SENIOR SIX TERM 2

TOPIC 4/5: Iterative Methods

Competency: The learner applies iterative numerical methods to approximate solutions to problems while critically analysing accuracy and limitations of methods used in real life situations

Contents

Linear interpolation and extrapolation to estimate the unknown tabular values of a function using the gradient method

Linear interpolation

Deals with computations of values that lie within a given range

Example 1

The table below shows values of a function f(x)

x	1.8	2.0	2.2	2.4
f(x)	0.532	0.484	0.436	0.384

Find values of (i) f(1.88) (ii) x corresponding to f(x) = 0.4

Solution

1.8	1.88	2.0
0.532	y	0.484

$$\frac{y-0.532}{1.88-1.8} = \frac{0.884-0.532}{2.0-1.8}$$

y = 0.513

(ii)

2.2	x_0	2.4
0.436	0.4	0.384

$$\frac{x_0-2.2}{0.4-0.436} = \frac{2.4-2.2}{0.384-0.436}$$

Example 2

Given the table below

x	9	10	11	12
f(x)	2.66	2.42	2.18	1.92

Using linear interpolation find

(i) f(x) when x = 10.15 (ii) f^{-1} (2.02)

Solution

10	10.15	11
2.42	y	2.18

$$\frac{y-2.42}{10.15-10} = \frac{2.18-2.42}{11-10}$$

y =2.384

11	x_0	12
2.18	2.02	1.92

$$\frac{x_0-11}{2.02-2.18} = \frac{12-11}{1.92-2.18}$$

Example 3

Given table below

x^0	40.0^0	40.4^0	40.8^0	50.4^0
$\sin x^0$	0.6428	0.6481	0.6534	0.7705

Find (i) $\sin 40.5^0$ (ii) $\sin^{-1} 0.6445$

Solution

40.4^0	40.5^0	40.8^0
0.6481	y	0.6534

$$\frac{y-0.6481}{40.5-40.4} = \frac{0.6534-0.6481}{40.8-40.4}$$

y = 0.6494

40.0^0	x_0	40.4^0
0.6428	0.6445	0.6481

$$\frac{x_0-40.0}{0.6445-0.6428} = \frac{40.4-40.0}{0.6481-0.6428}$$

Linear extrapolation

This deals with computation of values that lie outside given values

Example 4

Given the table below

x	2.2	2.6	3.1
x^3	10.648	17.576	29.791

Find 3.4^3

2.6	3.1	3.4
17.576	29.791	y

$$\frac{y-29.791}{3.4-3.1} = \frac{29.791-17.576}{3.1-2.6}$$

y = 37.12

Example 5

The table below is an extract from table of sec x

$x = 60^0$	0’	12’	24’	36’	48’
$\sec x$	2.0000	2.0122	2.0245	2.0371	2.0498

Use linear interpolation to determine

(i) $\sec 60^0 15'$

(ii) angle whose secant is 2.0436 [$60^0 42'$]

Solution

(i)

12’	15’	24’
2.0122	y	2.0245

$\frac{y-2.0122}{15-12} = \frac{2.0245-2.0122}{24-12}$

y = 2.03065

(ii)

36’	x	48’
2.0371	2.0436	2.0498

$\frac{x-36}{2.0436-2.0371} = \frac{48-36}{2.0498-2.0371}$

Example 6

The table below shows the values of a function f(x)

x	1.8	2.0	2.2	2.4
f(x)	0.532	0.484	0.436	0.384

Use linear interpolation to find the value of

(i) F(2.08)

1.8	2.08	2.0
0.532	f(x)	0.484

$\frac{f(x)-0.436}{2.08-2.0} = \frac{0.436-0.484}{2.2-2.0}$

$\frac{f(x)-0.436}{0.08} = \frac{-0048}{0.2}$

f(x) = 0.4648 or 0.465 (3D)

(ii) x corresponding to f(x) = 0.5 (05marks)

1.8	x	2.0
0.532	0.5	0.484

$\frac{0.5-0.532}{x-1.8} = \frac{0.484-0.532}{2.0-18}$

$\frac{-0.032}{x-1.8} = \frac{-0.048}{0.2}$

$x =$ 1.9333 or 1.9 (1D)

Example 7

Given the table below,

x	0	10	20	30
y	6.6	2.9	-0.1	-2.9

Use linear interpolation to find

(a) y when x = 16

Extract

x	10	16	20
y	2.9	y_0	-0.1

$\frac{y_0-2.9}{16-10} = \frac{-0.1-2.9}{20-10}$

$\frac{y_0-2.9}{6} - \frac{-3.0}{10}$

y_0 = 1.1

hence when x = 16, y =1.1

(b) x when y = -1

Extract

x	20	x_0	30
y	-0.1	-1	-2.9

$\frac{x_0-20}{-1-(-0.1)} = \frac{30-20}{-2.9-(-0.1)}$

$x_0 = 23.2$

Hence when y = -1; x = 23.2

Example 8

The table below shows the values of a function f(x) for given values of x.

x	9	10	11	12
f(x)	2.66	2.42	2.18	1.92

Use linear interpolation or extrapolation to find

(a) f(10.4)

Extract

10	10.4	11
2.42	f(x)	2.1

Using gradient approach

$\frac{2.18-f(x)}{11-10.4} = \frac{2.18-2.42}{11-10}$

$\frac{2.18-f(x)}{0.6} = \frac{-0.24}{1}$

f(x) = 2.18 + 0.24 x 0.6 = 2.324

(b) the value of x, corresponding to f(x) = 1.46 (05marks)

Extract

x	12	11
1.46	1.92	2.18

Using gradient approach

$\frac{2.18-1.46}{11-x} = \frac{2.18-1.92}{11-12}$

$\frac{0.72}{11-x} = \frac{0.26}{-1}$

X = 11 - $\frac{-1\ x\ 0.72}{0.26}$ = 13.769

Revision exercise

1. Table below is an extract from the table of cos x

x	0^0	10^0	20^0	30^0	40^0	50^0
Cos x	0.1736	0.1708	0.1679	0.1650	0.1622	0.1593

Use linear interpolation to determine: (i) cos 80^0 36' [0.1633] cos $^{-1}$(0.1685) [$80^0$18']

2. The table below shows variation of temperature with time in a certain experiment.

Time (s)	0	120	240	360	480	600
Temperature (^{0}C)	100	80	76	65	50	48

Use linear interpolation to determine

(i) value of ^{0}C corresponding to 400s [62^0C]

(ii) time at which the temperature is 77^0C [192s]

3. The table below shows the value of a function In(x) for given values of x

x	1.4	1.5	1.6	1.7
In(x)	0.3365	0.4055	0.4700	0.5306

Using linear interpolation or extrapolation, find

(i) In(1.66) [0.5064] (ii) find value of x corresponding to In(x) = 0.400 [1.492]

4. The table below shows variation of temperature with time in certain experiment.

Time (s)	0	10	15	20	30
Temperature (^{0}C)	80	70.2	65.8	61.9	54.2

Use linear interpolation to determine

(i) value of θ^0 corresponding to T= 18s [63.5^0C]

(ii) Time T at which the temperature θ^0 = 60^0C [22.5s]

5. Given the table below

x	-1.0	-0.5	-1.4
y	-1.0	-2.2	-3.7

Using linear interpolation or linear extrapolation to find

(i) y when x = 0.5 [-4.6] (ii) x when y = -4.5 [0.458]

6. In an examination, scaling is done such that candidate A who originally scored 35% gets 50% and candidate B with 40% gets 65%, determine the original mark for candidate C whose new mark is 80% [45%]
7. The table below is an extract of $\log_{10} x$

x	80.00	80.20	80.50	80.80
$\log_{10} x$	1.9031	1.9042	1.9058	1.9074

Using linear interpolation find

(i) $\log_{10} 80.759$[1.9072]

(ii) the number whose logarithm is 1.90388 [80.14]

8. The table below shows the values of a function f(x) for given values of x

x	2	3	4	5
$f(x)$	3.88	5.11	8.14	11.94

Use linear interpolation to determine

(i) f(2.15) [4.06]

(ii) the value of x corresponding to f(10.6) [4.68]

9. The table below shows distance in km a truck moves with a given amount of fuel in litres (l)

Distance (km)	20	28	33	42
$Fuel\ (l)$	10	13	21	24

Use linear interpolation or extrapolation to find

(i) How far the truck can move on 27.5l of fuel [52.5km]

(ii) the amount of fuel required to cover29.8km [15.88l]

10. The table below shows the values of a continuous f(t) with respect to t

t	0	0.3	0.6	1.2	1.6
$f(t)$	2.72	3.00	3.32	4.06	4.95

Use linear interpolation or extrapolation, find

(i) f(t) when t = 0.9 [3.69]

(ii) the value of t corresponding to f(t) = 4.48 [1.48]

11. The table below shows the delivery charges by courier company

Mass (g)	200	400	600
$charges\ (shs.)$	700	1200	300

Use linear interpolation or extrapolation, find

(i) the delivery charge of a parcel weighing 352g [1080]

(ii) mass of a parcel whose delivery charge is shs. 3,300 [633.33kg]

12. The table below shows the cost y shillings for hiring a motor cycle for a distance x kilometres.

Distance x (km)	10	20	30	40
Cost (shs.)	2800	3600	4400	5200

Use linear interpolation or extrapolation, find

(i) the cost of hiring the motor cycle for distance of 45km [shs. 5600]

(ii) distance travelled if he paid shs. 4000 [25km]

13. The table below shows the values of a function f(x) for given values of x

x	0.4	0.6	0.8
f(x)	-0.9613	-0.5108	-0.2231

Use linear interpolation to determine f^{-1}(-0.4308} correct to 2 decimal places [0.66]

14. The table below shows how T caries with S

T	-2.9	-0.1	2.9	3.1
S	30	20	12	9

Use linear interpolation/extrapolation to estimate values of

(a) T when S = 26 [-1.78] (ii) S when T = 3.4 [4.5]

15. The table below shows the commuter bus fare from stages A to B, C, D and E

Stage	A	B	C	D	E
Distance (km)	0	12	16	19	23
Fare (shs)	0	1300	1700	2200	2500

(a) Jane boarded from A and stopped at a place 2km after E. How much did she pay? (03marks) [shs. 2650]

(b) Okello paid shs 2000. How far from A did the bus leave him? (02marks) [17.km]

16. The table below shows the value of x and corresponding values of a function f(x)

The table below shows how T caries with S

x	0.3	0.6	0.9	1.2
f(x)	3.00	3.22	3.69	4.06

Use linear interpolation/extrapolation to estimate values of

(i) f(x) when x = 0.4 [3.0733] (ii) x when f(x) = 3.82 [1.0054]

Location of real roots

Using sign change and graphical methods to determine points between which roots a function are found.

The range where the roots of an equation lie can be located using the following methods

(i) sign change
(ii) Graphical method

(a) Sign change method

Example 9

Show that equation $x^3 + 6x^2 + 9x + 2 = 0$ has a root between -1 and 0

Solution

$f(x) = x^3 + 6x^2 + 9x + 2$

$f(-1) = (1)^3 + 6(1)^2 + 9(-1) + 2 = -14$

$f(0) = (0)^3 + 6(0)^2 + 9(0) + 2 = 2$

Since there is a sign change the root lies between 0 and -1.

Example 10

Show that the equation $e^{2x}\sin x - 1 = 0$ has a root between 0 and 1

Solution

Note that in trigonometric function the calculator must be in radian mode

$f(x) = e^{2x}\sin x - 1$

$f(0) = e^{2(0)}\sin 0 - 1 = -1$

$f(1) = e^{2}\sin 1 - 1 = 5.2177$

Since there is a sign change the root lies between 0 and 1.

(b) Using graphical method

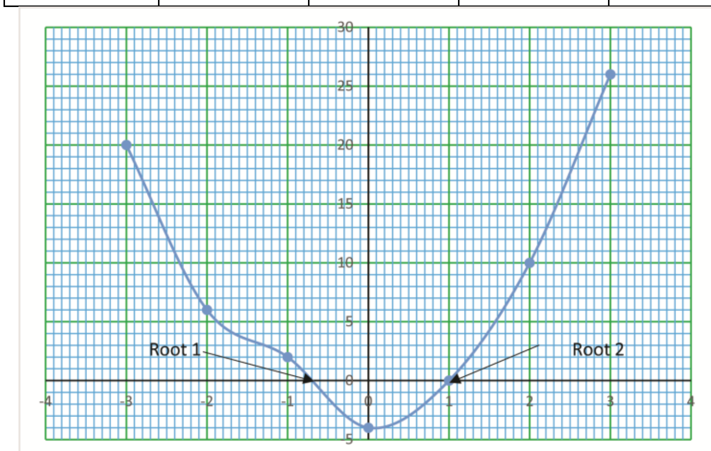
One or two graph(s) can be drawn to locate the root.

(i) Single graph method
When one graph is drawn then the root lies between the two points where the curve crosses the x-axis.

Example 3

Using a suitable graph locate the interval over which the root of the equation $3x^2 + x - 4 = 0$ lie.

x	-3	-2	-1	0	1	2	3
f(x)	20	6	-2	-4	0	10	26

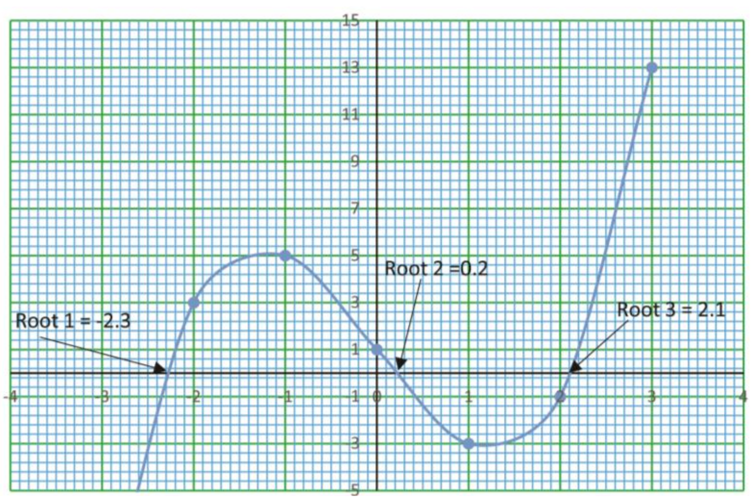


The root lies between -1 and 1

Example 11

Show graphically that there is positive real root of equation $x^3 - 5x + 1 = 0$

x	-3	-2	-1	0	1	2	3
f(x)	-11	3	5	1	-3	-1	13



(ii) Double graph method

When two graphs are drawn, the root lies between the points where the two curves meet.

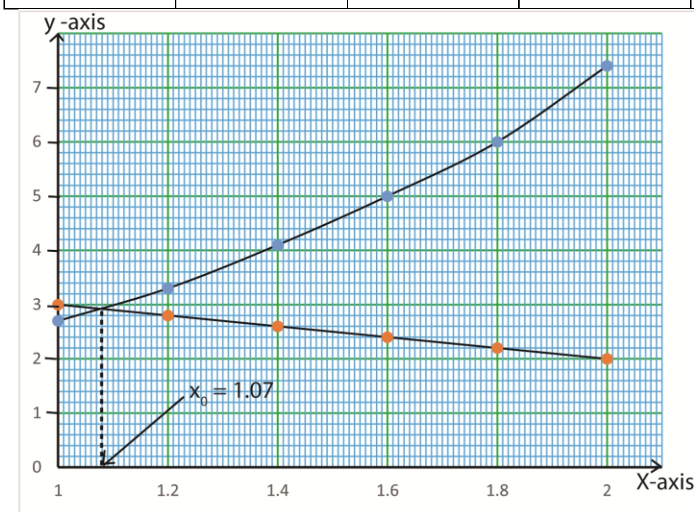
Note

(i) Both curves must have a consistent scale and should be labelled.
(ii) A line must be drawn using a ruler while a curve must be drawn using a freehand
(iii) Both graphs must be labelled
(iv) The initial approximation of the root must be located and indicated in the graph

Example 12

By plotting graph of y = e^x and y = 4 - x on the same axes, show the root of the equation e^x +x - 4 = 0 lie between 1 and 2

x	1	1.2	14	1.6	1.8	2.0
y = e^x	2.7	3.3	4.1	5.0	6.0	7.4
y = 4 - x	3.0	2.8	2.6	2.4	2.2	2

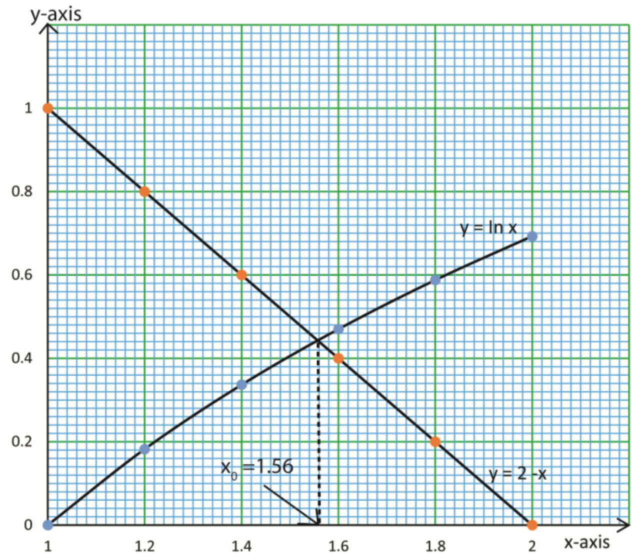


Therefore the root (1.07) lies between 1 and 2.

Example 13

Show that the equation lnx + x – 2 = 0 has a real root between x = 1 and x = 2

x	1	1.2	1.4	1.6	1.8	2.0
y = lnx	0	0.1823	0.3365	0.4700	0.5878	0.6731
y = 2-x	1	0.8	0.6	0.4	0.2	0

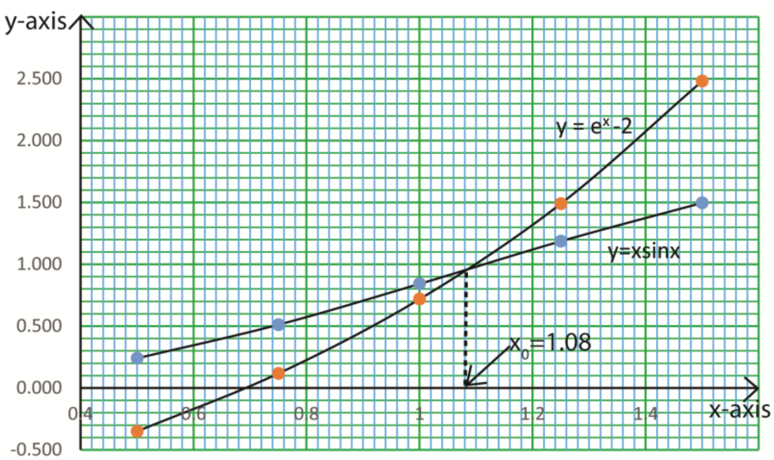


Therefore the root lies between x =1 and x =2

Example 14

By plotting graphs y = e^x -2 an y = x sin x on the same axis show that the root of the equation e^x – 2 –xsinx = 0 lies between x = 0.5 and x = 1.5

x	0.5	0.75	1.00	1.25	1.5
y = xsinx	0.240	0.511	0.841	1.186	1.496
y = e^x-2	-0.351	0.117	0.718	1.490	2.481



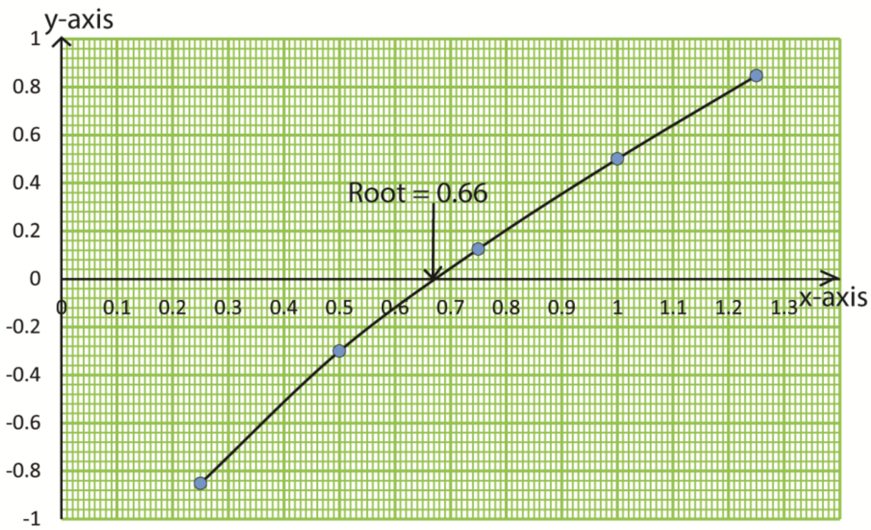
Example 15

Show graphically the equation x + logx = 0.5 has only one real root that lie between 0.5 and 1.

Solution

let y = x + logx - 0.5

x	0.25	0.5	0.75	1.00	1.25
y	-.852	-0.301	0.125	0.5	0.847



Therefore the root (0.66) lies between 0.5 and 1

Revision exercise 2

1. By sketching graphs of y = 2x and y = tanx show that the equation 2x = tanx has only one root between x = 1.1 and 1.2. Use linear interpolation to find the value of the root correct to 2dp.
2. Given the equation y = sinx - $\frac{x}{3}$, show by plotting two suitable graphs on the same axis that positive root lies between $\frac{2\pi}{3}$ and $\frac{5\pi}{6}$.
3. Show graphically that the positive real root of the equation $2x^2 + 3x - 3 = 0$ lies between 0 and 1 [0.7]
4. Use a graphical method to show that the equation $e^x - x - 2 = 0$ has only one real root between 2 and -1 by drawing two graphs $y = e^x$ and y = x + 2 [-1.8]
5. On the same axes, draw graphs of y = 3 – 3x and $y = 2x^2$ to show that the root of the equation $2x^2 + 3x\ \ 3 = 0$ lies between -3 and -2 [-2.2]
6. Show graphically that the positive real root of the equation $x^3 - 3x - 1 = 0$, lies between 1 and 2 [1.6]
7. on the same axes, draw graph y = 3x -1 and $y = x^3$ to show that the root of the equation $X^3 - 3x - 1 = 0$lies between 0 and 1.[0.35]

8. Using suitable graphs and plotting them on the same axes. Find the root of the equation $e^{2x}\sin x - 1 = 0$, in the interval x = 0.1 and x = 0.8. [0.44]
9. Show graphically that equation $e^{-x} = x$ has only one real root between 0.5 and 1. [0.56]
10. Show graphically that equation $e^x = -2x + 2$ has only one real root between 0 and 1.0.
11. on the same axes, draw graphs of y = 9x -4 and y = x^3 show that the root of equation $x^3 - 9x + 4 = 0$ lie between 2.5 and 3
12. Show that the positive real root of equation $4 + 5x^2 - x^3 = 0$ lies between 5 and 6.
13. On the same axes, draw graphs of y = x +1 and y = tanx to show that the equation tanx - x -1 = 0 lie between 1 and 1.5.
14. Using suitable graphs and plotting them on the same axes, find the roots of the equation $5e^x = 4x + 6$ in the interval x = 2 and x = -1.
15. On the same axes, draw graphs of y = 2x + 1 and y = $\log_e(x+2)$ to show that the root of equation $\log_e(x+2) - 2x - 1 = 0$ lies between 1 and 0.
16. Using suitable graphs and plotting them on the same axes, find the real root of the equation $9\log_{10} x$ = 2(X -1) in the interval x = 3 and x =4.

Method of solving for roots

The following methods can be used

Using Interpolation to solve roots of a functions

Example 16

Show that the equation $x^4 - 12x^2 + 12 = 0$ has root between 1 and 2. Hence use linear interpolation to get the first approximation of the root.

Solution

$f(x) = x^4 - 12x^2 + 12$

$f(1) = 1^4 - 12(1)^2 + 12 = 1$

$f(2) = 2^4 - 12(2)^2 + 12 = -20$

Since there is a sign change,

then the root lies between

1 and 2.

x	1	x_0	2
f(x)	1	0	-20

$\frac{x_0-1}{0-1} = \frac{2-1}{-20-1}$

$x_0 = 1.05$

Example 17

Show that the equation $2x - 3\cos\left(\frac{x}{2}\right) = 0$ has a root between 1 and 2.Hence use linear interpolation twice to get the approximation of the root.

solution

Note: for trigonometric functions the calculator must be strictly in radian mode

$f(x) = 2x - 3\cos\left(\frac{x}{2}\right)$

$f(1) = 2 \times 1 - 3\cos\left(\frac{1}{2}\right) = -0.633$

$f(2) = 2 \times 1 - 3\cos\left(\frac{2}{2}\right) = 2.379$

Since there is a sign change, then the root lies between 1 and 2

x	1	x_0	2
f(x)	-0.633	0	2.379

$$\frac{x_0 - 1}{0 - -0.633} = \frac{2-1}{2.379 - -0.633}$$

$x_0 = 1.2102$

x	1.2102	x_0	2
f(x)	-0.047	0	2.379

$$\frac{x_0 - 1}{0 - -0.047} = \frac{2-1}{2.379 - -0.047}$$

$x_0 = 1.226$

Example 18

Show that the equation $3x^2 + x - 5 = 0$ has a real root between x = 1 and x = 2. Hence use linear interpolation twice to calculate the root to 2 dp.

Solution

$f(x) = 3x^2 + x - 5$

$f(1) = 3(1)^2 + 1 - 5 = 1$

$f(2) = 3(2)^2 + 2 - 5 = 9$

Since there is a sign change,

then the root lies between

x	1	x_0	2
f(x)	-1	0	9

$$\frac{x_0 - 1}{0 - -1} = \frac{2-1}{9 - -1}$$

$x_0 = 1.1$

x	1.1	x_0	2
f(x)	-0.27	0	9

$$\frac{x_0 - 1.1}{0 - -0.27} = \frac{2-1.1}{9 - -0.27}$$

$x_0 = 1.13$

General iterative method

This involves generating equation by splitting the original equation into several equations by making x the subject.

Example 19

Given $x^2 + 4x - 2 = 0$. Find the possible equations for estimating the roots

Solution

Let x_{n+1} be a better approximation

x_n be the next approximation

$x_{n+1} = \frac{2}{x_n} - 4$ | $x_{n+1} = \sqrt{(2 - 4x_n)}$ | $x_{n+1} = \frac{2-x^2}{4}$

Example 20

Given f(x) = $x^3 - 3x - 12 = 0$. Generate equations in form of $x_{n+1} = g(x_n)$ that can be used to solve the equation f(x) =0

Solution

Let x_{n+1} be a better approximation

x_n be the next approximation

$x_{n+1} = \frac{x_n^3-12}{3}$ | $x_{n+1} = \sqrt[2]{(3x_n + 12)} = \frac{12}{x_n^2-3}$ | $x_{n+1} = \sqrt{\left(3 + \frac{12}{x_n}\right)} = \frac{3x_n+12}{x_n^2}$

Testing for convergence

From the several iterative equations obtained, the equation whose $|f^1(x_n| < 1$ is the one which converges the correct root.

Example 21

Given the two iterative formulas

(i) $x_{n+1} = \frac{x_n^3-1}{5}$ (ii) $x_{n+1} = \sqrt{\left(5 + \frac{1}{x_n}\right)}$

Using x_0 = 2 deduce a more suitable formula for solving the equation. Hence find the root correct to 2dp

$x_{n+1} = \frac{x_n^3-1}{5}$

f(x_n) = $x_{n+1} = \frac{x_n^3-1}{5}$; $f^1(x_n) = \frac{3x_n^2}{5}$

$f^1(2) = \frac{3(2)^2}{5}$ = 2.4

since $|f^1(2)|>1$ it will not converge

$x_{n+1} = \sqrt{\left(5 + \frac{1}{x_n}\right)}$

f(x_n) = $\sqrt{\left(5 + \frac{1}{x_n}\right)}$; $f^1(x_n) = -\frac{1}{2}x_n^{-2}\left(5 + \frac{1}{x_n}\right)$

$f^1(2) = -\frac{1}{2}(2)^{-2}\left(5 + \frac{1}{2}\right)$ = -0.0533 since $|f^1(2)|$<1 it will converge so this equation gives the root

$x_{n+1} = \sqrt{\left(5 + \frac{1}{x_n}\right)}$, $|e|$= 0.005, x_0 = 2

$x_1 = \sqrt{\left(5 + \frac{1}{2}\right)}$ = 2.3452

$|x_1 - x_0|$ = 2.3452 – 2 = 0.3452 > 0.005

$x_2 = \sqrt{\left(5 + \frac{1}{2.3452}\right)}$ = 2.3295

$|x_2 - x_1|$ = 2.3452 – 2.3295 = 0.0157 > 0.005

$x_3 = \sqrt{\left(5 + \frac{1}{2.3295}\right)}$ = 2.3301

$|e|$=$|2.3301 - 2.3295|$= 0.0006 < 0.005

Hence root is 2.33

Example 22

Show that the iterative formula for solving the equation $x^3 = x + 1$ is $x_{n+1} = \sqrt{\left(1 + \frac{1}{x_n}\right)}$ starting with x_0 = 1 find the solution of the equations to 3sf.

Solution

$x_{n+1} = \sqrt{\left(1 + \frac{1}{x_n}\right)}$ $|e|$= 0.005, x_0 = 1

$x_1 = \sqrt{\left(1 + \frac{1}{1}\right)}$ =1.41421

$|x_1 - x_0|$ = $|1.41421 - 1|$ = 0.41421 > 0.005

$x_2 = \sqrt{\left(1 + \frac{1}{1.41421}\right)}$ =1.30656;

$|x_2 - x_1|$ = $|1.30656 - 1.41421|$ = 0.10765 > 0.005

$x_3 = \sqrt{\left(1 + \frac{1}{1.30656}\right)}$ = 1.32869

$|x_3 - x_2|$ = $|1.32869 - 1.30656|$ =0.03691 > 0.005

$x_4 = \sqrt{\left(1 + \frac{1}{1.32869}\right)}$ =1.32389

$|e|$=$|1.32389 - 1.32869|$= 0.0048 < 0.005

Hence the root is 1.32

Example 23

Given two iterative formulae I and II (shown below) for calculating the positive root of the quadratic equation f(x) = 0

$x_{n+1} = \frac{1}{2}(x_n^2 - 1)$ and $x_{n+1} = \frac{1}{2}\left(\frac{x_n^2+1}{x_n-1}\right)$ for n = 1, 2, 3 …………

Taking x_0 = 2.5, use each formula thrice to two decimal places to decide which is the more suitable formula. Give a reason for your answer.

Solution

Iterative formula $x_{n+1} = \frac{1}{2}(x_n^2 - 1)$

x_0 = 2.5

$x_1 = \frac{1}{2}(2.5^2 - 1)$ = 2.625

$|x_1 - x_0|$ = 0.125

$x_2 = \frac{1}{2}(2.625^2 - 1)$ 2.99453125

$|x_2 - x_1|$ = 0.3200125

$x_3 = \frac{1}{2}(2.99453125^2 - 1)$ = 3.837432861

$|x_3 - x_2|$ = 0.89212036

Iterative formula $x_{n+1} = \frac{1}{2}\left(\frac{x_n^2+1}{x_n-1}\right)$

x_0 = 2.5

$x_1 = \frac{1}{2}\left(\frac{2.5^2+1}{2.5-1}\right)$ = 2.416666667

$|x_1 - x_0|$ = 0.083333

$x_2 = \frac{1}{2}\left(\frac{2.416666667^2+1}{2.416666667-1}\right)$ = 2.414215686

$|x_2 - x_1|$ = 0.002450781

$x_2 = \frac{1}{2}\left(\frac{2.414215686^2+1}{2.414215686-1}\right)$ = 2.414215686

$|x_3 - x_2|$ = 0.000002124

The more suitable formula is $x_{n+1} = \frac{1}{2}\left(\frac{x_n^2+1}{x_n-1}\right)$.

Because the absolute difference between $x_3 - x_2$ is less than absolute error, where as in the first formula the absolute difference between$x_3 - x_2$ is greater than absolute error. In all the 2nd formula converge whereas the first formula diverges.

Example 24

(a) (i) Show that the equation $e^x - 2x - 1 = 0$ has a root between x = 1 and x = 1.5.
(ii) Use linear interpolation to obtain an approximation for the root

(b) (i) Solve the equation in (a)(i), using each formula below twice

Take the approximation in (a)(i) as the initial value

Formula I: $x_{n+1} = \frac{1}{2}(e^{x_n} + 1)$.

Formula II: $x_{n+1} = \frac{e^{x_n}(x_n-1) + 1}{e^{x_n}-2}$

(ii) Deduce with a reason which of the two formulae is appropriate for solving the given equation in (a)(i). Hence write down a better approximate root, correct to two decimal places

Solution

(a) (i) using sign change method
let $f(x) = e^x - 2x - 1$
$f(1) = e^1 - 2(1) - 1 = -2.817$
$f(1.5) = e^{1.5} - 2(1.5) - 1 = 0.4817$

Since f(1).f(1.5)< 0, the root lies between x = 1 and x = 1.5

(a)(ii) Extract

1	x_0	1.5
-0.2817	0	0.4817

$\frac{x_0-1}{0--0.2817} = \frac{1.5-1}{0.4817--0.2817}$; $x_0 = 1.1845$

Hence the approximation to the root is 1.18 (2 dp)

(b)(i)

Solution

formula 1: $x_{n+1} = \frac{1}{2}(e^{x_n} + 1)$

$x_0 = 1.18$

$x_1 = \frac{1}{2}(e^{1.18} + 1)$= 2.1272

$|x_1 - x_0|$ = 0.9472

$x_2 = \frac{1}{2}(e^{2.127187} + 1)$= 4.6956

$|x_2 - x_1|$ = 2.5684

formula 2: $x_{n+1} = \frac{e^{x_n}(x_n-1)+1}{e^{x_n}-2}$

$x_0 = 1.18$

$x_1 = \frac{e^{1.18}(1.18-1)+1}{e^{1.18}-2}$= 1.2642

$|x_1 - x_0|$ = 0.0842

$x_2 = \frac{e^{1.2642}(1.2642-1)+1}{e^{1.2642}-2}$= = 1.2565

$|x_2 - x_1|$ = 0.0077

Formula 1, the sequence 1.18, 2.1272, 4.6956 diverge, hence the formula is not suitable

Formula 2, the sequence 1.18, 1.2642, 1.2565 converge, hence the formula is suitable solving the equation

A better approximation = 1.26 (2 dp)

Revision exercise 3

1. Given the following iterative formula

   (i) $x_{n+1} = 5 - \frac{3}{x_n}$ (ii) $x_{n+1} = \frac{1}{5}(x_n^2 + 3)$

   Taking x_0 = 5 deduce a more suitable iterative formula for solving the equation

2. Show that the iterative formula for solving the equation x^2 -5x + 2 = 0 can be written in two ways as $x_{n+1} = 5 - \frac{2}{x_n}$ or $x_{n+1} = \frac{x_n^2+2}{5}$.
   Starting with x_0 = 4, deduce the more suitable formula for the equation and hence find the root correct to 2 dp [4.56]

3. Show that the iterative formula for solving the equation x^3 – x – 1 = 0 is $x_{n+1} = \sqrt{\left(1 + \frac{1}{x_n}\right)}$.

   Starting with xo = 1 find the root of the equation correct to 3 s.f. [1.33]

4. (a) Show that the iterative formula for solving the equation $2x^2$ – 6x - 3 = 0 is $x_{n+1} = \frac{2x_n^2+3}{4x_n+6}$

   (b) Show that the positive root for $2x^2$ – 6x - 3 = 0 lies between 3 and 4. find the root correct to 2 decimal places [3.44]

5. (a) If b is the first approximation to the root of equation x2 = a, show that the second approximation to the root is given by $\frac{b+\frac{a}{b}}{2}$. Hence taking b = 4, estimate $\sqrt{17}$ correct to 3 dp [4.123]

   (b) Show that the positive real root of the equation x^2 – 17 = 0 lies between 1.5 and 1.8. Hence use the formula in (a) above to determine the root to 3 dp

Newton Raphson's Method to solve roots of a function

It is given by $x_{n+1} = x_n - \left[\frac{f(x_n)}{f^1(x_n)}\right]$ n = 1, 2, 3 …

Example 25

Use Newton Raphson's method to find the root of equation x^3 + x – 1 = 0 using x_0 = 0.5 as the initial approximation, correct your answer to 2 decimal places

Solution

f(x) = x^3 + x – 1, $f^1(x)$ = $2x^2$ + 1

$x_{n+1} = x_n - \left[\frac{(x^3+x-1)}{3x^2+1}\right]$

$x_1 = 0.5 - \left[\frac{((0.5)^3 + 0.5-1)}{3(0.5)^2+1}\right]$ =0.7142

$|x_1 - x_0| = 0.7142 - 0.5 = 0.2142 > 0.005$

$x_2 = 0.7142 - \left[\frac{((0.7142)^3 + 0.7142-1)}{3(0.7142)^2+1}\right]$ =0.6831

$|x_2 - x_1| = |0.6831 - 0.7142|$

$= 0.0311 > 0.005$

$x_2 = 0.6831 - \left[\frac{((0.6831)^3 + 0.6831-1)}{3(0.6831)^2+1}\right]$ =0.6824

$|3 - x_2| = |0.6824 - 0.6831|$

$= 0.0007 < 0.005$

∴ Root = 0.68

Example 25

Show that the equation 5x - 3cos2x = 0 has a root between 0 and 1. Hence use Newton Raphson's method to find the root of equation correct to 2 decimal places using x_0 = 0.5 .

Solution

Using sign change method to locate The roots. Note for trigonometric functions the calculator is used in radians mode

f(x) =5x - 3cos2x

f(0) = 5(0) – 3cos2(0) = -3

f(1) = 5(1) – 3cos2(1) = 2.455

Since there is change sign the root lies between x = 0 and x = 1

f(x) = 5x - 3cos2x, $f^1(x)$ = 5 + 6sin2x

$$x_{n+1} = x_n - \left[\frac{(5x_n - 3\cos 2x_n)}{5 + 6\sin 2x_n}\right]$$

x_0 = 0.5, $|e| = 0.005$

$$x_1 = 0.5 - \left[\frac{(5(0.5) - 3\cos 2(0,5))}{5 + 6\sin 2(0.5)}\right] = 0.4125$$

$$|x_1 - x_0| = |0.4125 - 0.5| = 0.0875 > 0.005$$

$$x_1 = 0.4125 - \left[\frac{(5(0.4125) - 3\cos 2(0.4125))}{5 + 6\sin 2(0.4125)}\right] =0.4096$$

$$|x_2 - x_1| = |0.4096 - 0.4125| = 0.0029 < 0.005$$

∴ Root = 0.41

Example 27

Use Newton Raphson's iterative formula to show that the cube root of a number N is given by

$\frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$. Hence taking x_0 = 2.5 determine $\sqrt[3]{10}$ correct to 3 dp.

Solution

$x = N^{\frac{1}{3}}$

$x^3 - N = 0$

$f(x) = x^3 - N$; $f^1(x) = 3x^2$

$x_{n+1} = x_n - \left[\frac{(x_n{}^3 - N)}{3x_n{}^2}\right] = \frac{x_n(3x_n{}^2) - (x_n{}^3 - N)}{3x_n{}^2}$

$= \frac{2x_n{}^3 + N}{3x_n{}^2} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$.

$x_0 = 2.5$, N = 10, $|e| = 0.005$

$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$

$x_1 = \frac{1}{3}\left(2(2.5) + \frac{N}{2.5^2}\right) = 2.2$

$|x_1 - x_0| = |2.2 - 2.5| = 0.3 > 0.005$

$x_2 = \frac{1}{3}\left(2(2.2) + \frac{N}{2.2^2}\right) = 2.1554$

$|x_2 - x_1| = |2.1554 - 2.2| = 0.0446 > 0.005$

$x_3 = \frac{1}{3}\left(2(2.1554) + \frac{N}{2.1554^2}\right) = 2.1544$

$|x_3 - x_2| = |2.1544 - 2.1554| = 0.001 < 0.005$

∴ Root = 2.154

Example 28

(a) Show that the equation x -3sinx = 0 has a root between 2 and 3. (03marks)

f(x) = x – 3sinx
f(2) = 2 – 3sin 2 = -0.7279
f(3) = 3 – 3sin3 = 2.5766
since f(2).f(3) = -1.8755<0
there exist a root of x-3sinx =0 between 2 and 3

(b) Show that Newton- Raphson iterative formula for estimating the root of the equation in (a) is given by

$$X_{n+1} = \frac{3(\sin x_n - x_n \cos x_n)}{1 - 3\cos x_n}, n = 0, 1, 2 \ldots.$$

Hence find the root of the equation corrected to 2 decimal places (09 marks)

f'(x) = 1 – 3cos x

$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$

$= x_n - \frac{x_n - 3\sin x_n}{1 - 3\cos x_n}$

$= \frac{x_n - 3x_n\cos x_n - x_n + 3\sin x_n}{1 - 3\cos x_n}$

$x_{n+1} = \frac{3(\sin x_n - x_n\cos x_n}{1 - 3\cos x_n}$

Taking $x_0 = \frac{2+3}{2} = 2.5$

$x_1 = \frac{3(\sin 2.5 - 2.5\cos 2.5}{1 - 3\cos 2.5} = 2.293$

Error = $|2.293 - 2.5| = 0.207 > 0.005$

$x_2 = \frac{3(\sin 2.293 - 2.5\cos 2.293}{1 - 3\cos 2.293} = 2.279$

Error = $|2.279 - 2.293| = 0.014 > 0.005$

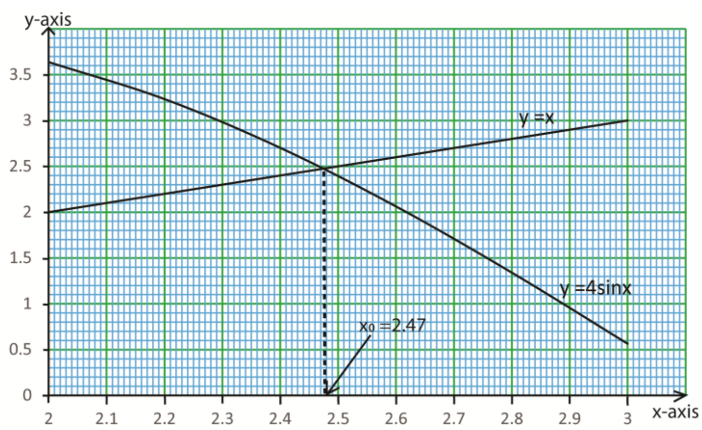
$x_3 = \frac{3(\sin 2.279 - 2.5\cos 2.279}{1 - 3\cos 2.279} = 2.279$

Error = $|2.279 - 2.279| = 0.000 < 0.005$

∴ root = 2.279 = 2.28(2D)

Example 29

(a) On the same axis, draw graphs of y = x and y = 4sinx to show that the root of the equation x-4sinx = 0 lies between x= 2 and x = 3



Therefore the root (2.47) lies between x = 2 and x = 3

(b) Use Newton Raphson's method to calculate the root of the equation x – 4sinx = 0, taking approximate root in (a) as the initial approximation to the root. correct your answer to 3 decimal places.

f(x) = x – 4sinx

f'(x) = 1 – 4cos x

$x_{n+1} = x_n - \frac{x_n - 4\sin x_n}{1 - 4\cos x_n}$

$Taking\ x_0 = 2.47$

$x_1 = 2.47 - \frac{2.47 - 4\sin 2.45}{1 - 4\cos 2.47} = 2.4746$

$Error = |2.4746 - 2.47| = 0.0046 > 0.0005$

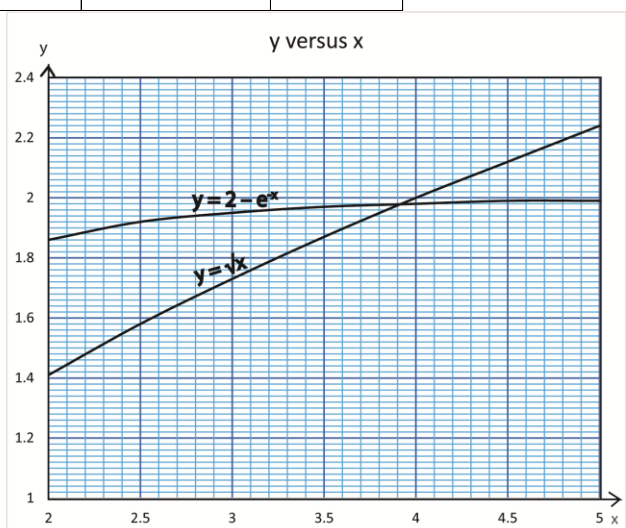
$x_2 = 2.4746 - \frac{2.4746 - 4\sin 2.4546}{1 - 4\cos 2.4746} = 2.4746$

$Error = |2.4746 - 2.4746| = 0.000 < 0.0005$ ∴ the root = 2.475 (3D)

Example 30

(a) Draw on the same axes the graphs of the curves y = 2 – e^{-x} and y = $\sqrt{x}$values 2≤ x ≤ 5. (04marks)

x	y = 2 – e^{-x}	y = $\sqrt{x}$
2.0	1.86	1.41
2.5	1.92	1.58
3.0	1.95	1.73
3.5	1.97	1.87
4.0	1.98	2.00
4.5	1.99	2.12
5.0	1.99	2.24



(b) Determine from your graph the interval within which the roots of the equation

$e^{-x} + \sqrt{x-2=0}$ lies

Hence, use Newton-Raphson's method to find the root of the equation correct to 3 decimal places (07marks)

Root lies between 3.9 and 4

f(x) = 2 – e^{-x} - $\sqrt{x}$

f'(x) = $e^{-x} - \frac{1}{2\sqrt{x}}$

f(x_n) = $e^{-x_n} - \frac{1}{2\sqrt{x_n}}$

x_{n+1} =x_n - $\frac{2-e^{x_n}-\sqrt{x_n}}{2e^{-x_n}\sqrt{x_n}-1}$

$x_0 = \frac{3.9+4}{2} = 3.95$

$x_1 = 3.95 - \frac{2\sqrt{3.95}(2 - e^{-3.95} - \sqrt{3.95})}{2e^{-3.95}\sqrt{3.95} - 1}$ = 3.9211

$Error = |3.9211 - 3.95| = 0.0289$

$x_2 = 3.9211 - \frac{2\sqrt{3.9211}(2 - e^{-3.9211} - \sqrt{3.9211})}{2e^{-3.9211}\sqrt{3.9211} - 1}$ = 3.9211

∴ Root = 3.921 (3dp)

Example 31

Given the equation $X^3 - 6x^2 + 9x + 2 = 0$

(a) Show that the equation has a root between -1 and 0.

Let $f(x) = X^3 - 6x^2 + 9x + 2$

$f(-1) = (-1)^3 - 6(-1)^2 + 9(-1) + 2$

$= -1 - 6 - 9 + 2 = -14$

$f(0) = 0 + 0 + 0 + 2$

$= 2$

$f(-1).f(0) = -14 \times 2 = -28$

since $f(-1).f(0) < 0$; the root exist between -1 and 0.

(b) (i) Show that the Newton Raphson formula approximating the root of the equation is given by

$X_{n+1} = \frac{2}{3}\left[\frac{x_n^3 - 3x_n^2 - 1}{x_n^2 - 4x_n + 3}\right]$

$f(x) = X^3 - 6x^2 + 9x + 2$

$f(x_n) = x_n^3 - 6x_n^2 + 9x_n + 2$

$f'(x_n) = 3x_n^2 - 12x_n + 9$

$x_{n+1} = x_n - \left(\frac{x_n^3 - 6x_n^2 + 9x_n + 2}{3x_n^2 - 12x_n + 9}\right)$

$= \frac{x_n(3x_n^2 - 12x_n + 9) - (x_n^3 - 6x_n^2 + 9x_n + 2)}{3x_n^2 - 12x_n + 9}$

$= \frac{(3x_n^3 - 12x_n^2 + 9x_n) - (x_n^3 - 6x_n^2 + 9x_n + 2)}{3x_n^2 - 12x_n + 9}$

$= \frac{2x_n^3 - 6x_n^2 - 2}{3x_n^2 - 12x_n + 9}$

$= \frac{2}{3}\left[\frac{x_n^3 - 3x_n^2 - 1}{x_n^2 - 4x_n + 3}\right]$

(ii) Use the formula in (b)(i) above, with initial approximation of $x_0 = -0.5$, to find the root of the given equation correct to two decimal places

Taking x = -0.5

$x_1 = \frac{2}{3}\left[\frac{(-0.5)^3 - 3(-0.5)^2 - 1}{(-0.5)^2 - 4(-0.5) + 3}\right]$ = -0.2381

$|e| = |0.2381 - (-0.5)|$ = 0.2619

$x_2 = \frac{2}{3}\left[\frac{(-0.2381)^3 - 3(-0.2381)^2 - 1}{(-0.2381)^2 - 4(-0.2381) + 3}\right]$ = -0.1968

$|e| = |0.1968 - (-0.2381)|$ = 0.0.0413

$x_3 = \frac{2}{3}\left[\frac{-0.1968^3 - 3(-0.1968)^2 - 1}{(-0.1968)^2 - 4(-0.1968) + 3}\right]$ = -0.1958

$|e| = |-0.1958 - (-0.1968)|$ = 0.001< 0.005

Hence the root = -0.20 (2D)

Revision Exercise 4

1. Using the Newton Raphson's formula, show that the reciprocal of a number N is $x_n(2 - Nx_n)$
2. Use Newton Raphson's iterative formula to show that the cube root of a number N is given by $\frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$. Hence use the iterative formula to find $\sqrt[3]{96}$ correct to3 decimal places. use $x_0 = 5$. [4.579]
3. (a) Show that the equation 3x3 + x -5 = 0 has real root between x = 1 and x = 2.
   (b) Using linear interpolation, find the first approximation for this root to 2dp. [1.04]
   (c) Using Newton Raphson's method twice find the value of this root correct to 2 dp. [1.09]
4. (a) Show graphically that there is a positive real root of equation $xe^{-x} - 2x + 5 = 0$ between x = 2 and x = 3
   (b) Using Newton Raphson's method, find this root correct to 1 dp. [2.6]
5. Using the iterative formula for NRM, show that the fourth root of a number N is $\frac{3}{4}\left(x_n + \frac{N}{3x_n^3}\right)$. Starting with $x_0 = 2.5$ show that $(45.7)^{\frac{1}{4}} = 2.600$ (3dp)
6. On the same axes, draw graphs of $y = x^3$ and $y = 2x + 5$. Using NRM twice find the positive root of the equation $x^3 - 2x – 5 = 0$ correct to 2 decimal places. [2.09]
7. (a) Show that the Newton Raphson's formula for finding the smallest positive root of the equation 3tanx + x = 0 is $\frac{6x_n - 3\sin 2x_n}{6 + 2\cos 2x_n}$
   (b) By sketching the graphs of y = tans, y = $\frac{-x}{3}$ 0r otherwise, find the first approximation to the required root and use it to find the actual root correct to 3 dp . [2.456]
8. (a) Show that the root of the equation $f(x) = e^x + x^3 – 4x = 0$ has a root between x =1 and x = 2
   (b) Use the Newton Raphson's method to find the root of equation in (a) correct to 2 decimal places. [x_0 = 1, root =1.12]
9. (a) Show that the iterative formula for approximation of the root of f(x) = 0 by NRM process for the equation $xe^x + 5x – 10 = 0$ is $x_{n+1} = \frac{x_n^2 e^{x_n} + 10}{x_n e^{x_n} + e^{x_n} + 5}$.
   (b) Show that the root of the equation in (i) above lies between x = 1 and x = 2. Hence find the root of the equation correct to 2 dp. [1.20]
10. (a) Use a graphical method to find a first approximation to the real root of $x^3 + 2x - 2 = 0$.
    (b) Use the Newton Raphson's method to find the root of the equation in (a) correct to 2 dp. [0.77]
11. (a) Show that equation x = In (8-x) has a root between x= 1 and x = 2.
    (b) Use the Newton Raphson's method to find the root of the equation in (a) correct to 2 decimal places [1.82]
12. (a) Use graphical method to find the first approximation to the root of $x^3 – 3x + 4 = 0$. [-2]
    (b) Use NRM to find the root of the equation in (a) correct to 2 d.p. [-2.20]
13. Show graphically that equation $e^x + x - 4 = 0$ has only one root between x = 1 and x = 2. Use NRM to find the approximation of the equation correct to 3dp. [1.07]

14. Show that the NRM for approximating the Kth root of a number N is given by $\frac{1}{K}\left((K-1)x_n + \frac{N}{x_n^{K-1}}\right)$. Hence use your formula to find the positive square root of 67 correct to 4 s.f. [8.185].
15. (a) Show that equation $x^3 + 3x - 9 = 0$ has a root between x= 1 and x = 2.
    (b) Use the Newton Raphson's method to find the root of the equation in (a) correct to 2 0ne places [1.6]
16. (a) Show graphically that there is a positive real root of equation $xe^{-x} - 2x - 1 = 0$ between x = 1 and x = 2
    (b) Using Newton Raphson's method, find this root for the equation in (a) correct to 2 dp. [1.26]
17. (a) Show that equation $2x - 3\cos\left(\frac{x}{2}\right) = 0$ has a root between x= 1 and x = 2.
    (b) Use the Newton Raphson's method to find the root of the equation in (a) correct to one places [1.23]
18. (a) If a is the first approximation to the root of the equation $x^5 - b = 0$, show that the second approximation is given by $\frac{4a+\frac{b}{a^4}}{5}$.
    (b) Show that the positive real root of the equation $x^5 - 17 = 0$ lies between 1.5 and 1.8. Hence use the formula in (a) above to determine the root to 3 decimal places. [1.762]
19. (a)(i) On the same axes, draw graphs of $y = x^2$ and y = cox x for $0 \le x \le \frac{\pi}{2}$ at intervals of $\frac{\pi}{8}$.
    (ii) Use your graphs, to find to 1 decimal place an approximate root of the equation $x^2 - \cos x = 0$ [0.8]
    (b) Use the NRM to calculate the root of the equation $x^2 - \cos x = 0$ taking the approximate root in (a) as the initial approximation. Correct your answer to 3 dp. [0.824]
20. (a) (i) Draw on same axes the graphs of equation y = xsinx and $y = e^x - 2$ for $0 \le x \le 1.5$.
    (ii) Use your graphs to find an approximate root of the equation $2 - e^x + x\sin x =$ [1.1]

(c) Use the Newton Raphson's method to find the root of the equation in (a)(ii) correct to three decimal places [1.085]

21. Show graphically that equation $e^x + x - 8 = 0$ has only one real root between x = 1 and x = 2. Use NRM to find approximation of x = In(x 8) correct to 3 dp [1.821]
22. Draw using the same axes, graphs of $y = x^2$ and y = sin2x for $0 \le x \le \frac{\pi}{2}$. From the graphs obtain to one decimal place an approximation of the non-zero root of the equation $x^2 - \sin 2x = 0$. Using NRM, calculate to 2 dp a more suitable approximation. [0.97]
23. Given the equation In(1 + 2x) – x = 0.
    (i) show the root of the equation above lies between x = 1 and x = 1.5
    (ii) Use NRM twice to estimate the root of the equation, correct to 2 dp. [1.26]

Advantages and limitations of Newton Raphson method to linear interpolation methods

Compared to linear interpolation methods, **Newton–Raphson** has the following advantages and disadvantages

Advantages of Newton–Raphson over Linear Interpolation

- **Speed of convergence**:
    - Newton–Raphson converges *quadratically* near the root (error shrinks very fast).
    - Linear interpolation methods (bisection, secant) converge only *linearly* or *superlinearly* (slower).
- **Efficiency**:
    - Less iteration are needed to reach high accuracy.
    - Especially useful in engineering/scientific problems where precision matters.
- **General applicability**:
    - Works well for smooth, differentiable functions.
    - Can be extended to systems of nonlinear equations, not just single-variable problems.

Limitations of Newton–Raphson compared to Linear Interpolation

- **Derivative requirement**:
    - Needs f′(x)). If the derivative is hard to compute or unavailable, this is a major drawback.
    - Linear interpolation methods only require function values.
- **Initial guess sensitivity**:
    - A poor starting point may cause divergence or convergence to the wrong root.
    - Linear interpolation methods (like bisection) guarantee convergence if the root is bracketed.
- **Problems near flat slopes**:
    - If $f'(x)\approx 0$, the method can fail or produce huge jumps.
    - Interpolation methods are more stable in such cases.
- **No global guarantee**:
    - Newton–Raphson is powerful locally but not globally convergent.
    - Bisection always converges, though slowly.

Side-by-Side Comparison

Feature	Newton–Raphson	Linear Interpolation (Bisection/Secant)
Convergence speed	Quadratic (fast)	Linear/superlinear (slower)
Derivative requirement	Yes (f′(x)f'(x))	No, only f(x)f(x)
Reliability	Sensitive to guess	Guaranteed if root is bracketed
Robustness	Can diverge	Always converges (bisection)
Efficiency	High (few steps)	Lower (many steps)
Handling flat slopes	Problematic	More stable

Thank you
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