2x\ 6.2\\4\\9-6.2\end{pmatrix} = \begin{pmatrix}-1.4\\4\\2.8\end{pmatrix}$$

$\left|R_{PQ(t=6.2)}\right| = \sqrt{(-1.4)^2+4^2+2.8^2}$ = 5.08m

(ii) How far each has travelled

$$R_{P(t=t)} = R_{P(t=0)} + (V_P)t$$

$$R_{P(t=6.2)} = \begin{pmatrix}0\\2\\2\end{pmatrix} + 6.2\ x\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\8.2\\8.2\end{pmatrix}$$

$$\left|R_{P(t=6.2)}\right| = \sqrt{0^2+8.2^2+8.2^2} = 11.6m$$

$$R_{Q(t=6.2)} = \begin{pmatrix}-11\\-2\\-7\end{pmatrix} + 6.2\ x\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}1.4\\4.2\\5.4\end{pmatrix}$$

$$\left|R_{Q(t=6.2)}\right| = \sqrt{1.4^2+4.2^2+5.4^2} = 6.8m$$

Method II

$$\frac{d}{dt}\left|R_{AB(t=t)}\right|^2 = 0.$$

$$\left|R_{PQ(t=t)}\right|^2 = \begin{pmatrix}11-2t\\4\\9-t\end{pmatrix}^2$$

$$\left|R_{PQ(t=t)}\right|^2 = (11-2t)^2+4^2+(9-t)^2$$

$$\left|R_{PQ(t=t)}\right|^2 = 218-62t+5t^2$$

$$\frac{d}{dt}\left|R_{PQ(t=t)}\right|^2 = \frac{d}{dt}(218-62t+5t^2)$$

$$\frac{d}{dt}\left|R_{PQ(t=t)}\right|^2 = -62+10t = 0$$

t = 6.2s

$$R_{PQ(t=6.2)} = \begin{pmatrix}11-2x\ 6.2\\4\\9-6.2\end{pmatrix} = \begin{pmatrix}-1.4\\4\\2.8\end{pmatrix}$$

$\left|R_{PQ(t=6.2)}\right| = \sqrt{(-1.4)^2+4^2+2.8^2}$ = 5.08m

Example 17

At 12 noon the position vectors r and velocity vectors v of ship A and ship B are as follows

r_A = (-9i + 6j)km, v_A = (3i + 12j) kmh^{-1} and r_B = (16i + 6j), v_B = (-9i+ 3j)kmh^{-1} respectively

(i) Find how far apart the ships are at noon

$$R_{AB(t=0)} = \left(R_{A(t=0)} - R_{B(t=0)}\right)$$

$$R_{AB(t=0)} = \begin{pmatrix}-9\\6\end{pmatrix} - \begin{pmatrix}16\\6\end{pmatrix} = \begin{pmatrix}-25\\0\end{pmatrix}$$

$$\left|R_{AB(t=0)}\right| = \sqrt{(-25)^2+0^2} = 25km\ apart$$

(ii) Assuming velocities do not change, find the least distance between the ships in the subsequent motion

$V_{AB} = V_A - V_B = \begin{pmatrix}3\\12\end{pmatrix} - \begin{pmatrix}-9\\3\end{pmatrix}$=$\begin{pmatrix}12\\9\end{pmatrix}$

$R_{AB(t=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

$R_{AB(t=t)} = \begin{pmatrix}-25\\0\end{pmatrix} + \begin{pmatrix}12\\9\end{pmatrix}t = \begin{pmatrix}-25+12t\\9t\end{pmatrix} km$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

$\begin{pmatrix}12\\9\end{pmatrix}.\begin{pmatrix}-25+12t\\9t\end{pmatrix} = 0$

-300 + 144t + 81t = 0

t = $\frac{4}{3}hours$

Shortest distance, d = $\left|R_{PQ(t=\frac{4}{3})}\right|$

$R_{AB(t=\frac{4}{3})} = \begin{pmatrix}-25+12\ x\ \frac{4}{3}\\ 9x\ \frac{4}{3}\end{pmatrix}$ = $\begin{pmatrix}-9\\12\end{pmatrix} km$

$\left|R_{AB(t=\frac{4}{3})}\right| = \sqrt{(-9)^2+12^2}$ = 15km

(iii) Find when their distance of closest approach occurs and the potion vectors of A and B

It occurs at 12.00 + $\frac{4}{3}\ x\ 60$ = 1.20pm

how far each travelled

$R_{A(t=t)} = R_{A(t=0)} + V_At$

$R_{A(t=\frac{4}{3})} = \begin{pmatrix}-9\\6\end{pmatrix}$ + $\begin{pmatrix}3\\12\end{pmatrix} x \frac{4}{3}$ = $\begin{pmatrix}-5\\22\end{pmatrix} km$

$R_{B(t=t)} = R_{B(t=0)} + V_Bt$

$R_{B(t=\frac{4}{3})} = \begin{pmatrix}16\\6\end{pmatrix}$ + $\begin{pmatrix}-9\\3\end{pmatrix} x \frac{4}{3}$ = $\begin{pmatrix}4\\10\end{pmatrix} km$

Example 18

At a certain time, the position vectors r and velocity vectors v of ship A and ship B are as follows

r_A = (20j)km V_A = (9i -2j)kmh^{-1} at 14.00hrs

r_B = (i + 4j)km V_B = (4i + 8j)kmh^{-1} at 15.00hrs

Assuming velocities do not change, find

(a) the position vector of A at 15.00hrs

$$R_{A(t=t)} = R_{A(t=0)} + V_At$$

At 16.00hrs: $R_{A(t=1)} = \begin{pmatrix}0\\20\end{pmatrix} + \begin{pmatrix}9\\-2\end{pmatrix} x\ 1 = \begin{pmatrix}9\\18\end{pmatrix} km$

(b) the least distance between A and B in the subsequent motion

$R_{AB(t=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix}9\\18\end{pmatrix} - \begin{pmatrix}1\\4\end{pmatrix}\right] + \left[\begin{pmatrix}9\\-2\end{pmatrix} - \begin{pmatrix}8\\8\end{pmatrix}\right]$t

$R_{AB(t=t)} = \begin{pmatrix}8+5t\\14-10t\end{pmatrix}$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

t = 0.8hrs

Shortest distance, d = $|R_{PQ(t=0.8)}|$

$R_{AB(t=0.8)} = \begin{pmatrix}8+5\ x\ 0.8\\14-10\ x\ 0.8\end{pmatrix} = \begin{pmatrix}12\\6\end{pmatrix} km$

$R_{AB(t=0.8)} = \sqrt{12^2+6^2}$ = 13.42km

$\begin{pmatrix}5\\-10\end{pmatrix}\cdot\begin{pmatrix}8+5t\\14-10t\end{pmatrix} = 0$

(c) time at which this least separation occurs.
15.00 + 0.8 x 60 = 15.48hrs

Example 19

At a certain time, the position vectors r and velocity vectors v of ship A and ship B are as follows

r_A = (-2i + 3j)km v_A =(12i -4j)kmh-1 at 11.45am

r_B = (8i + 7j)km v_B = (2i – 14j)kmh-1 at 12.00 noon

Assuming velocities do not change, find

(a) The least distance between A and B in the subsequent motion

OA = $\begin{pmatrix}-2\\3\end{pmatrix}$ and v_A = $\begin{pmatrix}12\\-4\end{pmatrix} kmh^{-1}$

$R_{A(t=t)} = R_{A(t=0)} + V_A t$

12.00: $R_{A(t=\frac{1}{4})} = \begin{pmatrix}-2\\3\end{pmatrix} + \frac{1}{4}\begin{pmatrix}12\\-4\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} km$

$V_{AB} = V_A - V_B = \begin{pmatrix}12\\-4\end{pmatrix} - \begin{pmatrix}2\\-14\end{pmatrix} = \begin{pmatrix}10\\10\end{pmatrix}$

$R_{AB(t=t)} = \left(R_{A(t=0)} - R_{B(t=0)}\right) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix}1\\2\end{pmatrix} - \begin{pmatrix}8\\7\end{pmatrix}\right] + \begin{pmatrix}10\\10\end{pmatrix} t$

$R_{AB(t=t)} = \begin{pmatrix}-7+10t\\-5+10t\end{pmatrix}$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

$\begin{pmatrix}10\\10\end{pmatrix}\cdot\begin{pmatrix}-7+10t\\-5+10t\end{pmatrix} = 0$

t = 0.6hrs

$R_{AB(t=0.6)} = \begin{pmatrix}-7+10\ x\ 0.6\\-5+10\ x\ 0.6\end{pmatrix}$ =$\begin{pmatrix}-1\\1\end{pmatrix} km$

$\left|R_{AB(t=0.6)}\right| = \sqrt{(-1)^2+1^2}$ =1.4142km

(b) length of time for which A is within range, if ship B has guns within a range of up to 2km

$R_{AB(t=t)} = \begin{pmatrix}-7+10t\\-5+10t\end{pmatrix}$

$\left|R_{AB(t=t)}\right| = 2km$

$100t^2$ -12t + 35 = 0

t = 0.7hrs or t = 0.5hrs

$\begin{pmatrix}-7+10t\\-5+10t\end{pmatrix}^2 = 2^2$

$(-7+10t)^2 + (-5+10t)^2 = 4$

Time for which they are in range

= 0.7 – 0.5 = 0.2h

Example 20

At 10am, ship A moves with a constant velocity (4i+ 20j) kmh^{-1} and ship B due north of A moves with a constant velocity (-3i – 4j) kmh^{-1}.

(a) Find the velocity of A relative to B

$V_{AB} = V_A - V_B = \begin{pmatrix}4\\20\end{pmatrix} - \begin{pmatrix}-3\\-4\end{pmatrix} = \begin{pmatrix}7\\24\end{pmatrix} kmh^{-1}$

(b) If the shortest distance between the two ships is 4.2km. Find the
   (i) time to the nearest minute when they are closest together
   (ii) original distance apart at 10am
   (iii) the bearing of B from A when they are closest together.

Solution

(ii) Let a km be the distance apart at 10am

$R_{AB(t=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}0\\a\end{pmatrix}\right] + \begin{pmatrix}7\\24\end{pmatrix}t = \begin{pmatrix}7t\\-a+24t\end{pmatrix} km$

$R_{AB(t=t)} = \begin{pmatrix}7t\\-a+24t\end{pmatrix} km$

$|R_{AB(t=t)}| = 4.2km$

$\begin{pmatrix}7t\\-a+24t\end{pmatrix}^2 = (4.2)^2$

$(7t)^2 + (-a+24t)^2 = 17.64$

$625t^2 - 48at + a^2 = 17.64$ ………….(i)

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

$\begin{pmatrix}7\\24\end{pmatrix}.\begin{pmatrix}7t\\-a+24t\end{pmatrix} = 0$

$49t - 24a + 576t = 0$

$a = \frac{625t}{24}$ ………………………………….(ii)

Substituting for a in eqn. (i)

$625t^2 - 48\left(\frac{625t}{24}\right)t + \left(\frac{625t}{24}\right)^2 = 17.64$

$53.1684t^2 = 17.67$

$t = \pm 0.57h = 0.576 \times 60 = 35 minute$

(ii) $a = \frac{625t}{24} = \frac{625 \times 0.576}{24} = 15km$

(iii) $R_{AB(t=0.576)} = \begin{pmatrix}7\times 0.576\\-15+24 \times 0.576\end{pmatrix} km$

$R_{AB(t=0.576)} = \begin{pmatrix}4.032\\-1.176\end{pmatrix}$

$\theta = \tan^{-1}\left(\frac{1.176}{4.032}\right) = 16.3^0$

$Direction: E16.3^0S$

(c) length of time for which A is within range, if the visibility of ship B is within 12km

$R_{AB(t=t)} = \begin{pmatrix}7t\\-a+24t\end{pmatrix} = \begin{pmatrix}7t\\-15+24t\end{pmatrix} km$

$|R_{AB(t=t)}| = 12km$

$(7t)^2 + (-a+24t)^2 = 144$

$625t^2 - 720t + 81 = 0$

t = 1.026h or t = 0.126h

Time for which they are in range

= 1.026 – 0.126 = 0.9h

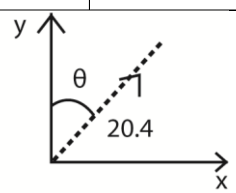
Example 21

At 12 noon a ship A is moving with constant velocity of 20.4kmh^{-1} in the direction Nθ^0E where tan θ = $\frac{1}{5}$. A second ship B is 15km due to north of A. Ship B is moving with constant velocity of 5kmh^{-1} in the direction Sα^0W, where tan α= ¼. If the shortest distance between the ships is 4.2km, find the time to the nearest minute when the distance between the ships is shortest. (12mars)

Table of results

Vector	Magnitude	Direction
V_A	$20.4kmh^{-1}$	NθE
V_B	$5kmh^{-1}$	SαW

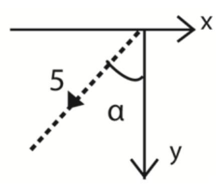
V_A ≡



$\tan\theta = \frac{1}{5}$; $\theta = 11.3^0$

$$V_A = \begin{pmatrix} 20.4\sin 11.3^0 \\ 20.4\cos 11.3^0 \end{pmatrix} = \begin{pmatrix} 4 \\ 20 \end{pmatrix}$$

V_B ≡



$\tan\alpha = \frac{1}{5}$; $\alpha = 36.87^0$

$$V_{-B} = \begin{pmatrix} -5\sin 36.87^0 \\ -5\cos 36.87^0 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

$r_A = \int V_A dt = \int \begin{pmatrix} 4 \\ 20 \end{pmatrix} dt = \begin{pmatrix} 4t \\ 20t \end{pmatrix} + c$

At t = 0, $r_A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ => c = $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Hence $r_A = \begin{pmatrix} 4t \\ 20t \end{pmatrix}$

$R_B = \int V_B dt = \int \begin{pmatrix} -3 \\ -4 \end{pmatrix} dt = \begin{pmatrix} -3t \\ -4t \end{pmatrix} + c$

At t = 0, $r_B = \begin{pmatrix} 0 \\ 15 \end{pmatrix}$ => c = $\begin{pmatrix} 0 \\ 15 \end{pmatrix}$

Hence $r_B = \begin{pmatrix} -3t \\ 15 - 4t \end{pmatrix}$

${}_Ar_B = r_A - r_B = \begin{pmatrix} 4t \\ 20t \end{pmatrix} - \begin{pmatrix} -3t \\ 15 - 4t \end{pmatrix} = \begin{pmatrix} 7t \\ 24t - 15 \end{pmatrix}$

$d_s = |{}_Ar_B| = \sqrt{(7t)^2 + (24t - 15)^5}$

but ds = 4.2

⇨ $\sqrt{(7t)^2 + (24t - 15)^5} = 4.2$

$\left(\sqrt{(7t)^2 + (24t - 15)^5}\right)^2 = 4.2^2$

$(7t)^2 + (24t - 15)^5 = 4.2^2$

$47t^2 + 576t^2 - 720t + 225 = 17.64$

$625t^2 - 720t + 207.36 = 0$

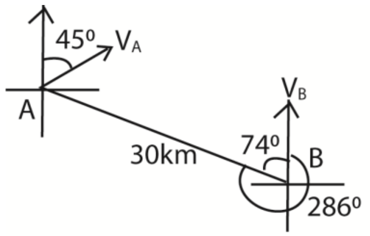
$t = \frac{720 \pm\sqrt{(-720)^2 - 4(625)(207.36)}}{2(625)}$ = 0.576hours

= 0.576 x 60 = 35minutes

Hence the time at which the distance is shortest is 12:35pm

Example 22

At noon a boat A is 30km from boat B and its direction from B is 2860. A is moving in the North-East direction at 16kmh-1 and B is moving in the north direction at 10kmh-1. Determine when they are closest to each other. What is the distance between them?



$R_{A(t=0)} = \begin{pmatrix}0\\0\end{pmatrix}, R_{B(t=0)} = \begin{pmatrix}30sin74\\-30cos74\end{pmatrix} km$

$V_{AB} = \begin{pmatrix}16sin45\\16cos45\end{pmatrix} - \begin{pmatrix}0\\10\end{pmatrix} = \begin{pmatrix}11.314\\1.314\end{pmatrix} km/h$

$R_{AB(t=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}30sin74\\-30cos74\end{pmatrix}\right] + \begin{pmatrix}11.314\\1.314\end{pmatrix}t$

$R_{AB(t=t)} = \begin{pmatrix}-28.838 + 11.314t\\8.269 + 1.314t\end{pmatrix} km$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

$\begin{pmatrix}11.314\\1.314\end{pmatrix}.\begin{pmatrix}-28.838 + 11.314t\\8.269 + 1.314t\end{pmatrix} = 0$

t = 2.43h

$R_{AB(t=2.43)}$

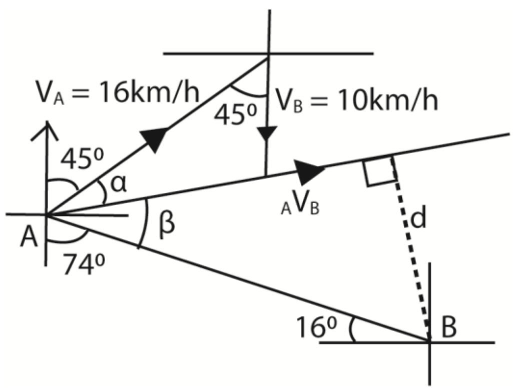
$= \begin{pmatrix}-28.838 + 11.314 \, x \, 2.43\\8.269 + 1.314 \, x \, 2.43\end{pmatrix} km$

$R_{AB(t=2.43)} = \begin{pmatrix}-1.345\\11.462\end{pmatrix} km$

$|R_{AB(t=2.43)}| = \sqrt{(-1.345)^2 + 11.462^2}$

= 11.54km

Alternatively



$V_{AB}{}^2 = V_A{}^2 + V_B{}^2 - 2V_AV_Bcos45$

$V_{AB} = \sqrt{16^2 + 10^2 - 2x16x10cos45}$

= 11.39km/h

$\frac{10}{sin\alpha} = \frac{11.39}{sin45}$

α = 38.38^0

45 + α + β + 74 = 180^0

β = 22.62^0

d = ABsinβ = 30sin22.62 = 11.54km

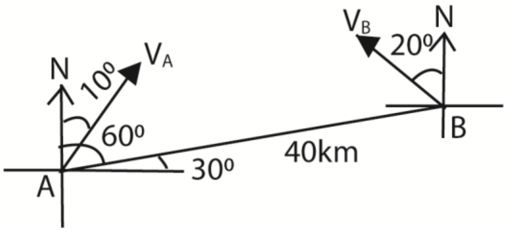
Time, t = $\frac{ABcos\beta}{V_{AB}} = \frac{30cos22.62}{11.39}$ = 2.43h

Time is 2.43h from noon or 2hour and 25.8minutes

It occurs 2.26pm at a distance 11.54km

Example 23

Two planes A and B are both flying above the Pacific Ocean. Plane A is flying on a course of 010^{0}at a speed 300kmh^{-1} and plane B is flying on a course of 340^0 at 200kmh^{-1}. At a certain instant, plane B is 40km from A. Plane A is then on a bearing of 060^0. After what time will they come closest together and what will be their minimum distance apart.



$V_{AB} = V_A - V_B$

$V_{AB} = \begin{pmatrix} 300sin10 \\ 300cos10 \end{pmatrix} - \begin{pmatrix} -200sin20 \\ 200cos20 \end{pmatrix} = \begin{pmatrix} 120.4985 \\ 107.5038 \end{pmatrix}$

$R_{A(t=0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} km, R_{B(t=0)} = \begin{pmatrix} 40cos30 \\ 40sin30 \end{pmatrix} km$

$R_{AB(t=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 40cos30 \\ 40sin30 \end{pmatrix}\right] + t\begin{pmatrix} 120.4985 \\ 107.5038 \end{pmatrix}$

$R_{AB(t=t)} = \begin{pmatrix} -34.641 + 120.4985t \\ -20 + 107.5038t \end{pmatrix}$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0$

$\begin{pmatrix} 120.4985 \\ 107.5038 \end{pmatrix}\begin{pmatrix} -34.641 + 120.4985t \\ -20 + 107.5038t \end{pmatrix} = 0$

-6324.2645 + 26076.9555t = 0

t = 0.2425h

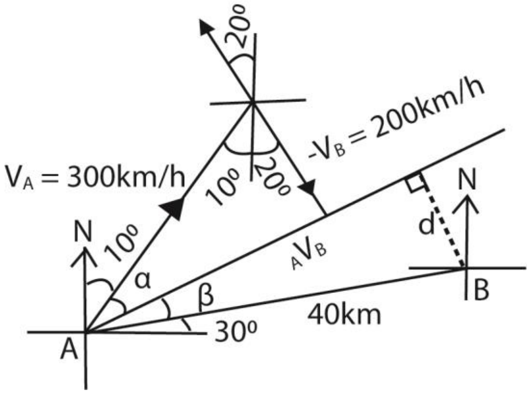
Least distance = $|R_{AB(t=0.2425)}|$

$R_{AB(t=\,0.2425)} = \begin{pmatrix} -34.641 + 120.4985x0.2425 \\ -20 + 107.5038x0.2425 \end{pmatrix}$

$R_{AB(t=\,0.2425)} = \begin{pmatrix} 5.4202 \\ 6.0692 \end{pmatrix} km$

$|R_{AB(t=0.2425)}| = \sqrt{(5.4202)^2 + (6.0692)^2} = 8.14m$

Alternatively



$V_{AB}{}^2 = V_A{}^2 + V_B{}^2 - 2V_AV_Bcos30$

$V_{AB} = \sqrt{300^2 + 200^2 - 2x300x200cos30}$

= 161.484km/h

$\frac{200}{sin\alpha} = \frac{161.484}{sin30}$

α = 38.26^0

α + β =50

β = 11.74^0

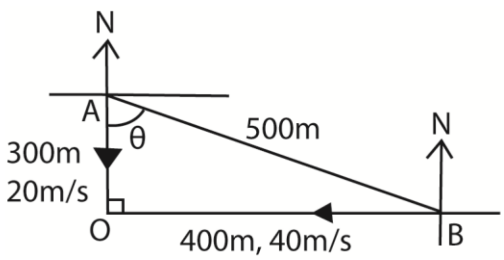
d = ABsinβ = 40sin11.74 = 8.14km

Time, t = $\frac{ABcos\beta}{V_{AB}} = \frac{40cos11.74}{161.484} = 0.2425h$

Example 24

At a given instant two cars are at a distance 300m and 400m from a point of intersection O of two roads crossing at right angles and are approaching O at uniform speeds of 20m/s and 40m/s respectively. Find

(i) Initial distance between the two cars
(ii) shortest distance between the cars
(iii) time taken to reach this point



$AB = \sqrt{300^2 + 400^2} = 500m$

$\theta = \tan^{-1}\left(\frac{400}{300}\right) = 53.1^0$

$R_{A(t=0)} = \begin{pmatrix}0\\0\end{pmatrix} km, R_{B(t=0)} = \begin{pmatrix}50cos53.1\\50sin53.1\end{pmatrix} km$

$V_{AB} = V_A - V_B$

$V_{AB} = \begin{pmatrix}0\\-20\end{pmatrix} - \begin{pmatrix}-40\\0\end{pmatrix} = \begin{pmatrix}40\\-20\end{pmatrix} ms^{-1}$

$R_{AB(t=t)} = \left(R_{A(t=0)} - R_{B(t=0)}\right) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}500cos53.1\\500sin53.1\end{pmatrix}\right] + t\begin{pmatrix}40\\-20\end{pmatrix}$

$R_{AB(t=t)} = \begin{pmatrix}-399.842 + 40t\\300.21 - 20t\end{pmatrix}$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0$

$\begin{pmatrix}40\\-20\end{pmatrix}.\begin{pmatrix}-399.842 + 40t\\300.21 - 20t\end{pmatrix} = 0$

-21,997.88 +2000t = 0

t = 11s

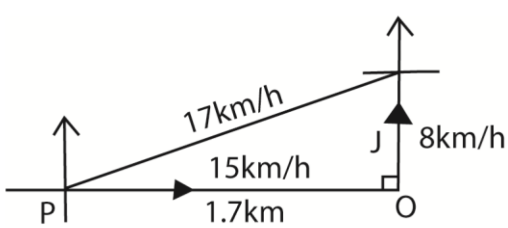
$R_{AB(t=11)} = \begin{pmatrix}-399.842 + 40x11\\300.21 - 20\ x\ 11\end{pmatrix} = \begin{pmatrix}40.158\\80.21\end{pmatrix} m$

$R_{AB(t=11)} = \sqrt{(40.158)^2 + (80.21)^2}$ =89.701m

Example 25

A road running north-south crosses a road running east-west at a junction O. Initially Paul is on the east-west, 1.7km west of O and is cycling towards O at15mk/h. At the same time John is at O cycling due north at 8km/h. If Paul and John do not alter their velocities, find the

(i) relative velocity of Paul to John
(ii) shortest distance between Paul and John



$R_{P(t=0)} = \begin{pmatrix}0\\0\end{pmatrix} km, R_{J(t=0)} = \begin{pmatrix}1.7\\0\end{pmatrix} km$

$V_{PJ} = V_P - V_J$

$V_{PJ} = \begin{pmatrix}15\\0\end{pmatrix} - \begin{pmatrix}0\\8\end{pmatrix} = \begin{pmatrix}15\\-8\end{pmatrix} ms^{-1}$

$|V_{PJ}| = \sqrt{15^2 + (-8)^2}$ = 17km/h

$R_{PJ(t=t)} = \left(R_{P(t=0)} - R_{J(t=0)}\right) + (V_{PJ})t$

$R_{PJ(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}1.7\\0\end{pmatrix}\right] + \begin{pmatrix}15\\-8\end{pmatrix} t$

$R_{PJ(t=t)} = \begin{pmatrix}-1.7 + 15t\\-8t\end{pmatrix}$

For minimum distance: $V_{PJ}.R_{PJ(t=t)} = 0$

$\begin{pmatrix}15\\-8\end{pmatrix}.\begin{pmatrix}-1.7 + 15t\\-8t\end{pmatrix}$ =0

-25.5 +289t = 0

t = 0.088h

Least distance = $|R_{PJ(t=0.088)}|$

$R_{PJ(t=0.088)} = \begin{pmatrix}-1.7 + 15\ x\ 0.088\\-8x\ 0.88\end{pmatrix} = \begin{pmatrix}-0.38\\-0.704\end{pmatrix}$

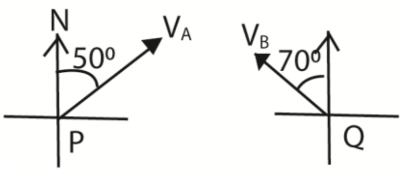
$|R_{PJ}| = \sqrt{(-0.38)^2 + (-0.704)^2}$ = 0.8km

Example 26

Two airship P and Q are 100km apart, P being west of Q. Two Helicopters A and B fly simultaneously from P and Q respectively, at 11.00a.m. Helicopter A is flying with a constant speed of 400km/h in the direction N50^{0}E. Helicopter B flying at a constant speed of 500km in the direction N70^{0}W. Find the

(i) Time when the helicopters are closest together.

(ii) closest distance between the helicopters



$R_{A(t=0)} = \begin{pmatrix}0\\0\end{pmatrix} km, R_{B(t=0)} = \begin{pmatrix}100\\0\end{pmatrix} km$

$V_{AB} = V_A - V_B$

$V_{AB} = \begin{pmatrix}400sin50\\400cos50\end{pmatrix} - \begin{pmatrix}-500sin70\\500cos70\end{pmatrix}$

$V_{AB} = \begin{pmatrix}776.264\\86.105\end{pmatrix} kms^{-1}$

Time when they are closest

= 11.00 + 8= 11.08am

Least distance = $|R_{AB(t=0.1273)}|$

$R_{AB(t=0.1273)} \begin{pmatrix}-100 + 776.264\ x\ 0.1273\\86.105\ x\ 0.1273\end{pmatrix} km$

$R_{AB(t=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}100\\0\end{pmatrix}\right] + t\begin{pmatrix}776.264\\86.105\end{pmatrix}$

$R_{AB(t=t)} = \begin{pmatrix}-100 + 776.264t\\86.105t\end{pmatrix} km$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0$

$\begin{pmatrix}776.264\\86.105\end{pmatrix}\begin{pmatrix}-100 + 776.264t\\86.105t\end{pmatrix}.$

609999.869t =77624.4

t =

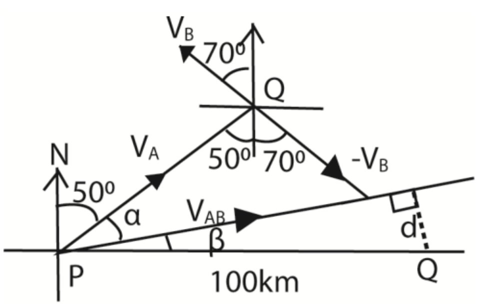
$R_{AB(t=0.1273)} = \begin{pmatrix}-1.182\\10.961\end{pmatrix} km$

$|R_{AB(t=0.1273)}| = \sqrt{(-1.182)^2 + 10.961^2}$

=11.0247km

Closest distance =11.025km

Alternatively



$V_{AB}{}^2 = V_A{}^2 + V_B{}^2 - 2V_AV_Bcos120$

$V_{AB} = \sqrt{400^2 + 500^2 - 2x400x500cos120}$

= 781.025km/h

$\frac{500}{sin\alpha} = \frac{781.025}{sin120}$

α = 33.67^0

α + β + 50 =90

β = 6.33^0

d = PQsinβ = 100sin6.33 = 11.025km

Time, t = $\frac{PQcos\beta}{V_{AB}} = \frac{100cos6.33}{781.025}$

t = 0.1273h = 0.1273 x 60= 8minutes

Time when they are closest

= 11.00 + 8= 11.08am

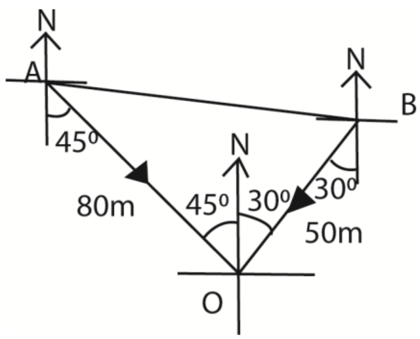
Example 27

Car A is 80m North-West of point O, Car B is 50m N300E of O. Car A is moving at 20m/s on a straight road towards O. Car B is moving at 10m/s on another straight road towards O. Determine the

(i) Initial distance between the cars

(ii) Velocity of A relative to B

(iii) shortest distance between the two cars as they approach O



$R_{A(t=0)} = \begin{pmatrix} -80cos45 \\ 80sin45 \end{pmatrix}, R_{B(t=0)} = \begin{pmatrix} 50sin30 \\ 50cos30 \end{pmatrix}$

$R_{AB(t=0)} = \begin{pmatrix} -80cos45 \\ 80sin45 \end{pmatrix} - \begin{pmatrix} 50sin30 \\ 50cos30 \end{pmatrix} = \begin{pmatrix} -81.5685 \\ 13.2673 \end{pmatrix}$

$[R_{AB(t=0)}] = \sqrt{(-81.5685)^2 + 13.2673^2}$ = 82 .6404m

(ii) $V_A = \begin{pmatrix} 20sin45 \\ -20cos45 \end{pmatrix}, V_B = \begin{pmatrix} -10sin30 \\ -10cos30 \end{pmatrix}$

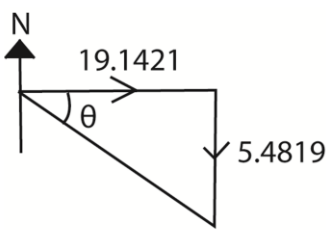
$V_{AB} = V_A - V_B$

$V_{AB} = \begin{pmatrix} 20sin45 \\ -20cos45 \end{pmatrix} - \begin{pmatrix} -10sin30 \\ -10cos30 \end{pmatrix}$

$V_{AB} = \begin{pmatrix} 19.1421 \\ -5.4819 \end{pmatrix}$

$|V_{AB}| = \sqrt{(19.1421)^2 + (-5.4819)^2}$

= 19.9116m/s



θ=$\tan^{-1}\frac{5.4819}{19.1421}$ =15.98^0

Direction: E15.98^0S

(iii)

$R_{AB(t=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

$R_{AB(t=t)} = \begin{pmatrix} -81.5685 \\ 13.2673 \end{pmatrix} + t\begin{pmatrix} 19.1421 \\ -5.4819 \end{pmatrix}$

$R_{AB(t=t)} = \begin{pmatrix} -81.5685 + 19.1421t \\ 13.2673 - 5.4821t \end{pmatrix} km$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0$

$\begin{pmatrix} 19.1421 \\ -5.4819 \end{pmatrix}.\begin{pmatrix} -81.5685 + 19.1421t \\ 13.2673 - 5.4821t \end{pmatrix} = 0$

t = 4.1216s

Least distance = $|R_{AB(t=4.1216s)}|$

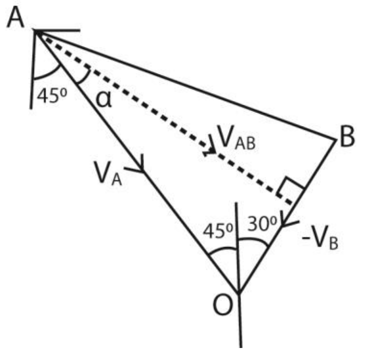
$R_{AB(t=t)} = \begin{pmatrix} -81.5685 + 19.1421\ x\ 4.1216 \\ 13.2673 - 5.4821\ x\ 4.1216 \end{pmatrix} m$

$R_{AB(t=t)} = \begin{pmatrix} -2.672 \\ -9.328 \end{pmatrix} m$

$|R_{AB(t=t)}| = \sqrt{(-2.672)^2 + (-9.328)^2}$ =9.728m

Alternatively

Method II: Using geometric approach



Using cosine rule

$|V_{AB}|^2 = 20^2 + 10^2 - 2\ x\ 20\ x10 \cos 75^0$

$|V_{AB}| = 19.912ms^{-1}$

$\frac{|V_{AB}|}{\sin 75^0} = \frac{10}{\sin \alpha}$

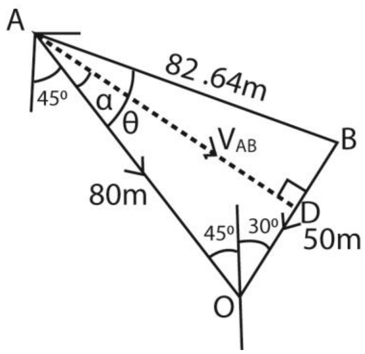
$\alpha = 29.02^0$

$45^0 + 29.02^0 = 74.02^0$

∴ The velocity of A relative to B is $19.912ms^{-1}$ due $S74.02^0E$

The shortest distance between the two cars as they approach O (04marks)

Using geometrical approach



$\frac{50}{\sin \theta} = \frac{82.64}{\sin 75^0}$; $\theta = 35.76^0$

$< BAD = 35.76 - 29.02 = 6.74^0$

$\sin 6.74^0 = \frac{|r_{AB}|}{AB}$

${}_Ar_B = 82.64 \sin 6.74^0 = 9.699m$

∴ The shortest distance between the two cars they approach O is 9.699m

Revision exercise 3

1. 1.At 8am ship A and ship Bare 11km apart with B due west of A. a and B move with a constant velocities (-4i +3j)kh/h and (2i + 4j)km/h respectively. Find the the

(i) least distance between the two ship in the subsequent motion[1.81km]

(ii) time to the nearest minute at which this situation occurs[9.47am]

2. 2. At 7.30am, two ship A and B are 8km apart with B due north of A. A and B move with constant velocities (12j)km/h and (-5i)km/h respectively. Find the

(i) least distance between the two ship in the subsequent motion[3.08km]

(ii) time to the nearest minute at which this situation occurs[8.04am]

3. A and B are two tankers at 13.00hrs, tanker B bas position vector of (4i+8j)km relative to A. A and B move with constant velocities (6i + 9j)km/h and (-3i + 6j)km/h respectively. Find the
   (i) least distance between the two ship in the subsequent motion[6.32km]
   (ii) time to the nearest minute at which this situation occurs[13.40hrs]

4. At 12 noon the position vectors r and velocity vectors v of two ship A and B are as follows
   r_A = (5i + j)km, V_A = (7i + 3j)km/h and r_B = (8i + 7j)km, V_B = (2i –j)km/h
   (i) Assuming velocities do not change , find the least distance between the ships in subsequent motion [2.81]
   (ii) Find the time when their distance of closest approach occur [12.57pm]
5. At a certain time, the position vectors r and velocity vectors V of two ship A and B are as follows

   r_A = (3i + j)km, V_A = (2i + 3j)km/h at 11.00am

   r_B = (2i - j)km, V_B = (3i + 7j)km/h at 12.00noon

   Assuming velocities do not change; find the

   (i) The position vector of A at noon [5i + 4j]
   (ii) Distance between the ships at 12.00 noon [5.83km]
   (iii) The least distance between A and B in the subsequent motion[1.7km]
   (iv) Time at which the least separation occurs [1.21pm]

6. At 12 noon, the position vectors r and velocity vectors V of two battle ship A and battle B are as follows

   r_A = (13i + 5j)km, V_A = (3i - 10j)km/h

   r_B = (3i - 5j)km, V_B = (15i + 14j)km/h

   (i) Assuming the velocities do not change, find the least distance between the ships in subsequent motion [4.47km]
   (ii) The battle ship has guns with a range of up to 5km, find the length of time during which the cruiser is within range of the battle ships [10minutes]
7. At tine t = 0 the position vectors r and velocity vectors V of two battle ship A and battle B are as follows
   $r_A = \begin{pmatrix}1\\2\\3\end{pmatrix}m, V_A = \begin{pmatrix}-6\\0\\1\end{pmatrix}m/s \; and \; r_B = \begin{pmatrix}4\\-14\\1\end{pmatrix}m, V_A = \begin{pmatrix}-5\\1\\7\end{pmatrix}m/s$
   Assuming velocities do not change, find
   (i) The potion vectors of B relative to A at time t seconds $\left[\left(\begin{pmatrix}7\\2\\2\end{pmatrix} + \begin{pmatrix}9\\13\\-7\end{pmatrix}t\right)km\right]$
   (ii) The least distance between the ships in the subsequent motion [15.9m]
   (iii) The time taken to the closest distance $\left[\frac{25}{33}s\right]$
8. At time t = 0 the position vectors r and velocity vectors V of two battle ship A and battle B are as follows
   r_A = (3i + j + 5k)m, V_A = (4i + j – 3k)m/s
   r_B = (i - 3j+ 2k) m, V_B = (i + 2j + 2k)m/s
   Assuming velocities do not change, find
   (i) The position vector of B relative to A at time t second $\left[\begin{pmatrix}2+3t\\4-t\\3-5t\end{pmatrix}m\right]$

(ii) The value of t when A and B are closed $\left(\frac{13}{35}\right)$

(iii) Least distance between A and B [4.917m]

9. At tine t = 0 the position vectors r and velocity vectors V of two battle ship A and battle B are as follows

$$r_A = (\beta)m, V_A = \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} m/s \; and \; r_B = (2\beta)m, V_A = \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} m/s$$

Where β is a constant, assuming velocities do not change show that the least distance between the ships in the subsequent motion is $\frac{\beta}{73}$ and their distance of closest approach is $\frac{6\beta\sqrt{2}}{\sqrt{73}}$.

10. A lizard on a wall at point A, has a position vector $r_A = \begin{pmatrix} 65 \\ 40 \\ 0 \end{pmatrix} cm$. At time t= 0 seconds a fly has a position vector $r_F = \begin{pmatrix} 37 \\ 16 \\ 22 \end{pmatrix} cm$ and velocity vector $V_F = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} cm/s$

If the fly were to continue with this velocity, find the closest distance it would come to the lizard and the value of t when it occurs [$\sqrt{374}m$, 7s]

11. A particle P move with constant velocity (2i + 3j + 8k)m/s passes a point with position vector (6i – 11j + 4k)m. At the same instant particle Q passes through a point whose position vector is (i -2j + 5k)m moving at constant velocity of (3i + 4j- 7k)m/s. Find
    (a) Position of Q relative to P at that instant. [10.344m]
    (b) Shortest distance between the particles [10.32m]
    (c) Time that elapses before the particles are nearest to each other [0.0485s]

12. Two particles P and Q move with constant velocities (4i + j – 2k)m/s and (6i + 3k)m/s respectively. Initially P is at a point whose position vector is (i – 20j + 21k)m and Q is at a point whose position vector is (i + 3k)m. find
    (a) Time for which the distance between P and Q is least [2.2s]
    (b) Distance of P from the origin at the time when the distance between P and Q is least [28.8m]
    (c) Least distance between P and Q [24.14m]

Course of closest approach

If A is to pass as close as possible to B, then velocity of A must be perpendicular to the relative velocity

$V_{AB}.V_A = 0$

Example 28

Two particles P and Q initially at positions (3i + 2j)m and (13i + 2j)m respectively begin moving. Particle P moves with a constant velocity (2i + 6j)m/s. A second particle Q moves with a constant velocity of (5j)m/s

(a) Find
  (i) Time when the particles are closest together.
  (ii) Bearing of particle O from Q when they are closest to each other.

(b) Given that half the time, the particle are moving closest to each other, particle P reduces its speed to half its original speed, in the direction to approach particle Q and the velocity of particle Q remains unchanged, find the direction of particle P.

Solution

$r_{P(t=0)} = \binom{3}{2}m, r_{Q(t=0)} = \binom{13}{2}m$

$V_{PQ} = V_P - V_Q = \binom{2}{6} - \binom{0}{5} = \binom{2}{1}m/s$

$r_{PQ(t=t)} = (r_{P(t=0)} - r_{Q(t=0)}) + (V_{PQ})t$

$r_{PQ(t=t)} = \binom{3}{2} - \binom{13}{2} + \binom{2}{1}t = \binom{-10+2t}{t}m$

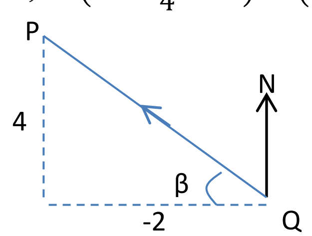
For minimum distance $V_{PQ}.r_{PQ(t=t)}$=0

$\binom{2}{1}.\binom{-10+2t}{t} = 0$

-20 +4t +t = 0

t = 4s

(ii) $r_{PQ(t=4)} = \binom{-10+2\ x\ 4}{4} = \binom{-2}{4}$



$\beta = \tan^{-1}\frac{4}{2}$ =63.43^0

The bearing of P from Q = (270 + 63.43)

=333.43^0

(b) At t = 2s, $V_P = \binom{1}{3}m/s$

Let P move at angle θ to x-axis

$V_{PQ} = \sqrt{10}\binom{cos\theta}{sin\theta} - \binom{0}{5} = \binom{\sqrt{10}co\theta}{\sqrt{10}sin\theta - 5}$

If P is to approach Q; $8sin\theta$

$\binom{\sqrt{10}co\theta}{\sqrt{10}sin\theta - 5}.\binom{\sqrt{10}cos\theta}{\sqrt{10}sin\theta}$ = 0

10cos^{2}θ + 10sin^{2}θ -5$\sqrt{10}sin\theta$ =0

θ =sin$^{-1}\frac{10}{5\sqrt{10}}$ = 39.2^0

Direction: N50.8^{0}E

Example 29

A motor boat B is travelling at a constant velocity of 10m/s due east and motor boat A is travelling at a constant speed of 8m/s. Initially A and B are 600m apart with A due south of B. Find

(a) course that A should set to get close as possible to B

Let A move at an angle θ to x-axis

$V_{AB} = \binom{8cos\theta}{8sin\theta} - \binom{10}{0} = \binom{8cos\theta - 10}{8sin\theta}$

If A is to approach B; $V_{AB}.V_A = 0$

$\binom{8cos\theta}{8sin\theta}.\binom{8cos\theta - 10}{8sin\theta}$ =0

θ = cos$^{-1}\frac{64}{80}$ =36.9^0

Direction: N53.1^{0}E or E36.9^{0}N

(ii) Closest distance and time taken for the situation to occur

$r_{A(t=0)} = \begin{pmatrix}0\\0\end{pmatrix} m, r_{B(t=0)} = \begin{pmatrix}0\\600\end{pmatrix} m$

$r_{AB(t=t)} = (r_{A(t=0)} - r_{B(t=0)}) + (V_{AB})t$

$r_{AB(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}0\\600\end{pmatrix}\right] + \begin{pmatrix}8cos36.9 - 10\\8sin36.9\end{pmatrix} t$

$r_{AB(t=t)} = \begin{pmatrix}-3.603t\\-600 + 4.803t\end{pmatrix}$

For minimum distance: $V_{AB}.r_{AB(t=t)} = 0$

$\begin{pmatrix}-3.603\\4.803\end{pmatrix}.\begin{pmatrix}-3.603t\\-600 + 4.803t\end{pmatrix}$ =0

t = 80s

$r_{AB(t=80)} = \begin{pmatrix}-3.603\ x\ 80\\-600 + 4.803x80\end{pmatrix}$

$= \begin{pmatrix}-288.24\\-215.76\end{pmatrix}$

Least distance,d = $|r_{AB(t=80)}|$

$|r_{AB(t=80)}| = \sqrt{(-288.24)^2 + (-215.76)^2}$

d = 360m

Example 30

A motor boat B is travelling at constant velocity of 14km/h due north and a motor boat A is travelling at constant speed 12km/h. Initially A and B are 5.2km apart with A due west of B. Find

(i) Couse that A should set in order to get as close as possible to B

Let A move at an angle θ to x-axis

$V_{AB} = \begin{pmatrix}12cos\theta\\12sin\theta\end{pmatrix} - \begin{pmatrix}0\\13\end{pmatrix} = \begin{pmatrix}12cos\theta\\12sin\theta - 13\end{pmatrix}$

If A is to approach B; $V_{AB}.V_A = 0$

$\begin{pmatrix}12cos\theta\\12sin\theta - 13\end{pmatrix}.\begin{pmatrix}12cos\theta\\12sin\theta\end{pmatrix}$=0

θ = $\sin^{-1}\frac{12}{13}$ =67.4^0

Direction: N22.6^0E or 67.4^0N

(ii) Closest distance and time taken for the situation to occur

$r_{A(t=0)} = \begin{pmatrix}0\\0\end{pmatrix} m, r_{B(t=0)} = \begin{pmatrix}5.2\\0\end{pmatrix} m$

$r_{AB(t=t)} = (r_{A(t=0)} - r_{B(t=0)}) + (V_{AB})t$

$r_{AB(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}5.2\\0\end{pmatrix}\right] + \begin{pmatrix}12cos67.4\\12sin67.4 - 13\end{pmatrix} t$

$r_{AB(t=t)} = \begin{pmatrix}-5.2 + 4.612t\\-1.921t\end{pmatrix}$

For minimum distance: $V_{AB}.r_{AB(t=t)} = 0$

$\begin{pmatrix}4.612\\-1.921\end{pmatrix}.\begin{pmatrix}-5.2 + 4.612t\\-1.921t\end{pmatrix}$ =0

t = 0.961h

$r_{AB(t=0.961)} = \begin{pmatrix}-5.2 + 4.612\ x\ 0.961\\-1.921\ x\ 0.961\end{pmatrix}$

$= \begin{pmatrix}-0.7725\\-1.8442\end{pmatrix}$

Least distance, d = $|r_{AB(t=0.961)}|$

$|r_{AB(t=0.961)}| = \sqrt{(-0.7725)^2 + (-1.8442)^2}$

d = 2km

Revision exercise 4

1. A ship A is moving with constant velocity of 18km/h in a direction N55^{0}E and is initially 6km from a second ship B, the bearing of A from B being N25^{0}W. If B moves with a constant speed of 15km/h. Find
(a) Course that B should set in order to get as close as possible to A [N21.4^{0}E]
(b) Closest distance and time taken for the situation to occur.[4.135km, t=0.437h]
2. Two aircraft A and B are flying at the same altitude with A initially 10km due north of B and flying at constant speed of 300m/s on a bearing of 060^0. If B flies at constant speed of 200m/s, find
   (a) Course that B should set in order to get as close as possible to A [E78.4^{0}N]
   (b) Closest distance and time taken for the situation to occur.[9.79km, t= 9.12s]
3. At 8am two boats A and B are 5.2km apart with A due west of B, and B travelling due north at a steady speed 13km/h. If A travels with a constant speed of 12km/h, show that for A to get as close as possible to B, A should set a course of Nθ0E where sinθ = $\frac{5}{13}$. Find the closest distance and time at which it occurs. [2km, 8.57 am]
4. Two aircraft A and B are flying at the same altitude with A initially 5km due north of B and B flying at constant speed of 300m/s on bearing of 060^0.If A flies at constant speed of 200m/s, find
   (a) Course that A should set in order to get as close as possible to B [108.2^0]
   (b) Time taken for the situation to occur.[5.4min]

5. A ship A moving with a constant speed of 24km/h in the direction N40^{0}E and is initially 10km from a second ship B, the bearing of A from B being N300W. If B moves with a constant speed of 22km/h; find
   (a) Course that B should set in order to get as close as possible to A [N16.4^{0}E]
   (b) Closest distance and time taken for the situation to occur.[6.89km, 45min]

Interception and collision

Consider two bodies A and B moving with V_A and V_B from points with position vectors OA and OB

Position of A after time t is

$r_{A(t=t)} = r_{A(t=0)} + t\ x\ V_A$

Position of B after time t is

$r_{B(t=t)} = r_{B(t=0)} + t\ x\ V_B$

For collision to occur $r_{A(t=t)} = r_{B(t=t)}$

$r_{A(t=0)} + t\ x\ V_A = r_{B(t=0)} + t\ x\ V_B$

$(r_{A(t=0)} - r_{B(t=0)}) + t(V_{AB})=0$

Hence $r_{AB(t=t)} = 0$

Example 31

The position vectors $r_A = (5i - 3j + 4k)m$ and $r_B = (7i + 5j - 2k)m$ are for two particles with velocities $V_A = (2i + 5j + 3k)m/s$ and $V_B = (-3i - 55j + 18k)m/s$ respectively. Show that if the velocities remain constant, a collision will occur

Solution

$r_{A(t=0)} + t\ x\ V_A = r_{B(t=0)} + t\ x\ V_B$

$$\begin{pmatrix}5\\-3\\4\end{pmatrix} + t\begin{pmatrix}2\\5\\3\end{pmatrix} = \begin{pmatrix}7\\5\\-2\end{pmatrix} + t\begin{pmatrix}-3\\-15\\18\end{pmatrix}$$

$$\begin{pmatrix}-2\\-8\\6\end{pmatrix} = \begin{pmatrix}-5\\-20\\15\end{pmatrix}t$$

Along i direction: -2 = -5t; t = 0.4s

Along j direction: -8= -20t; t = 0.4s

Along k direction: 6 = 15t; t = 0.4s

Since t is the same in all directions, collision occurred

Example 32

At 12 noon the position vectors r and velocity vectors V of two ships A and B are as follows

$r_A = (i + 7j)m, V_A = (6i + 2j)m/s,\ r_B = (6i + 4j)m\ and\ V_B = (-4i + 8j)m/s$

Assuming velocities do not change

(i) Show that collision will occur
(ii) Find the time at which collision occurs
(iii) Find the position vector of the location during collision

Solution

$r_{A(t=0)} + t\ x\ V_A = r_{B(t=0)} + t\ x\ V_B$

$$\binom{1}{7} + t\binom{6}{2} = \binom{6}{4} + t\binom{-4}{8}$$

$$\binom{-5}{3} = \binom{-10}{6}t$$

Along the i direction: -5 = -10t; t = 0.5h

Along the j direction: 3 = 6t; t = 0.5h

Since t is the same in all directions

Collision occurred

(ii) time it occurred = 12:00 + 0.5 x 60

= 12:30pm

(iii) How far each had travelled

$r_{A(t=t)} = r_{A(t=0)} + t\ x\ V_A$

$$r_{A(t=0.5)} = \binom{1}{7} + 0.5\binom{6}{2} = \binom{4}{8}km$$

Example 33

At 11:30am a battle ship is at a place with position vector (-6i + 12j)km and is moving with velocity vector (16i – 4j)km/h. At 12:00 noon a cruiser is at a place with position vector (12i – 15j) and is moving with velocity vector (8i + 16j)km/h. Assuming velocities do not change

(i) Show that collision will occur
(ii) Find the time at which collision occurs
(iii) Find the position vector of the location of collision

$r_{A(t=t)} = r_{A(t=0)} + t\ x\ V_A$

At 12:00: $r_{A(t=0.5)} = \binom{-6}{12} + 0.5\binom{16}{-4} = \binom{2}{10}$

For collision to occur

$r_{A(t=0)} + t\ x\ V_A = r_{B(t=0)} + t\ x\ V_B$

$\binom{2}{10} + t\binom{16}{-4} = \binom{12}{-15} + t\binom{8}{18}$

$\binom{-10}{25} = \binom{-8}{20}t$

Along the i direction: -10 = -8t; t = 1.25h

Along the j direction: 15 = 20t; t = 1.25h

Since t is the same in all directions collision occurred

(ii) time it occurred = 11:30 + 1.25 x 60

= 12:45pm

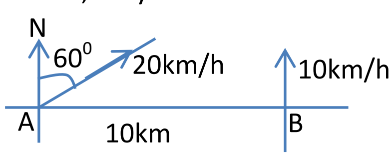
(iii) How far each had travelled

$r_{A(t=t)} = r_{A(t=0)} + t\ x\ V_A$

$r_{A(t=0.5)} = \binom{2}{10} + 1.25\binom{16}{-4} = \binom{22}{5}km$

Example 34

At 12:30 noon two ships A and B are 10km apart with B due east of A. A is travelling N600E at a speed of 12km/h and ship B is travelling due north at 10km/h. Show that, is the two ships do not change their velocities, they collide and fins to the nearest minute when collision occurs.



For collision to occur

$r_{A(t=0)} + t\ x\ V_A = r_{B(t=0)} + t\ x\ V_B$

$\binom{0}{0} + t\binom{20sin60}{20cos60} = \binom{10}{0} + t\binom{0}{10}$

$\binom{10\sqrt{3}}{10}t = \binom{10}{0} + t\binom{0}{10}$

Along the i direction: $10\sqrt{3}t$ = 10;

t = 0.5774h

Along the j direction: 10t = 10t

∴ t =0.5774 x 60 = 35minute

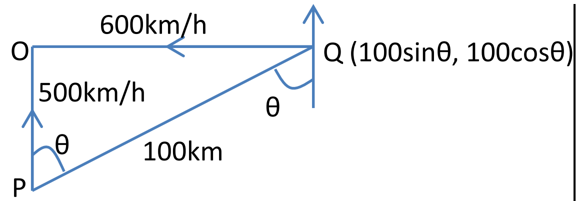
Collision occurred at 12:35minutes

Example 35

Two aircraft P and Q are flying at the same height. P is flying due north at 500km/h while Q is flying due west at 600km/h. When the aircrafts are 10km apart, the pilots realize that they are about to collide. The pilot of P changes Course to 345^0 and maintains the speed of 500km/h. The pilot Q maintains his course but increases speed. Determine the

(i) Distance each aircraft would have travelled if the pilots had not realized that they were about to collide

Solution



θ = $\tan^{-1}\frac{600}{500}$ = 50.2^0

For collision to occur

$r_{P(t=0)} + t\ x\ V_P = r_{Q(t=0)} + t\ x\ V_Q$

$\binom{0}{0} + t\binom{0}{500} = \binom{100sin\theta}{100cos\theta} + t\binom{-600}{0}$

$\binom{0}{0} + t\binom{0}{500} = \binom{100sin50.2}{100cos50.2} + t\binom{-600}{0}$

Along i direction: 0 = 100sin50.2 – 600t

t = 0.128h

Distance moved by P

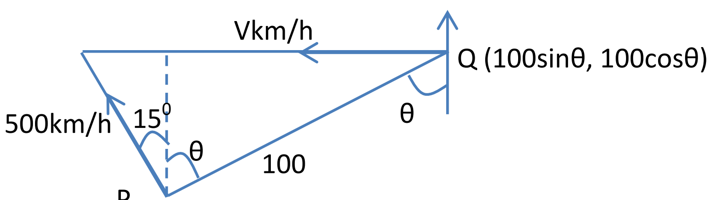
$\binom{0}{0} + 0.128\binom{0}{500} = \binom{0}{64} = 64km$

Distance moved by Q

$\binom{100sin\theta}{100cos\theta} + 0.128\binom{-600}{0} = \binom{0.0284}{64.011}$

=64.011km

(ii) New speed beyond which the aircraft Q must fly in order to avoid collision



For collision to occur

$r_{P(t=0)} + t\ x\ V_P = r_{Q(t=0)} + t\ x\ V_Q$

$\binom{0}{0} + t\binom{-500\sin 15}{500cos15} = \binom{100sin\theta}{100cos\theta} + t\binom{-V}{0}$

$\binom{0}{0} + t\binom{-500\sin 15}{500cos15} = \binom{100sin50.2}{100cos50.2} + t\binom{-V}{0}$

Along j direction: 500cos15t = 64.011

t = 0.1325h

Along i direction:

-500sin15t = 76.8284 – Vt

V x 0.1325 = 76.8284 + 500sin15 x 0.1325

V =709.2837km/h

Course of interception

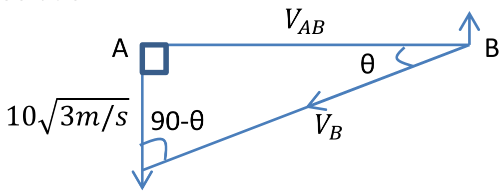
Suppose particle A with speed V_A is to intercept particle B with speed V_B, then

- Draw a sketch diagram showing the initial position and velocities of the two particles
- For interception to occur, the relative velocity must be in the direction of the initial displacement of the particles.

Example 36

At an instant a body A travelling south at $10\sqrt{3}$m/s is 150m west of B. Show that B will intercept A if B is travelling $S30^0W$ at 20m/s and find the time that elapses before collision occurs.

Solution



$A = \begin{pmatrix}0\\0\end{pmatrix}m \quad B = \begin{pmatrix}150\\0\end{pmatrix}m$

$V_A = \begin{pmatrix}0\\-10\sqrt{3}\end{pmatrix}m/s \quad V_B = \begin{pmatrix}-20cos\theta\\-20sin\theta\end{pmatrix}m/s$

OA + t x v_A = OB + t x v_B

$\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}0\\-10\sqrt{3}\end{pmatrix}t = \begin{pmatrix}150\\0\end{pmatrix} + \begin{pmatrix}-20cos\theta\\-20sin\theta\end{pmatrix}t$

j: 20sinθ =$10\sqrt{3}$

θ = $\sin^{-1}\frac{\sqrt{3}}{2}$ = 60^0

Bearing S30^0W

i: t = $\frac{150}{20cos\theta} = \frac{150}{20\cos 60} = 15s$

Alternatively

$\frac{sin90}{20} = \frac{sin\theta}{10\sqrt{3}}$

θ = 60^0: bearing S30^0W

Also, $\frac{sin90}{20} = \frac{\sin(90-\theta)}{V_{AB}}$

$V_{AB} = 10m/s$

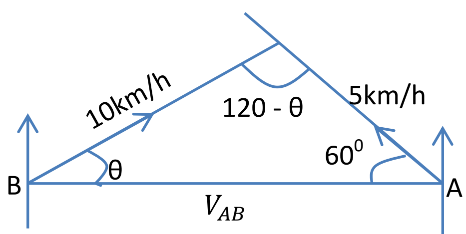
t = $\frac{AB}{VA_{AB}} = \frac{150}{10} = 15s$

Example 37

At 9:00am two ships A and B are 15km apart with B on a bearing of 270^0 from A. Ship A moves at 5km/h on a bearing of 330^0. If the maximum speed of B is 10km/h. Find the

(i) Direction B should set in order to intercept A as soon as possible
(ii) Time taken for the interception to occur.

Solution



$B = \begin{pmatrix}0\\0\end{pmatrix}km \quad A = \begin{pmatrix}15\\0\end{pmatrix}km$

$V_B = \begin{pmatrix}10cos\theta\\10sin\theta\end{pmatrix}km/h \quad V_A = \begin{pmatrix}-5cos60\\5sin60\end{pmatrix}km/h$

OB + t x v_B = OA + t x v_A

$\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}10cos\theta\\10sin\theta\end{pmatrix}t = \begin{pmatrix}15\\0\end{pmatrix} + \begin{pmatrix}-5cos60\\5sin60\end{pmatrix}t$

j: 10sinθ =$5sin60$

θ = $\sin^{-1}\frac{5sin60}{10}$ = 25.7^0

Bearing E25.7^0N

i: t = $\frac{15}{10cos\theta+2.5} = \frac{15}{2.5+10\cos 25.7} = 1.303h$

t = 1.303 x 60 = 78mins

Alternatively

$\frac{sin60}{10} = \frac{sin\theta}{5}$

θ = 25.7^0: bearing E25.7^0N

Also, $\frac{sin90}{10} = \frac{\sin(120-\theta)}{V_{AB}}$

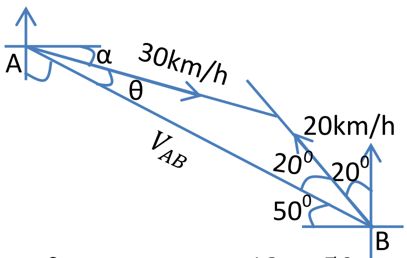
$V_{AB} = 11.515km/h$

t = $\frac{AB}{VA_{AB}} = \frac{15}{11.515}\ x\ 60 = 78minutes$

Example 38

At 12:00 noon two ship A and B are 12km apart with B on a bearing of 140^0 from A. Ship A moves at 30km/h to intercept B which is travelling at 20km/h on a bearing of 340^0. Find the

(i) Direction A should set in order to intercept B (ii) time taken for the interception to occur.



$A = \begin{pmatrix}0\\0\end{pmatrix} km \quad B = \begin{pmatrix}12cos50\\-12sin50\end{pmatrix} km$

$V_A = \begin{pmatrix}30cos\alpha\\30sin\alpha\end{pmatrix} km/h \quad V_B = \begin{pmatrix}-20cos20\\20sin20\end{pmatrix} km/h$

OA + t x v_A = OB + t x v_B

$\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}30cos\alpha\\30sin\alpha\end{pmatrix} t = \begin{pmatrix}12cos50\\-12sin50\end{pmatrix} + \begin{pmatrix}-20cos20\\20sin20\end{pmatrix} t$

i: $30tcos\alpha = 12cos50 + -20tcos20$

$t = \frac{7.713}{30cos\alpha+6.84}$ .... (i)

J: $30tsin\alpha = -12sin50 + 20tsin20$

$t = \frac{9.193}{30sin\alpha+18.794}$ .... (ii)

(i) = (ii): $\frac{7.713}{30cos\alpha+6.84} = \frac{9.193}{30sin\alpha+18.794}$

231.39sinα – 275.79cosα = 82.078

But sinα= $\frac{2T}{1+T^2}$ and cosα= $\frac{1-T^2}{1+T^2}$

$231.39\left(\frac{2T}{1+T^2}\right) - 275.79\left(c\frac{1-T^2}{1+T^2}\right) = 82.078$

$357.868T^2 + 462.78T - 193.712 = 0$

T = -1.626 or T = 0.333

∴ T = 0.333

sinα= $\frac{2T}{1+T^2}$

α $=\sin^{-1}\frac{2\ x\ 0.333}{1+0.333^0}$ $=36.8^0$

Bearing: $E36.8^0S$

$t = \frac{7.713}{30cos\alpha+6.84} = \frac{7.713}{30cos36.8+6.84}$

T = 0.25h=0.25x 60 =15minutes

Alternatively

$\frac{sin20}{30} = \frac{sin\theta}{20}$

θ = 13.2^0:

Bearing $(50 – 13,2)^0S$

$E36.8^0S$ or $S53.2^0E$

Also, $\frac{sin20}{30} = \frac{\sin(180-(20+13.2))}{V_{AB}}$

$V_{AB} = 48.03km/h$

t = $\frac{AB}{VA_{AB}} = \frac{12}{48.03}\ x\ 60 = 15minutes$

Revision exercise 5

1. At 12:00 noon two ships A and B are 12km apart with B on a bearing of 250^0 from A. Ship A moves at 4km/h on a bearing of 320^0. If the maximum speed of B is 7km/h, find the
   (a) Direction B should set in order to intercept A [$N37.6^0E$]
   (b) Time taken for interception to occur. [99minutes]

2. Initially two particles A and B are 48m apart with B due north of A. A has a constant velocity of (5i + 4j)m/s and B a constant speed of 13m/s. Find the velocity of Y if it is to intercept A and find the time taken to do so [5i – 12j)m/s, 3s]
3. At 12 noon the position vectors, r and velocity vectors, V of two ships are
   $r_A = \begin{pmatrix}5\\2\end{pmatrix} km, V_A = \begin{pmatrix}15\\10\end{pmatrix} km/h \ and \ r_B = \begin{pmatrix}7\\7\end{pmatrix} km, V_A = \begin{pmatrix}9\\-5\end{pmatrix} km/h$
   Show that if the ships do not alter their velocities, a collision will occur and find the time at which it occurs and the position vector of its location [12.30pm, (10i + $\frac{16}{3}j$)km]
4. At 11.30 a jumbo jet has position vectors (-100i + 220j)km and it is moving with velocity vectors (300i + 400J)km/h. At 11:45am a cargo plane has a position vectors (-60i + 355j)km and is moving with velocity vectors (400i + 300j)km/h. Assuming velocities do not change
   (a) Show that the planes will crash
   (b) Find the time of the crash. [12.06pm]
   (c) Find the position vector of the crash [(80i + 460j)km]
5. At 2pm the position vectors, r and velocity vectors, V of three ships are as follows
   $r_A = (5i + j)km \quad V_A = (9i + 18j)km/h$
   $r_B = (12i + 5j)km \quad V_A = (-12i + 6j)km/h$
   $r_C = (13i - 3j)km \quad V_A = (9i + 12j)km/h$
   Assuming velocities do not change
   (a) Show that Ship A and B will collide and fins when and where collision occur. [2.20pm, (8i + 7j)km]
   (b) Find the position vector of C when A and B collide and find how far C is from the point of collision. [(16i + j)km, 10km]
   (c) When the collision occurs, C immediately changes its course but not its speed and streams directly to the scene. When does C arrive[3.00pm]
6. At 12 noon the position vectors, r and velocity vectors, V of three ships A, B and C are as follows
   $r_A = (10.5i + 6j)km \quad V_A = (9i + 18j)km/h$
   $r_B = (7i + 20j)km \quad V_A = (12i + 6j)km/h$
   $r_C = (10i + 15j)km \quad V_A = (6i + 12j)km/h$
   Assuming velocities do not change
   (a) Show that Ship A and B will collide and fins when and where collision occur. [1:10pm, (21i + 27j)km]
   (b) When the collision occurs, C immediately changes its course but not its speed and streams directly to the scene. When does C arrive[1:300pm]
7. In gulf water, a battleship streaming at 16km/h is5km southwest of a submarine. Find the course which the submarine should set in order to intercept the battle ship, if its speed is 12km/h. [N15^{0}W]
8. A boy hits a ball at 15m/s in a direction S80^{0}W. A girl 45m and S65^{0}W from the boy run at 6m/s to intercept the ball. Find in what direction the girl must rum to intercept the ball as quickly as possible and how long does it take her. [N24.7^{0}E, 2.35s]
9. A helicopter sets off from its bas and flies at 50m/s to intercept a ship which, when the helicopter sets off, is at a distance of 5km on a bearing 335^0 from the base. The ship is travelling at 10m/s on a bearing 095^0. Find the course that the helicopter pilot should set if he is to intercept the ship as quickly as possible and the time interval between the helicopter taking off and its reaching the ship. [N15^{0}W, 92.2s]

10. A life boat sets out a harbour at 9:10pm to go for assistance of a yatch which is, at the time, 5km due north of the harbour and drifting due west at 8km/h. If the life boat travels at 20km/h find:
    (a) Course the life boat should set so as to reach the yatch as quickly as possible [S23.6^{0}W]
    (b) Time when the boat arrives [9:27pm]
11. A coast guard vessel wishes to intercept a yatch suspected of smuggling. At 1am the yatch is 10km due east of the coast guard vessel and travelling due north at 15km/h. If the coast guard vessel travels at 20km/h,
    (a) In which direction should it steer in order to intercept the yatch? [N41.4^{0}E]
    (b) When would this interception occur. [1:45am]
12. The driver of a speed boat travelling at 75km/h wishes to intercept a yatch travelling at 20km/h in a direction N40^{0}E. Initially the speed boat is 10km from the yatch on a bearing S30^{0}E. Find
    (a) Course the speed boat should set so as to reach the yatch as quickly as possible. [N15.5^{0}W]
    (b) Time when the interception occurs[9minutes and 7 seconds]
13. A jet fighter travelling at 30km/h wishes to intercept a plane travelling at 20km/h in a course of 200^0. Initially the plane is 40km away on a bearing of 11^0 from the jet fighter. Find
    (a) Course the jet fighter should set so as to reach the plane as quickly as possible. [S5^{0}E]
    (b) Time taken for interception to occur. [48minutes and 24 seconds]
14. A batman hits a ball at 15m/s in a direction S80^{0}W. A fielder, 45m and S65^{0}W from the batsman, runs at 6m/s to intercept the ball. Assuming the velocities remain unchanged,
    (a) Find what direction the fielder must take to intercept the ball as quickly as possible. [N24.7^{0}E]
    (b) How long did it take him. [2.4s}

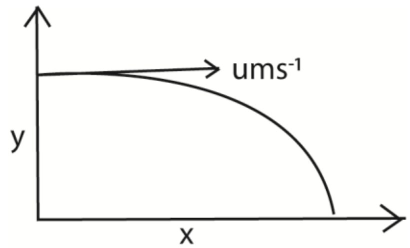
Projectile motion

This is the motion of a body which after being given an initial velocity moves freely under the influence of gravity.

Terms used in projectiles

1. ***Angle of projection:*** is the angle the initial velocity makes with the horizontal
2. ***Maximum height/ greatest height*** is the greatest height reached by projectile
3. ***Time of flight (T)*** is the time taken for projectile to complete motion. Note: the time of flight is twice the time to maximum height.
4. ***Range, R:*** is the horizontal distance covered by projectile
5. ***Maximum range (Rmax)*** is the greatest horizontal distance covered
6. ***Trajectory;*** is the path described by a projectile.

An object projected horizontally from a height above the ground.



Horizontal motion: Ux = u, a = 0

x = ut

Vertical motion: Uy = 0, a = $-9.8ms^{-2}$

$s = ut + ½ at^2$

$-y =- ½ gt^2$

V = u + at

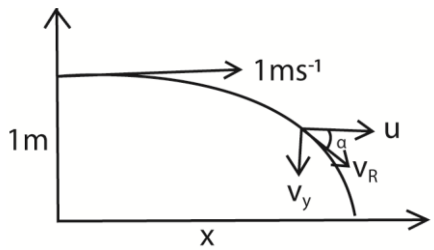
Vy = -gt

Example 39

A ball rolls of the edge of a table top 1m high above the floor with horizontal velocity 1ms-1. find:

(i) time taken to hit the floor
(ii) horizontal distance covered
(iii) the velocity when it hits the floor.

Solution



(i) vertical motion: u = 1ms^{-1}, θ = 0

y = 1m below the point of projection

y = ½ gt^2

-1 = $\frac{1}{2}$ $x - 9.8\, t^2$

t = 0.4518s

(ii) x = ut = 1 x 0.4518 = 0.4518m

(iii) v_x = 1ms^{-1}

v_y = -gt = -9.8 x 0.4518 = -4.428ms^{-1}

v_R =$\sqrt{1^2 + (-4.428)^2}$ = 4.54ms^{-1}

$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{4.428}{1}\right)$ = 77.3^0

Example 40

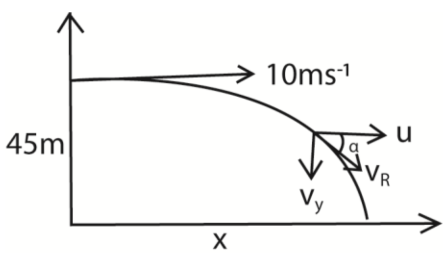
A ball is thrown forward horizontally from the top of a cliff with a velocity of 10ms^{-1}. The height of a cliff above the ground is 45m. Calculate

(i) time to reach the ground

(ii) distance from the cliff where the ball hits the ground

(iii) velocity and direction of the ball just before it hits the ground

Solution



(ii) vertical motion: u = 10ms^{-1}, θ = 0

y = 45m below the point of projection

y = ½ gt^2

-45 = $\frac{1}{2}$ $x - 9.8\, t^2$

t = 3.03s

(ii) x = ut = 10 x 3.03 = 30.3m

(iii) v_x = 10ms^{-1}

v_y = -gt = -9.8 x 3.03 = -29.694ms^{-1}

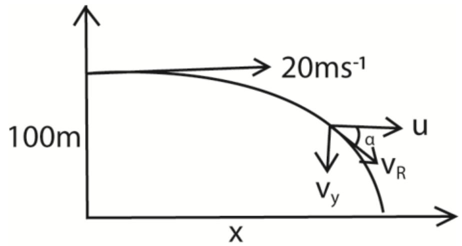
v_R =$\sqrt{10^2 + (-29.694)^2}$ = 31.33ms^{-1}

$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{29.694}{10}\right)$ = 71.4^0

Example 41

An object is projected horizontally at a speed of 20ms^{-1} from a height of 100m. Find

(i) the time of flight



$y = \frac{1}{2} gt^2$
$-100 = \frac{1}{2}\, x - 9.8\, t^2$
t = 4.52ms^{-1}

(ii) the horizontal range
x = ut = 20 x 4.52 = 90.4m

(iii) its velocity on reaching the ground

v_x = 20ms^{-1}
v_y = -gt = -9.8 x 4.52 = -32.7ms^{-1}

$v_R = \sqrt{20^2 + (-32.7)^2}$ =38 .33ms^{-1}

$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{32.7}{20}\right)$ = 58.5^0

Example 42

At time t = 0, a particle is projected with a velocity of 3ms-1 from a point with position vector (5i + 25j)m. Find the

(i) speed and direction of the particle when t = 2s
v_x = u = 3ms^{-1}
v_y = gt = -9.8 x 2 = -19.6ms^{-1}

$v_R = \sqrt{3^2 + (-19.6)^2}$ =19.83ms^{-1}

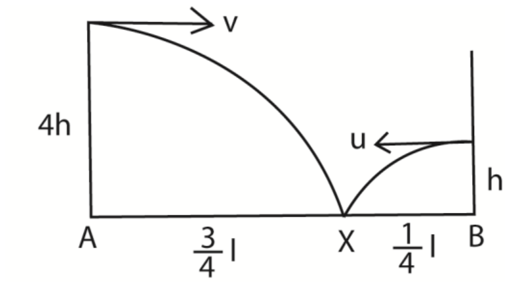
$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{19.6}{3}\right)$ = 81.3^0

(ii) Position vector of the particle when t = 2s

$$P_{(t=t)} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 25 \end{pmatrix} + \begin{pmatrix} ut \\ -\frac{1}{2}gt^2 \end{pmatrix} = \begin{pmatrix} 5 \\ 25 \end{pmatrix} + \begin{pmatrix} 3\ x\ 2 \\ -\frac{1}{2}x9.8\ x\ 2^2 \end{pmatrix} = \begin{pmatrix} 11 \\ 5.4 \end{pmatrix} m$$

Example 43

A and B are two points on a level ground. A vertical tower of height 4h has its base at A and vertical tower of height h has its bas at B. When a stone is thrown horizontally with speed v from the top the taller tower towards the smaller tower, it lands at point X where AX = $\frac{3}{4}AB$. When a stone is thrown horizontally with speed u from the top of the smaller tower towards the taller tower, it also lands at the point A. Show that 3u = 2V



For A Vertical motion: y = $\frac{1}{2}gt^2$

$$4h = \frac{1}{2}gt^2$$

$$t = \sqrt{\frac{8h}{g}}$$

Horizontal motion: x = vt

$$\frac{3}{4}l = vt$$

$$l = \frac{4}{3}vt = \frac{4}{3}v\sqrt{\frac{8h}{g}} \dots.(i)$$

For B Vertical motion: y = $\frac{1}{2}gt^2$

$$h = \frac{1}{2}gt^2$$

$$t = \sqrt{\frac{2h}{g}}$$

Horizontal motion: x = ut

$$\frac{1}{4}l = ut$$

$$l = \frac{4}{3}ut = \frac{4}{3}v\sqrt{\frac{2h}{g}} \dots.(ii)$$

Eqn. (i) and (ii)

$\frac{4}{3}v\sqrt{\frac{8h}{g}} = \frac{4}{3}u\sqrt{\frac{2h}{g}}$;

v = $\frac{3}{2}u$

2v = 3u

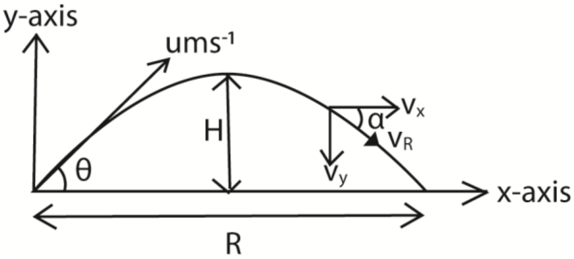
Revision exercise 6

1. A pencil is accidentally knocked off the edge of a horizontal desktop. The height of the desk is 64.8cm and the pencil hits the floor a horizontal distance of 32.4cm from the edge of the desk. What was the speed of the pencil as it left the desk? [0.9ms^{-1}]
2. A particle is projected horizontally at 20ms^{-1} from a point 78.4m above a horizontal surface. Find the time taken for the particle to reach the surface and the horizontal distance travelled in that time. [4s, 80m]
3. A particle is projected horizontally with a speed of 2ms^{-1}
   (i) Find the horizontal and vertical displacements of the particle from the point of projection, $2\frac{6}{7}$s after projection [60m, 40m below]
   (ii) Find how far the particle is then from the point of projection. [72.1m]
4. A particle is projected horizontally from a point 2.5m above the horizontal surface. The partial hits the surface at a point which is horizontally 10m from the point of projection. Find the initial speed of the particle.[14ms^{-1}]
5. At time t= 0, a particle is projected with a velocity of 2ims^{-1} from a point with position vector (10i + 150j)m. Find the
   (i) speed and direction of the particle when t = 5s [49.04ms^{-1}, at 87.6^0]
   (ii) position vector of the particle when t = 1s $\left[\begin{pmatrix}20\\27.5\end{pmatrix}m\right]$

6. At time t = 0, a particle is projected with a velocity of 5ims-1 from a point with position vector (20j)m. find the position vector of the particle when t = 2s. $\left[\begin{pmatrix}10\\0.4\end{pmatrix}m\right]$.
7. A batsman strikes a ball horizontally when it is 1m above the ground. The ball is caught 10cm above the grounder by a fielder standing 12m from the batsman. Find the speed with which the batsman hits the ball. [$28ms^{-1}$]
8. A darts player throws a dart horizontally with a speed of $14ms^{-1}$. The dart hits the board at a point 10cm below the level at which it is released. Find the horizontal distance travelled by the dart. [2m]
9. A tennis ball is served horizontally with initial speed of $21ms^{-1}$ from a height of 2.8m, by what distance does the ball clears a net 1m high situated 12m horizontally from the server? [20cm]
10. A fielder retrieves cricket ball and throws it horizontally with a speed of $28ms^{-1}$ to a wicket-keeper standing 12m away. If the fielder releases the ball at a height of 2m above level ground, find the height of the ball when it reaches the wicket-keeper. [110cm]
11. Initially a particle is at an origin and is projected with a velocity of ai ms^{-1}. After t second, the particle is at the point with position vector (30i – 21j)m. Find the value of t and a. [$1\frac{3}{7}, 21$]
12. Two vertical towers stand a horizontal ground level and are of height 40m and 30m. A ball is thrown horizontally from the top of the higher tower with a speed of $24.5ms^{-1}$ and just clears the short tower. Find the distance
    (i) between the two towers [35m]
    (ii) between the short and the point on the ground where the ball first lands. [35m]
13. The top of a vertical tower is 20m above ground level. When a ball is thrown horizontally from the top of this tower. By how much does the ball clears a vertical wall of height 13m situated 12m from the tower.[2m]
14. A stone is thrown horizontally with speed u from the edge of a vertical cliff of height h. The stone hits the ground at a point which is a distance d horizontally from the base of the cliff. Show that $2hu^2 = gd^2$.
15. A vertical tower stands with its base on a horizontal ground. Two particles A and B are both projected horizontally and in the same direction from the top of the tower with initial velocities of $14ms^{-1}$ and $17.5ms^{-1}$ respectively. If A and B hit the ground at two points 10cm apart, find the height of the tower[40m]
16. O, A and B are three points with O on level ground and A and B respectively 3.6 and 40m vertically above O. A particle is projected horizontally from B with a speed of $21ms^{-1}$ and 2seconds later, a particle is projected horizontally from A with a speed of $70ms^{-1}$. Show that the two particles reach the ground at the same distance from O, find this distance.
17. An aeroplane moving horizontally at 150ms-1 releases a bomb at height of 500m. The aeroplane hits the intended target. What was the horizontal distance of aeroplane from the target when the bomb was released? [1500m]
18. A projectile is fired horizontally from the top of a cliff 250m high. The projectile lands 144m from the bottom of the cliff. Find the
    (i) initial speed of the projectile [198ms-1]
    (ii) velocity of the projectile just before it hits the ground [$210ms^{-1}$ at 19.5^0]

Object projected upwards from the ground at an angle to the horizontal

Suppose an object is project with velocity u at an angle θ from a horizontal ground. H and R are the maximum height reached and range respectively.



Horizontally; $u_x = u\cos\theta$, a = 0

v = u + at

$v_x = u\cos\theta$

$x = u\cos\theta t$

$v_y = s = ut + \frac{1}{2}at^2$

$x = u\cos\theta t$

Vertically; $u_y = u\sin\theta$, $a = -9.8ms^{-2}$

v = u + at

$v_y = u\sin\theta - gt$

$s = ut + \frac{1}{2}at^2$

$y = u\sin\theta t - \frac{1}{2}gt^2$

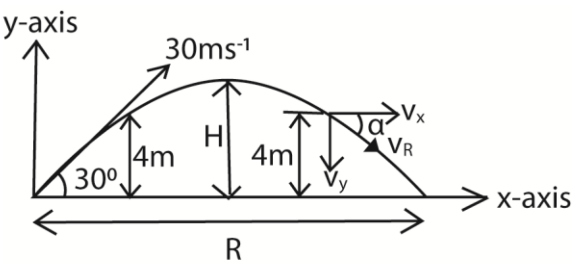
Example 44

A particle is projected with a velocity of $30ms^{-1}$ at an angle of elevation of 30^0. Find

(i) the greatest height reached

(ii) the time of flight

(iii) the velocity and direction of motion at a height of 4m on its way upwards.



(i) (↑) $v_y = u\sin\theta$, $a = -9.8ms^{-2}$

At maximum height, $v_y = 0$

From $v^2 = u^2 + 2as$

$H = \frac{(30\sin 30)^2}{2\ x\ 9.8} = 11,47m$

(→): $u_x = u\cos\theta$, a = 0

$s = ut + \frac{1}{2}at^2$

$R = u\cos\theta T$

R = (30cos 30) x 3.0612 = 79.5329m

(↑) $y = u\sin\theta t - \frac{1}{2}gt^2$

$4 = 30\sin30t - \frac{1}{2}x\ 9.8\ x\ t^2$

t = 0.30s or t = 2.76s

t = 0.30s is the correct time since it is smaller indicating that the body is moving upwards

(ii) at time of flight, s = 0

$s = ut + \frac{1}{2}at^2$

$0 = 30\sin30T - \frac{1}{2}x\ 9.8\ xT^2$

$0 = (30\sin30 - \frac{1}{2}x\ 9.8\ T)T$

Either T = 0

or $(30\sin30 - \frac{1}{2}x\ 9.8\ T) = 0$

T = 3.0612s

$u_x = u\cos\theta = 30\cos30 = 25.98ms^{-1}$

$v_y = u\sin\theta - gt$

$= 30\sin30 - 9.8 \times 0.30 = 12.06ms^{-1}$

$v = \sqrt{(v_x)^2 + (v_y)^2}$

$= \sqrt{25.98^2 + 12.06^2} = 28.64ms^{-1}$

direction $\alpha = \tan^{-1}\left(\frac{12.06}{25.98}\right) = 24.9^0$

Example 45

A particle is projected from the origin at a velocity of (10i + 20j)ms^{-1}. Find the position and velocity vectors of the particle 3s after projection. (Take g = 10ms^{-2})

Solution

$P_{t=t} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$

$P_{(t=3)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} ucos\theta t \\ usin\theta t - \frac{1}{2}gt^2 \end{pmatrix}$

$P_{(t=3)} = \begin{pmatrix} 10\ x\ 3 \\ 20\ x\ 3 - \frac{1}{2}\ x 9.8\ x\ 3^2 \end{pmatrix} = \begin{pmatrix} 30 \\ 15 \end{pmatrix} m$

$v_{(t=t)} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} ucos\theta \\ usin\theta - gt \end{pmatrix}$

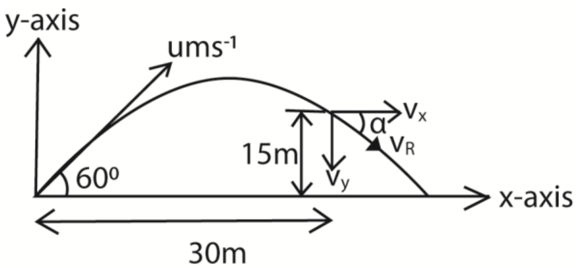
$v_{(t=3)} = \begin{pmatrix} 10 \\ 20 - 10x\ 3 \end{pmatrix}$

$= \begin{pmatrix} 10 \\ -10 \end{pmatrix} ms^{-1}$

Example 46

A projectile fired at an angle of 60^0 above the horizontal strikes a building 30m away at a point 15m above the point of projection. Find

(i) speed of projection (i) velocity when it strikes a building



(→) x = ucosθt

30 = utcos 60

t = $\frac{60}{u}$

(↑) y = $usin\theta t - \frac{1}{2}gt^2$

15 = usin60 x $\frac{60}{u}$ - $\frac{1}{2}$ x 9.8 x $\left(\frac{60}{u}\right)^2$

u = 21.86ms^{-1}

(ii) t = $\frac{60}{u}$ = $\frac{60}{21.86}$ = 2.75s

u_x = ucosθ

= 21.86x cos 60 = 10.93ms^{-1}

u_y = usinθ – gt

= 21.86x sin 60 – 9.8 x 2.75

= -8.09ms^{-1}

v = $\sqrt{(v_x)^2 + (v_y)^2}$

= $\sqrt{10.93^2 + (-8.93)^2}$ = 13.60ms^{-1}

direction α = $\tan^{-1}\left(\frac{8.09}{10.93}\right)$= 36.58^0

Example 47

A football player projects a ball at a speed of 8ms^{-1} at an angle of 30^0 with the ground. The ball strikes the ground at a point which is level with the point of projection. After impact with the ground, the ball bounces and the horizontal component of the velocity of the ball remains the same but the vertical component is reversed in direction and halved in magnitude. The player running after the ball. Kicks it again at appoint which is at a horizontal distance 1.0m from the point where it bounced, so that the ball continues in the same direction. Find the

(a) horizontal distance between the point of projection and the point at which the ball first strikes the ground. (Take g = $10ms^{-2}$)
(b) (i) the time interval between the ball striking the ground and the player kicking it again.
(ii) the height of the ball above the ground when it is kicked again (take g = $10ms^{-2}$)

Solution

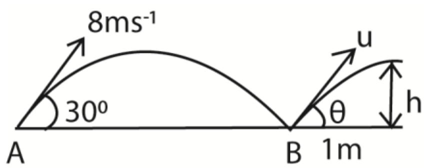
y = usinθt - $\frac{1}{2}gt^2$

At AB:

0 = 8sin30 x t - $\frac{1}{2}$ x $10\ t^2$

t = 0.8s

x = cosθt = 8cos30 x 0.8 = 5.543m



(b)(i) x = ucosθt

1 = 8cos (30)t

t = 0.1443s

(ii) y = usinθt - $\frac{1}{2}gt^2$

h = 4sin30 x 0.1443 - $\frac{1}{2}$ $x\ 10\ x\ (0.1443)^2$

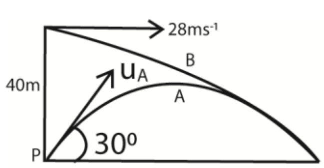
h = 0.185m

Example 48

Two objects A and B are projected simultaneously from different points. A is projected from the top of vertical cliff and A from the base. Particle B is projected horizontally with a speed $28ms^{-1}$ and A is projected at an angle θ above the horizontal. The height of the cliff is 40m and the particles hit the same point on the ground, find;

(a) time taken and the distance from P to where they hit
(b) speed and angle of projection of A

Solution



(a) For B: y = $\frac{1}{2}gt^2$

-40 = $\frac{1}{2}$ $x - 9.8t^2$

t = $\frac{20}{7}s$

x = ut = 28 x $\frac{20}{7}$ = 80m

For A: x =ucosθt

80 = $u_A$cosθ x $\frac{20}{7}$

$u_A$cosθ = 28 …….. (i)

y = usinθt - $\frac{1}{2}gt^2$

0 = $u_A$sinθ x $\frac{20}{7}$ - $\frac{1}{2}$ $x - 9.8\ x\ \left(\frac{20}{7}\right)^2$

$u_A$sinθ = 14………… (ii)

(i) ÷ (ii)

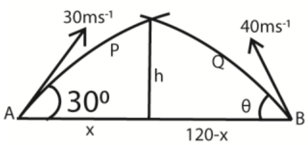
θ = $\tan^{-1}\frac{14}{28} = 26.6^0$

$u_A$sin26.6 = 14

u_A = $31.3ms^{-1}$

Example 49

A particle P is projected a point A with initial velocity 30ms-1at an angle of elevation 300 to the horizontal. At the same instant a particle Q is projected in the opposite direction with initial speed of 40ms-1 from a point at the same level with a and 120m from A. Given that the particles collide. Find (i) angle of projection of Q (ii) time when collision occur.



y = usinθt - $\frac{1}{2}gt^2$

For P

h = 30sin30 x t - $\frac{1}{2}$ $x9.8t^2$.... (i)

For Q

h = 40sinθ x t - $\frac{1}{2}$ $x9.8t^2$..... (ii)

(i) and (ii)

30sin30 x t = 40sinθ x t

θ = 24.5^0

For P

x – 30cos30t ……… (iii)

For Q

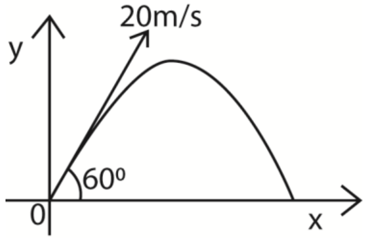
120 –x = 40cos24.5t … (iv)

(iii)and (iv)

t = 1.9s

Example 50

A particle is projected from a point O with speed20m/s at an angle 60^0 to the horizontal. Express in vector form its velocity v and its displacement, r, from O at any time t seconds . (05marks)



At time t = 0

V = $20\cos60^0$i + $20\sin60^0$j = 10i + $10\sqrt{3}$j

But A = -gj

At any time t,

V = $\int adt$ = -gj$\int dt$

= -gtj + c

At t = 0

10i + $10\sqrt{3}$j = 0 + cn

⇨ At time t

V= 10i + $(10\sqrt{3} - gt)$j = 10i + $(10\sqrt{3} - 9.8t)$j

Or

V = $\begin{pmatrix} 10 \\ 10\sqrt{3} - 9.8t \end{pmatrix}$

r = $\int vdt = \int \begin{pmatrix} 10 \\ 10\sqrt{3} - 9.8t \end{pmatrix} dt$ = $\begin{pmatrix} 10t \\ 10\sqrt{3}t - 4.9t^2 \end{pmatrix}$+ c

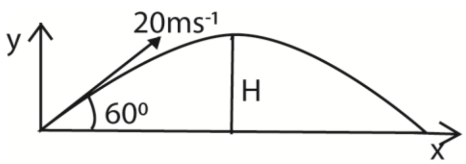
at t = 0, r = $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

⇨ $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + c$, c = $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Hence at time t, r = $\begin{pmatrix} 10t \\ 10\sqrt{3}t - 4.9t^2 \end{pmatrix}$

Example 51

A particle is projected at an angle 600 to the horizontal with velocity of $20ms^{-1}$. Calculate the greatest height the particle attains. [Use $g = 10ms^{-2}$]



Use $v^2 = u^2 + 2as$; at maximum height, v = 0

$0 = (20\sin 60)^2 - 2gH$

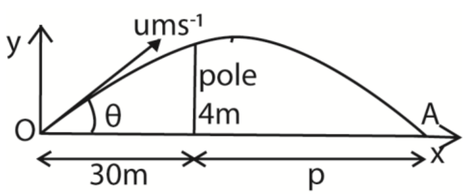
H = 15m

Examples 52

A particle is projected from ground level towards a vertical pole 4m high and 30m away from the point of projection. It just passes the pole in one second. Find

(i) its initial speed and angle of projection
(ii) the distance beyond the pole where the particle falls

Solution



(→) $x = u_x t$ but t = 1

30 = ucosθ x 1

30 = ucosθ ………………….. (i)

(↑) $y = u_y t - \frac{1}{2}gt^2$

$4 = u\sin\theta \times 1 - \frac{1}{2} x\ 9.8\ x\ 1^2$

4 = usinθ – 4.9

8.9 = usinθ ………. (ii)

(ii) ÷ (i)

$\theta = \tan^{-1}\left(\frac{8.9}{30}\right) = 16.5^0$

30 = ucos16.5

$u = 31.29ms^{-1}$

(b) At point O and A, y=0

$y = u\sin\theta t - \frac{1}{2}gt^2$

$0 = 31.29\sin 16.5T - \frac{1}{2}\ x\ 9.8xT^2$

T = 1.8136s

Range, R= ucosθT

R = 31.29cos 16.6 x 1.8136 = 54.36m

p = 54.36 – 30 = 24.36m

Revision exercise 7

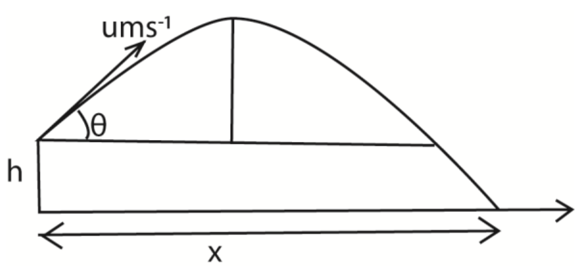
1. A particle projected at an angle of 30^0 to the horizontal with velocity of $60ms^{-1}$. Calculate
   (a) time taken for the particle to reach maximum height [3s]
   (b) maximum height [45m]
   (c) horizontal range of the particle [312m]

2. A particle is projected from an origin O and has an initial velocity of $30\sqrt{2}$ ms^{-1} at angle 450 above the horizontal. Find the horizontal and vertical components of displacement 2s after projection. [60m, 40.4m] Hence find the distance of motion of the particle at that time [72.3m]
3. A particle projected from a point on the level ground has horizontal range of 240m and time of flight of 6s. find the magnitude and direction of velocity of projection [$50ms^{-1}$, 36.9^0]
4. A particle is projected with a velocity of $30ms^{-1}$ at an angle of 40^0 above the horizontal plane. Find
   (i) the time for which the particle is in air [3.9s]
   (ii) the horizontal distance it travels [22.9m]
5. A body is projected with a velocity of 200ms-1 at an angle of 300 above the horizontal. Calculate
   (i) time taken to reach the maximum height[10.2s]
   (ii) its velocity after 16s [$183ms^{-1}$ at 19.1^0]
6. A football is kicked from O on a level ground. 2s later the football just clears a vertical wall of height 2.4m. If O is 22m from the wall, find the velocity with which the ball is kicked. [$15.6ms^{-1}$ at 45^0 above the horizontal]
7. A particle is projected from a level ground in such away that its horizontal and vertical components of velocity are $20ms^{-1}$ and $10ms^{-1}$ respectively. find
   (a) maximum height of the particle [5.0m]
   (b) its horizontal distance from the point of projection when it returns to the ground. [40m]
   (c) the magnitude and direction of the velocity on landing [$22.4ms^{-1}$ at 26.6^0 below the horizontal]
8. A particle is projected with a speed of 25ms-1 at 300 above the horizontal. Find
   (a) time taken to reach the height point of the trajectory. [1.5s]
   (b) the magnitude and direction of velocity after 2.0s.[$22.9ms^{-1}$ at 19.1^0 below the horizontal]
9. A particle is projected from the origin at a velocity (4i + 13j)ms^{-1}. Find the position vector and distance of the particle in 2s after projection (take g = $10ms^{-2}$) [(8i + 6j)m, 10m]
10. A particle is projected from the origin at a velocity of (4i + 11j)ms^{-1}. and passes a point P which has a position vector (8i + xj)m. Find the time taken for the particle to reach P from O and the value of x. [2s, 2.4m]
11. A Particle is projected from the origin at a velocity of (7i + 5j)ms^{-1} and passes a point P which has position vector (xi – 30j)m. Find the time taken for the particle to reach P from O and the value of x [3s, 21]
12. A particle is projected from the origin at velocity of (4i + 2j)ms^{-1}. Find
   (a) the direction in which it is moving after 1s[63.43^0]
   (b) Two second later after launch of the first particle, a second particle is projected from the same point with a velocity (8i – 26j)ms^{-1}. Show that the two particles collide and find the time and position at which this occurs [t =4s, (16i – 72j)m]
13. A particle is projected at $84ms^{-1}$ to hit a point 360m away and on the same horizontal level at the point as the point of projection. Find the two possible angles of projection. [15^0, 75^0]
14. A golfer hits a golf ball at $30ms^{-1}$ and wishes it to land at a point 45m away, on the same horizontal level as the starting point. Find the two possible angles of projection. [4.7^0, 75.3^0]
15. A particle is projected from a horizontal ground and has an initial speed of $35ms^{-1}$. When the ball is travelling horizontally, it strikes a vertical wall. If the wall is 25m from the point of projection, find the two possible angles of projection. [11.8^0, 8.2^0]

16. A particle is projected from a point O and passes through a point A when the particle is travelling horizontally. If A is 10m horizontally and 8m vertically from, find the magnitude and direction of the velocity of projection. [14.8ms^{-1}, 58^0 above the horizontal]
17. A stone is projected at an angle of 20^0 to the horizontal and just clears a wall which is 10m high and 30m from the point of projection. Find the
    (a) speed of projection [73.78ms^{-1}]
    (b) angle which the stone makes with the horizontal as it clears the wall [16.9^0]
18. A particle is projected from a point on a horizontal plane and has an initial speed of 28ms-1. If the particle passes through a point above the plane, 40m horizontally and 20m vertically from the point of projection, find the possible angles of projection. [45^0, 71.6^0]
19. Two objects A and B are projected simultaneously from different points. A is projected from the top of a vertical cliff and B from the base. Particle A is projected horizontally with a speed 3u ms^{-1} and B is projected at an angle θ above the horizontal with speed 5u ms^{-1}. The height of the cliff is 56m and the particles collide after 2s, find
    (a) horizontal and vertical distances from the point of collision to the base of the cliff. [42m, 36.4m]
    (b) value of angle u and θ. [7ms^{-1}, 53.1^0]

Objects projected upwards from a point above the ground at an angle to the horizontal

Suppose an object is projected with velocity u at an angle θ from a height h.



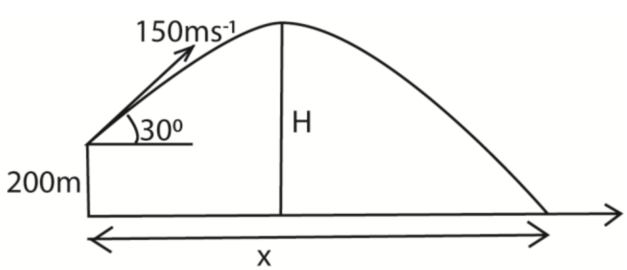
Horizontal: u_x = ucosθ, a = 0

Vertical: uy = usinθ, a = -g = -9.8

Example 53

A bullet is fired from a gun at a height of 200m with velocity150ms-1 at an angle of 300 to the horizontal. Find

(i) maximum height attained.



$v^2 = u^2 + 2as$

at maximum height, H, v =0

$0^2 = (150\sin30)^2 - 2 \times 9.8H$

H = 86.70m

(ii) Time taken for the bullet to hit the ground

y = usinθt - $\frac{1}{2}gt^2$

y = -200m since it's below the point of projection

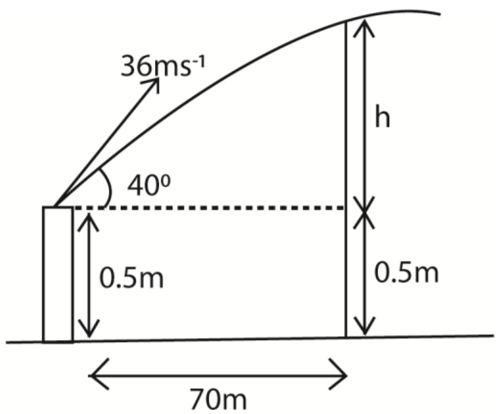
-200 = 150sin30t - $\frac{1}{2}x\ 9.8t^2$

t = 17.61s, or t = -2.32s

time taken = 17.61

Example 54

A particle I projected with a speed of 36ms^{-1} at an angle of 40^0 to the horizontal from a point 0.5m above the level ground. It just clears a wall which is 70 metres on the horizontal plane from a point of projection. Find the;



Vx = 36cos40^0

Vy = 36sin40^0

(a) (i) time taken for the particle to reach the wall.

Time take to clear the wall = time taken to cover a horizontal distance of 70m

Usin X= Vxt

t = $\frac{70}{36\sin 40^0}$ = 2.5384s

(ii) height of the wall (08marks)

Using h = usinθt - $\frac{1}{2}gt^2$

=36sin40^0 x 2.5384 - $\frac{1}{2}\ x\ 9.8\ x\ (2.5384)^2$

= 27.1664 + 0.5 = 27.6664m

(a) Maximum height reached by the particle from the point of projection. (04marks)

From v^2 = u^2 +2as

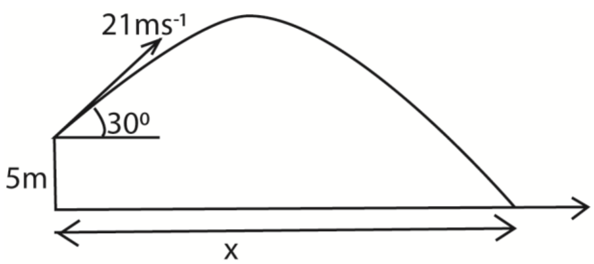
At maximum height vertical component of velocity is zero

⇨ 0 = $(36\sin 40^2)^2 - 2\ x\ 9.8H$

H = $\frac{(36\sin 40^2)^2}{2\ x\ 9.8}$ =27.32m

Example 55

A particle is projected at an angle of 30^0 with speed of $21ms^{-1}$. If the point of projection is 5m above the horizontal ground, find the horizontal distance that the particle travels before hitting the ground



y = usinθt - $\frac{1}{2}gt^2$

y = -5m since it's below the point of projection

-5 = 21sin30T - $\frac{1}{2} x\ 9.8T^2$

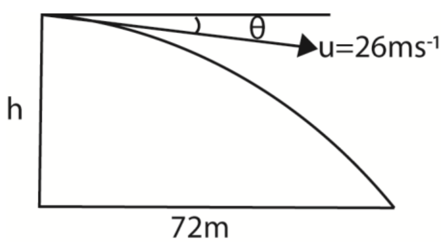
T = 2.54s

(→): x = ucosθ x T

= 21cos30 x 2.54 = 46.19m

Example 56

A stone is thrown from the edge of a vertical cliff and has initial velocity of $26ms^{-1}$ at an angle of $\tan^{-1}\left(\frac{5}{12}\right)$ below the horizontal. The stone hits the sea at a point level with the base of the cliff and 72m from it. Find the height of the cliff and the time for which the stone is in the air. Take g = $10ms^{-2}$.



y = xtanθ - $\frac{gx^2(1+\tan^2\theta)}{2u^2}$

h = 72 x$\left(-\frac{5}{12}\right)$ = $\frac{10\ x\ 72^2\left[1+\left(\left(-\frac{5}{12}\right)\right)^2\right]}{2\ x\ 26^2}$

= -75m

h= 75m below the point of projection

x = ucosθt

72 = 26$\left[cos\left(-\tan^{-1}\frac{5}{12}\right)\right]t$; t =3s

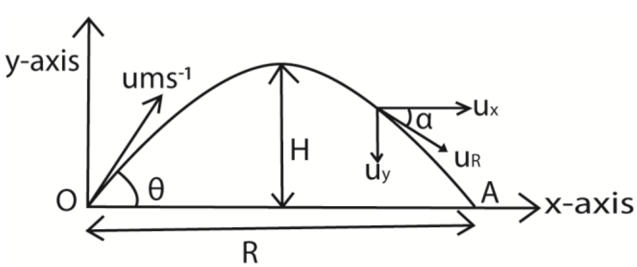
Revision exercise 8

1. A stone is thrown from the edge of a vertical cliff with velocity of $50ms^{-1}$ at an angle of $\tan^{-1}\frac{7}{24}$ above the horizontal. The stone strikes the sea at a point 240m from the foot of the cliff. Find the time for which the stone is in air and the height of the cliff. [5s, 52.5m]
2. A particle is projected with a velocity of 10ms-1 at an angle of 45^0 to the horizontal; it hits the ground at a point which is 3m below the point of projection. Find the time for which it is in the air and the horizontal distance covered the particle in this time. [1.76s, 12.42m]
3. A batsman hits a ball with velocity of $17ms^{-1}$ at an angle $\tan^{-1}\frac{3}{4}$ above the horizontal, the ball initially being60cm above the level ground. The ball is caught by a fielder standing 28m from the batsman. Find the time taken for the ball to reach the fielder and the height above the ground at which he takes the catch. [2s, 2m]

4. A stone is thrown from the edge of a vertical cliff 70m high at an angle of 300 below the horizontal. the stone hits the sea at a point level with the base of the cliff and 20m from it. Find the initial speed of the stone and the direction it is moving when it hits the sea. [6.69mms^{-1}, 81.2^{0}]
5. A vertical tower stands on a level ground. A stone is thrown from the top of the tower and has an initial velocity of 24.5ms^{-1} at an angle of $\tan^{-1}\frac{4}{3}$ above the horizontal. The stone strikes the ground at a point 73.5m from the foot of the tower. Find the time taken for the stone to reach the ground and the height of the tower. [5s, 24.5m]
6. A stone is thrown from the top of a vertical cliff, 100mabove sea level. The initial velocity of the stone is 13ms-1 at an angle of elevation of$\tan^{-1}\frac{5}{12}$. Find the time taken for the stone to reach the sea and its horizontal distance from the cliff at that time. Take g = 10ms^{-2}. [5s. 60m]
7. A golfer hits a golf ball with a velocity of 30ms^{-1} at an angle of $\tan^{-1}\frac{4}{3}$ above the horizontal. The ball lands on green 5m below the level from which it was struck. Find the horizontal distance travelled by the ball. Take g = 10ms^{-2}. [90m]
8. A pebble is thrown from the top of a cliff at a speed of 10ms^{-1} and at 30^{0} above the horizontal. It hits the sea below the cliff 6s later. Find
   (a) the height of the cliff [150m]
   (b) the distance from the base of the cliff at which the pebble falls into the sea. [52m]
9. An arrow is fired from a point at a height 1.5m above the horizontal. It has a velocity of 12ms^{-1} at an angle 30^{0} above the horizontal. The arrow hits the target at a height of 1m above the horizontal ground, find
   (i) time taken for the arrow to hit the target [1.3s]
   (ii) horizontal distance between where the arrow is fired and the target. [13.51m]
   (iii) speed of the arrow when the arrow hit the target [12.39ms^{-1}]

Standard equations of the projectile

Suppose an object is projected with velocity u at an angle θ from a horizontal ground



(a) Maximum height (greatest height), H
For vertical motion, at maximum height v = 0,
a = -g = -9,8ms^{-2}
$v^2 = u^2 + 2as$

$0 = (u\sin\theta)^2 - 2gH$

$H = \frac{(u\sin\theta)^2}{2g}$

(b) Time to reach maximum height

(↑): v = u_y + at, at maximum height, v = 0

0 = usinμ – gt

$t = \frac{usin\theta}{g}$

(c) Time of flight, T

(↑): s_y = u_yt - $\frac{1}{2}gt^2$

At A s_y = 0

0 = usinθT - $\frac{1}{2}gT^2$

$T = \frac{2usin\theta}{g}$

(d) Range

(→) x = ucosθt

R = ucosθT

= ucosθ. $\frac{2usin\theta}{g}$

= $\frac{2u^2cos\theta sin\theta}{g}$

R = $\frac{2u^2sin2\theta}{g}$

(e) Maximum range

for maximum range sin2θ = 1

2θ = $\sin^{-1} 1$

2θ =90

R_{max} = $\frac{2u^2sin90}{g} = \frac{u^2}{g}$

(f) Equation of trajectory

A trajectory is expresses zin terms of horizontal distance x and vertical distance y

(→): x = ucosθt

t= $\frac{x}{ucos\theta}$ ………… (i)

(↑): y= usinθt - $\frac{1}{2}gt^2$ …. (ii)

putting (i) into (ii)

y= usinθ$\left(\frac{x}{ucos\theta}\right)$ - $\frac{1}{2}$g$\left(\frac{x}{ucos\theta}\right)^2$

y = xtanθ - $\frac{gx^2}{2u^2cos^2\theta}$

= xtanθ - $\frac{gx^2sec^2\theta}{2u^2}$

y = xtanθ - $\frac{gx^2(1+tan^2\theta)}{2u^2}$

Example 57

A ball is projected from the horizontal ground and has an initial velocity of 20ms^{-1} at an angle of elevation$\tan^{-1}\frac{7}{24}$. When the ball is travelling horizontally it strike a vertical wall. How high above the ground does the impact occur.

Projectile travel horizontally at maximum height

H = $\frac{(usin\theta)^2}{2g}$ = $\frac{20^2 sin^1\left(\tan^{-1}\frac{7}{24}\right)}{2\ x\ 9.8}$ = 1.6m

Example 58

A particle is projected from a point on a horizontal ground at a speed of 84ms^{-1}. If the particle hits a point 300m away and on the same horizontal plane as the projection, find the

(i) angle of projection

R = $\frac{2u^2 sin2\theta}{g}$

360 = $\frac{2x\ 84^2\ x\ sin2\theta}{9.8}$; θ = 15^0 0r θ = 75^0

(ii) maximum height

H_1 = $\frac{(usin\theta)^2}{2g}$ = $\frac{84^2 sin^2(15)}{2\ x\ 9.8}$ = 24.1m

H_2 = $\frac{(usin\theta)^2}{2g}$ = $\frac{84^2 sin^2(75)}{2\ x\ 9.8}$ = 335.9m

(iii) times of flight

$T_1 = \frac{2usin\theta}{g}$ = $\frac{2x\ 84\ x\ sin15}{9.8}$ = 4.44s

$T_2 = \frac{2usin\theta}{g}$ = $\frac{2x\ 84\ x\ sin75}{9.8}$ = 16.56s

Example 59

A gun has its barrel set at an angle of elevation of 15^0. The gun fires a shell with initial speed of $210ms^{-1}$. Find the

(a) horizontal range of the shell

R = $\frac{2u^2 sin2\theta}{g}$ = $\frac{2x\ 84^2\ x\ sin30}{9.8}$ = 2250m

(b) maximum range

R = $\frac{2u^2}{g}$ = R = $\frac{2\ x\ 84^2}{g}$ = 4500m

Example 60

A stone thrown upwards at an angle θ to the horizontal with speed ums-1 just clears a vertical wall4m high and 10m from the point of projection when travelling horizontally. Find the angle of projection.

Solution

Projectile travel horizontally

at maximum height

H = $\frac{(usin\theta)^2}{2g}$

4 = $\frac{(usin\theta)^2}{2g}$

8g = $u^2sin^2\theta$ =……… (i)

Also, x = ucosθt and t = $\frac{ucos\theta}{g}$

x = ucosθ$\left(\frac{ucos\theta}{g}\right)$

10g = $u^2cos\theta sin\theta$ ……(ii)

(i)÷(ii)

θ = $tan^{-1}\frac{8}{10}$ = 38.7^0

Example 61

If the horizontal range of a particle with velocity u is r, show that the greatest height H is satisfied by the equation $16gH^2 - 8Hu^2 + gR^2 = 0$

$H = \frac{(usin\theta)^2}{2g}$

$sin^2\theta = \frac{2gH}{u^2}$.......... (i)

$R = \frac{2u^2 sin2\theta}{g}$

$gR = u^2(2cos\theta sin\theta)$

$cos\theta = \frac{gR}{2u^2 sin\theta}$

$cos^2\theta = \frac{(gR)^2}{4u^4 sin^2\theta}$ .... (ii)

(i) and (ii)

$cos^2\theta = \frac{(gR)^2}{4u^4\left(\frac{2gH}{u^2}\right)}$

$cos^2\theta = \frac{gR^2}{8u^2H}$

$Cos^2\theta + sin^2\theta = 1$

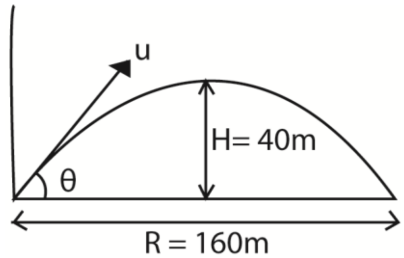
$\frac{2gH}{u^2} + \frac{gR^2}{8u^2H} = 1$

$16gH^2 - 8Hu^2 + gR^2 = 0$

Example 62

A ball is projected from point A and falls at point B which is in level with A at a distance of 160m from A. The greatest height of the ball attained is 40m. find the;

(a) angle and velocity at which the ball is projected (10marks)



Using $v^2 = u^2 + 2as$

$v^2 = u^2sin^2\theta - 2gh$

At maximum height $v_y = 0$

$0 = u^2sin^2\theta - 2gH$

$u^2 = \frac{2gH}{sin^2\theta}$ .... (i)

Range, $R = \frac{2u^2 sin\theta cos\theta}{g}$ .... (ii)

Eqn. (i) and eqn. (ii)

$R = 2 x \frac{2gH}{sin^2\theta} . \frac{sin\theta cos\theta}{g} = 4Hcot\theta$

.... (iii)

Substituting for R and H in eqn. (iii)

$160 = 4 \times 40cot\theta$

$\tan\theta = 1$

$\theta = \tan^{-1} 1 = 45^0$

Substituting for θ

$u^2 = \frac{2gH}{sin^2\theta} = \frac{2 x 40 x 9.8}{sin^2 45^0}$

$u = \sqrt{\frac{2 x 40 x 9.8}{sin^2 45^0}}$ = 39.60ms^{-1}

(b) time taken for the ball to attain the greatest height (02marks)

$t = \frac{usin\theta}{g} = \frac{39.60 x \sin 45^0}{9.8} = 2.8573s$

Example 63

A boy throws a ball at an initial speed of $40ms^{-1}$ at an angle of elevation θ. Taking g = $10ms^{-2}$, show that the times of flight corresponding to a horizontal range of 80m are positive roots of the equation $T^4 - 64T^2 + 256$.

$R = \frac{2u^2 cos\theta sin\theta}{g}$

$80 = \frac{2\ x\ 40^2 cos\theta sin\theta}{10}$

sinθcosθ =0.25

sinθ = $\frac{1}{4cos\theta}$ ……. (i)

x = ucosθt

80 = 40cosθT

cos θ = $\frac{2}{T}$ ………. (ii)

but $\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{1}{4x\frac{2}{T}}\right)^2 + \left(\frac{2}{T}\right)^2 = 1$

$\frac{T^2}{64} + \frac{4}{T^2} = 1$

$T^4 - 64T^2 + 256$

Example 64

A particle projected from a point O on a horizontal ground moves freely under gravity and it hits the ground again at A. Taking O as the origin, the equation of the path of the particle is $60y = 20\sqrt{3}x - x^2$ where x and y are measured in meters. Determine the

(a) initial speed and angle of projection

$60y = 20\sqrt{3}x - x^2$

$y = \frac{\sqrt{3}}{3}x - \frac{x^2}{60}$

Comparing with

$y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = 30^0$

$\frac{gx^2(1+\tan^2\theta)}{2u^2} = \frac{x^2}{60}$

$\frac{g(1+\tan^2\theta)}{2u^2} = \frac{1}{60}$

$\frac{9.8\left(1+\frac{3}{9}\right)}{2u^2} = \frac{1}{60}$; $u = 19.8ms^{-1}$

(b) Distance OA

At A, y = 0

$0 = 20\sqrt{3}x - x^2$

$0 = (20\sqrt{3} - x)x$

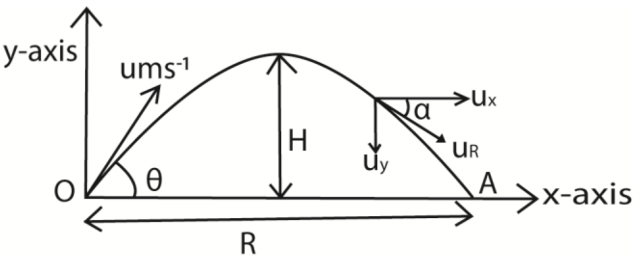
$x = 20\sqrt{3}m$

Example 65

A particle is projected from a point O on a level ground with initial speed 30ms^{-1} to pass through a point which is a horizontal distance 40m from O and a distance 10 vertically above the level of O.

(a) Show that there are two possible angles of projection
(b) If these angles are α and β, prove that tan (α+ β) = -4, take g = 10ms^{-2}

Solution



y = xtanθ $-\frac{gx^2(1+\tan^2\theta)}{2u^2}$

10 = 40tanθ $-\frac{10 \times 40(1+\tan^2\theta)}{2 \times 30^2}$

8tan$_2$θ – 36tanθ + 17 = 0

since it's quadratic equation in tanθ;

it has two roots and hence two values of θ < 90

(b) tanα + tanβ = $\frac{36}{8}$

tanα tanβ = $\frac{17}{8}$

tan(α + β) = $\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

= $\frac{\frac{36}{8}}{1-\frac{17}{8}}$

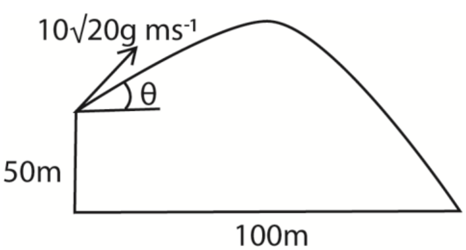
= $\frac{\frac{36}{8}}{\frac{8-17}{8}}$

= $\frac{36}{8} x \frac{8}{-9}$

tan(α + β) = -4

Example 66

A particle is projected with a speed $10\sqrt{2g}$ms-1from the top of a cliff 50m high. The particle hits the sea at a distance of 100m from the vertical through the point of projection. Show that there two possible directions of projection which are perpendicular. Determine the time taken from the point of projection in each case.



y = xtanθ $-\frac{gx^2(1+\tan^2\theta)}{2u^2}$

-50 = 100tanθ $-\frac{g \times 100^2(1+\tan^2\theta)}{2x\ 100\ x\ 2g}$

tan^{2}θ -4tanθ – 1 = 0

tanθ$_1$ = 2+$\sqrt{5}$ and tanθ$_2$ = 2-$\sqrt{5}$

tanθ$_1$ tanθ$_2$= (2+$\sqrt{5}$)(2-$\sqrt{5}$) = -1

Hence they are perpendicular

For horizontal motion

x = ucosθt

$\theta_1 = \tan^{-1}(2+\sqrt{5})$= 76.72^0

$t_1 = \frac{100}{10\sqrt{2g}\cos(76.72^0)}$ = 9.83s

$\theta_2 = \tan^{-1}(2-\sqrt{5})$= -13.36^0

$t_2 = \frac{100}{10\sqrt{2g}\cos(-13.38^0)}$ = 2.32s

Revision exercise 9

1. A golfer hits a ball with a velocity of 44.1ms-1 at an angle of $\sin^{-1}\frac{3}{5}$ above the horizontal. The ball lands on the green at a point which is level with the point of projection. Find the time for which the golf ball was in air. [5.4s]
2. a tennis ball is served horizontally from a point which is 2.5m vertically above a point A. The ball first strikes the horizontal ground through A at a distance 20m from A.
   (i) show that the ball is served with speed 28ms-1
   (ii) During its flight the ball passes over a net which is horizontal distance 12m from A. Find the vertical distance of the ball above the horizontal ground at the instant when it passes over the net. [1.6m]
3. An aircraft, at a height of 180m above horizontal ground and flying horizontally with speed of $70ms^{-1}$ releases emergency supplies. If these supplies are to land at a specific point, at what horizontal distance from this point must the aircraft release them? {take g = $10ms^{-2}$) {420m]
4. A stone is projected from top of a vertical cliff of height of h and the stone attains a maximum height (h+ b) above the ground. The stone strikes a sea at a distance, a from the foot of the cliff. Prove that the angle of elevation θ of the stone is given by $a^2\tan^2\theta - 4ab\tan\theta - 4bh = 0$
5. At time t = 0 a particle is projected from a point O on a horizontal plane with speed $14ms^{-1}$ in a direction inclined at an angle $\tan^{-1}\frac{3}{4}$ above the horizontal. The particle just clears the top of a vertical wall, the base of which is 8m from O. Find
   (a) time at which the particle passes over the wall [$\frac{5}{7}s$]
   (b) height of the wall [3.5m]
6. A particle P is projected from a point O with a speed of 60ms-1 at an angle $\cos^{-1}\frac{4}{5}$ above the horizontal. Find
   (a) time the particle takes to reach the point Q whose horizontal displacement from O is 96m. [2s]
   (b) height of Q above O [52.4m]
   (c) speed of the particle 2s after projection [$50.7ms^{-1}$]
7. A particle P is projected from a point O with a speed$50ms^{-1}$ at an angle $\sin^{-1}\frac{7}{25}$ above the horizontal. Find
   (a) height of P at the point where its horizontal displacement from O is 120m[4.375m]
   (b) speed of P 2s after projection. [$48.3ms^{-1}$]
   (c) times after projection at which P is moving at an angle of $\tan^{-1}\frac{1}{4}$ to the ground [0.204s, 2.65s]
8. A child throws a small ball from a height of 1.5m above level ground, aiming at a small target. The target is on top of a vertical pole of height 2m from the ground and horizontal displacement of the child from the pole is 6m. The initial velocity of the ball has magnitude ums^{-1} at an angle of elevation 40^0. The ball moves freely under gravity. (Take g = $10ms^{-2}$)
   (a) For u = 10, find the greatest height above the ground reached by the ball. [3.6m]
   (b) Calculate the value of u for which the ball hits the target[8.2]
9. A girl thrown a stone from a height of 1.5m above the ground with speed of $10ms^{-1}$ and hits a bottle standing on a wall 4m high and 5m from her. Take g = $10ms^{-1}$.

(a) Show that if α is the angle of projection of the stone as it leaves her hand then $1.25\tan^2\alpha - 5\tan\alpha + 3.75 = 0$

(b) the horizontal component of the stone's velocity has to be $6ms^{-1}$ for the bottle to be knocked off. By solving the above equation or otherwise, show that α has to be 45^0 for the bottle to be knocked off.

10. A basketball is released from player's hands with a speed of $8ms^{-1}$ at inclination of α^0 above the horizontal so as t land in the centre of the basket, which is 4m horizontally from the point of release and a vertical height of 0.5m above it. Take g =$10ms^{-2}$.

(a) show that α satisfies the quadratic equation; $5\tan^2\alpha - 16\tan\alpha + 7 = 0$

(b) Given that the player throws the ball at a large angle of projection find α for the ball to land in the basket. [70^0, 1.43s]

Thank you

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