
## SENIOR FIVE TERM 1

# TOPIC 2/6: Equations and Inequalities

**Competency:** The learner investigates equations and inequalities to acquire mathematical computational and analytical skills applicable in the real world.

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# Quadratic equations

These are equations expressed in the form $ax^2 + bx + c = 0$ where a, b and c are constants and $a \neq 0$

They have at most two roots which may be real or complex.

The latter roots are handled exclusively under complex numbers

Example of quadratic equations

$2y^2 + 3y + 5 = 0$; a = 2, b = 3 and c = 5

$x^2 + 4x – 10 = 0$; a = 1, b = 4, c = -10

## Forming quadratic equations

Suppose that the roots of a quadratic equation are α and β,

then x – α = 0 and x – β = 0

When forming a quadratic equation

(x – α)( x – β) = 0

$x^2 – (\alpha + \beta)x + \alpha\beta = 0$

This means that if the roots of a quadratic equation are given, its equation in terms of x is $x^2$-(sum of roots)x + product of roots = 0

**Example 1**

Form quadratic equation I terms of x with roots

(a) (2, 3)

Solution

Let x = 2 or x = 3

x – 2 = 0 x – 3 = 0

⇨ (x – 2)(x – 3) = 0

$x^2 – 5x – 6 = 0$

(b) (p, q)

Solution

Let x = p or x = q

x – p = 0 x – q = 0

⇨ (x – p)(x – q) = 0

$x^2 – (p + q)x – pq = 0$

(c) (-3, -2)

Solution

Let x = -3 or x = -2

x + 3 = 0 x + 2 = 0

⇨ (x + 3)(x + 2) = 0

$x^2 + 5x + 6 = 0$

**Example 2**

Given that the root of the equation $x^2 + px + q = 0$ are α and β, express $(\alpha - \beta^2)(\beta - \alpha^2)$ in terms of p and q. Deduce that for one root to be the square of another $p^3 – 3pq + q^2 + q = o$ must hold

Solution

For $x^2 + px + q = 0$

α + β = -p

αβ = q

$$(\alpha - \beta^2)(\beta - \alpha^2) = \alpha\beta - \alpha^3 - \beta^3 + (\alpha\beta)^2$$

$$= \alpha\beta - (\alpha^3 + \beta^3) + (\alpha\beta)^2$$

But $(\alpha^3 + \beta^3) = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$(\alpha - \beta^2)(\beta - \alpha^2)$

$$= \alpha\beta - [(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)] + (\alpha\beta)^2$$

$$= q - [(-p)^3 - 3q(-p)] + q^2$$

$$= q + p^3 - 3pq + q^2$$

Hence $(\alpha - \beta^2)(\beta - \alpha^2) = p^3 - 3pq + q^2 + q$

If $\alpha = \beta^2$

then $(\beta^2 - \beta^2)(\beta - \beta^4) = p^3 - 3pq + q^2 + q$

$p^3 - 3pq + q^2 + q = 0$ As required

***Example 3***

Given that the root of the equation $x^2 + px + q = 0$ are α and β, Form quadratic equations with roots

(a) $\frac{\alpha}{\beta} and \ \frac{\beta}{\alpha}$

Solution

Sum of roots $=\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$

$= \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$

Product of roots $=\frac{\alpha}{\beta} x \ \frac{\beta}{\alpha} = 1$

Required equation

$x^2 - \left(\frac{p^2-2q}{q}\right)x + 1 = 0$

Or $qx^2 - (p^2 - 2q)x + q = 0$

(b) pα + qβ and qα + pβ

Solution

Sum of roots

pα + qβ + qα + pβ = p(α +β) + q(α +β)

=(p + q)(α +β)

=p(p + q)

Product of roots= (pα + qβ)(qα + pβ)

$= pq\alpha^2 + p^2\alpha\beta + q^2\alpha\beta + pq\beta^2$

$= pq(\alpha^2 + \beta^2) + \alpha\beta(p^2 + q^2)$

$= pq[(\alpha + \beta)^2 - 2\alpha\beta] + \alpha\beta(p^2 + q^2)$

$=pq(p^2-2q) + q(p^2 + q^2)$

Required equation

$x^2 - p(p + q)x + pq(p^2 - 2q) + q(p^2 + q^2) = 0$

***Example 4***

Given the equation $x^3 + x – 10 = 0$.

(a) Show that x = 2 is a root of the equation

Let $f(x) = x^3 + x – 10$

Substituting for x = 2

$f(2) = 2^3 + x -10$

= 8 + 2 – 10

= 10 – 10 = 0

Hence x = 2 is a root of $x^3 + x –10 = 0$

(b) Deduce the values of α + β and αβ where α and β are roots of the equation.

Hence form a quadratic equation whose roots are $\alpha^2$ and $\beta^2$.

⇨ $x^3 + x –10 = (x-2)(x^2 + 2x + 5)$

Either x – 2 = 0

Or $(x^2 + 2x + 5) = 0$

α + β = 2

αβ = 5

Sum of roots = $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (-2)^2 – 2(5) = 4 – 10 = -6$

Product= $\alpha^2\beta^2 = (\alpha\beta)^2 = 5^2 = 25$

The equation become

$x^2 - (-6)x + 25 = 0$

$x^2 + 6x + 25 = 0$

## Solving quadratic equations

Quadratic equations may be solved by

(a) Factorization method
(b) Completing square method
(c) Graphical method

## Solving quadratic equations using Factorization method

It is used for quadratic equations that are easy to factorize

***Example 5***

(a) $4x^2 + 7x + 3 = 0$

| Solution | side work |
|---|---|
| $4x(x + 1) + 3(x + 1) = 0$ | Product = 4 x 3 = 12 |
| $(x + 1)(4x + 3) = 0$ | Sum = 7 |
| Either $x + 1 = 0$; $x = -1$ | Factor = (4, 3) |
| Or $4x + 3 = 0$; $x = -\frac{3}{4}$ | |

(b) $2x^2 + 5x + 3 = 0$

| Solution | side work |
|---|---|
| $2x(x + 1) + 3(x + 1) = 0$ | Product = 3 x 2 = 6 |
| $(x + 1)(2x + 3) = 0$ | Sum = 5 |
| Either $x + 1 = 0$; $x = -1$ | Factors 3, 2 |
| Or $2x + 3 = 0$; $x = -\frac{3}{2}$ | |

(c) $x^2 + x - 20 = 0$

| Solution | Side work |
|---|---|
| $x(x - 4) + 5(x - 4) = 0$ | Product = -20 |
| $(x - 4)(x + 5) = 0$ | Sum = 1 |
| Either $x - 4 = 0$; $x = 4$ | Factors ( 5, - 4) |
| Or $x + 5 = 0$; $x = -5$ | |

(d) $10x^2 + x - 3 = 0$

| Solution | Side work |
|---|---|
| $5x(2x - 1) + 3(2x - 1) = 0$ | product 10 x – 3 = -30 |
| $(5x + 3)(2x - 1) = 0$ | sum = 1 |
| Either $5x + 3 = 0$; $x = -\frac{3}{5}$ | Factors (6, -5) |
| Or $2x - 1 = 0$; $x = \frac{1}{2}$ | |

## Solving quadratic equations using Completing squares approach

The idea is to create a perfect square on one side of the equation:

Given the equation $ax^2 + bx + c = 0$

- Dividing the equation by and transposing the constant term to the RHS
  $x^2 + \frac{b}{a}x = -\frac{c}{a}$
- Marking the LHS a perfect square, add a half the coefficient of x squared on both sides
  $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$
- Factories the terms on the LHS
- $\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2-4ac}{4a^2}$
- Taking square root on both sides of the equation
- $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$
- Solving $x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$
- $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

This is the general quadratic equation formula for finding the square root of any quadratic equation. This formula is locally known as ***bull dozer formula***

***Example 6***

Solve the following equations by completing squares

(a) $2x^2 - x - 3 = 0$

Solution

$2x^2 - x - 3 = 0$

$x^2 - \frac{1}{2}x = \frac{3}{2}$

$x^2 - \frac{1}{2}x + \left(-\frac{1}{4}\right)^2 = \frac{3}{2} + \left(-\frac{1}{4}\right)^2$

$\left(x - \frac{1}{4}\right)^2 = \frac{25}{16}$

$x - \frac{1}{4} = \sqrt{\frac{25}{16}} = \pm\frac{5}{4}$

Either $x = \frac{5}{4} + \frac{1}{4} = \frac{6}{4} = \frac{3}{2}$

Or $x = -\frac{5}{4} + \frac{1}{4} = \frac{-4}{4} = -1$

Hence $x = -1$ and $x = \frac{3}{2}$

(b) $18x^2 + 7x - 1 = 0$

Solution

$18x^2 + 7x - 1 = 0$

$x^2 + \frac{7}{18}x = \frac{1}{18}$

$x^2 + \frac{7}{18}x + \left(\frac{7}{36}\right)^2 = \frac{1}{18} + \left(-\frac{7}{36}\right)^2$

$\left(x + \frac{7}{36}\right)^2 = \frac{1}{18} + \frac{49}{1296} = \frac{121}{1296}$

$x + \frac{7}{36} = \sqrt{\frac{121}{1296}} = \pm\frac{11}{36}$

Either $x = \frac{11}{36} - \frac{7}{36} = \frac{4}{36} = \frac{1}{9}$

Or $x = -\frac{11}{36} - \frac{7}{36} = -\frac{18}{36} = -\frac{1}{2}$

Hence x = $\frac{1}{9}$ or x =$-\frac{1}{2}$

(c) $3x^2 + 7x + 2 = 0$

Solution

$3x^2 + 7x + 2 = 0$

$x^2 + \frac{7}{3}x = -\frac{2}{3}$

$x^2 + \frac{7}{3}x + \left(\frac{7}{6}\right)^2 = -\frac{2}{3} + \left(\frac{7}{6}\right)^2$

$\left(x + \frac{7}{6}\right)^2 = -\frac{2}{3} + \frac{49}{36} = \frac{25}{36}$

$x + \frac{7}{6} = \sqrt{\frac{25}{36}} = \pm\frac{5}{6}$

Either x =$\frac{5}{6} - \frac{7}{6} = -\frac{2}{6} = -\frac{1}{3}$

Or x =$-\frac{5}{6} - \frac{7}{6} = -\frac{12}{6} = -2$

Hence x = -2 and x =$-\frac{1}{3}$

***Example 7***

Solve the following equations by using the quadratic formula

(a) $7x^2 - 5x - 2 = 0$

Solution

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{-5^2 - 4 \, x \, 7 \, x - 2}}{2 \, x \, 7} = \frac{5 \pm 9}{14}$

Either $x = \frac{14}{14} = 1$ or $x = \frac{-4}{14} = -\frac{2}{7}$

Hence $x = 1 \; and \; x = -\frac{2}{7}$

(b) $3x^2 - 7x - 6 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{7 \pm \sqrt{-7^2 - 4 \, x \, 3 \, x - 6}}{2 \, x \, 3} = \frac{7 \pm 11}{6}$

Either $x = \frac{18}{6} = 3$ or $x = \frac{-4}{6} = -\frac{2}{3}$

Hence $x = 3 \; and \; x = -\frac{2}{3}$

(c) $6x^2 - 5x - 6 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{-5^2 - 4 \, x \, 6 \, x - 6}}{2 \, x \, 6} = \frac{5 \pm 13}{12}$

Either $x = \frac{18}{12} = \frac{3}{2}$ or $x = \frac{-8}{12} = -\frac{2}{3}$

Hence $x = \frac{3}{2} \; and \; x = -\frac{2}{3}$

## Solving quadratic equations using Graphical approach

Here, the roots of the equations are established by plotting suitable graphs. It may done by:-

(i) Single graph plot
(ii) Two graph plot

**Single graph method**

Given the function y = f(x), tabulate the selected values of x closer to zero within a given range which must be substituted in the function given to obtain the corresponding values of y. These values are plotted on the same x and y axes and joined by a curve. The roots of the function f(x) = 0 are the values of x where the function y = f(x) crosses the x-axis.

**Two graphs method**

Here the function y = f(x) is split into two and the equations plotted separately on the same axes. The points of intersection of the two graphs are noted and the corresponding values of x are read off on the x – axis.

***Example 8***

Solve the following equation by using graphical approach

(a) $2x^2 + 3x - 3 = 0$

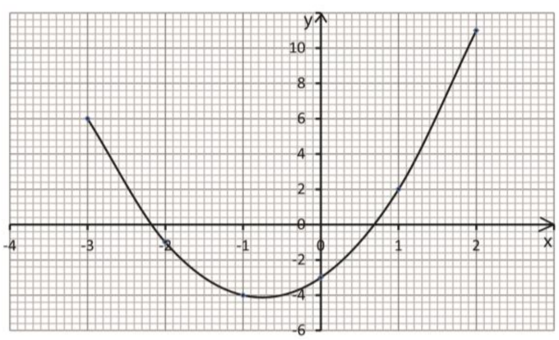
Solution

**Using a single graphs**

Let $y = 2x^2 + 3x - 3$ taking $-3 \leq x \leq 2$

Table of values

| X | -3 | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|---|
| y | 6 | -1 | -4 | -3 | 2 | 11 |




From the graph, the roots of the equation are x = -2.2 and x = 0.7

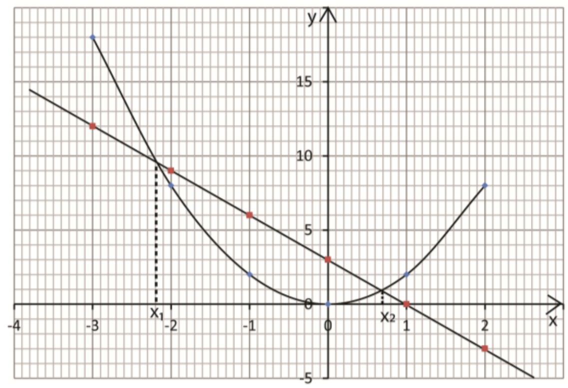
**Using two graphs**

By splitting the equation $2x^2 + 3x - 3 = 0$ into two we have $2x^2 = -3x + 3$

Let $y_1 = 2x^2$ and $y_2 = 3 - 3x$

Table of results

| X | -3 | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|---|
| $y_1$ | 18 | 8 | 2 | 0 | 2 | 8 |
| $y_1$ | 12 | 9 | 6 | 3 | 0 | -3 |




From the graph, the roots of the equation are x = -2.2 and x = 0.7

(b) $x^2 + x - 6 = 0$

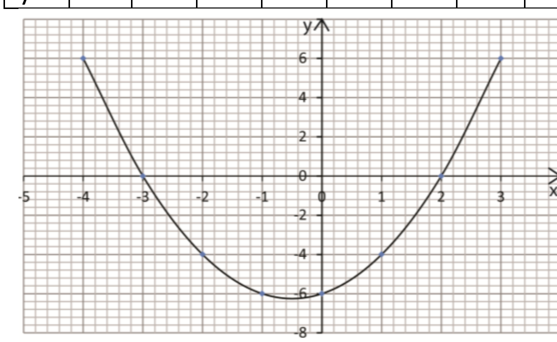
**Solution**

**Using a single graphs**

Let $y = x^2 + x - 6 = 0$, taking $-4 \leq x \leq 3$

Table of values

| X | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|
| y | 6 | 0 | -4 | -6 | -6 | -4 | 0 | 6 |




From the graph, the roots of the equation are x = -3 and x = 2

**Using two graphs**

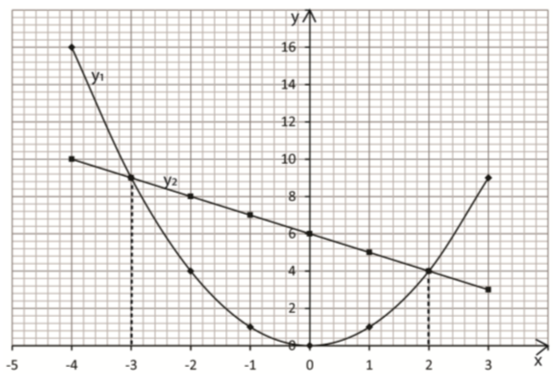
By splitting the equation $x^2 + x - 6 = 0$

Into two we have $x^2 = 6 - x$

Let $y_1 = x^2$ and $y_2 = 6 - x$

Table of results

| X | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|
| $y_1$ | 16 | 9 | 4 | 1 | 0 | 1 | 4 | 9 |
| $y_1$ | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 |




From the graph, the roots of the equation are x = -3 and x = 2

## Minimum and maximum values of quadratic expression

The methods of completing squares and graphing can be used to obtain minimum and maximum values of quadratic expression

Completing square method

The general form of quadratic equation

$y = ax^2 + bx + c$ can be expressed as

$y = ax^2 + bx + c$

$$y = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$y = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right)$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac - b^2}{4a}\right)$$

The minimum/maximum value is the constant $\frac{4ac-b^2}{4a}$ which is attained when $x + \frac{b}{2a} = 0$

Note

- if a < 0, the value of the function is maximum
- if a > 0, the value of the function is minimum

***Example 9***

Determine the minimum or maximum values of the following expressions using completing squares

(a) $2x^2 - x - 3 = 0$

Solution

Let $y = 2x^2 - x - 3 = 0$

$$y = 2\left(x^2 + \frac{2}{2}x - \frac{3}{2}\right)$$

$$y = 2\left(x^2 + x + \left(\frac{1}{2}\right)^2 - \frac{3}{2} - \left(\frac{1}{2}\right)^2\right)$$

$$y = 2\left(\left(x + \frac{1}{2}\right)^2 - \frac{7}{4}\right)$$

$$y = 2\left(x + \frac{1}{2}\right)^2 - \frac{7}{4}$$

Since a > 0, the function has got a minimum value at $x = -\frac{1}{2}$. Hence $y_{min} = -\frac{7}{4}$

(b) $-4 + 6x - x^2$

Let $y = -4 + 6x - x^2$

$y = -(x^2 - 6x + 4)$

$y = -(x^2 - 6x + (-3)^2 + 4 - (-3)^2)$

$y = -((x - 3)^2 + 4 - 9)$

$y = -((x - 3)^2 + -5)$

Since a = -1 < 0, the expression has got a maximum value when x = 3. Hence $y_{max} = 5$

**Using graphical method for minimum and maximum value**

After graphing, y =f(x) the minimum value of the expression is lowest point if a trough/valley like or curving upwards and the maximum value is the maximum point if downwards


***Example 10***

Determine the maximum or minimum values of the following expression using graphical method

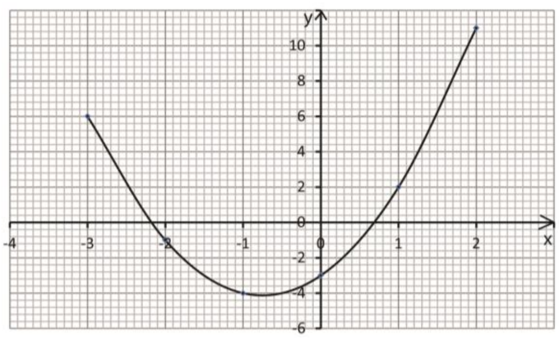
(a) $2x^2 + 3x - 3 = 0$

Solution

Let $y = 2x^2 + 3x - 3$ taking $-3 \le x \le 2$

Table of values

| | | | | | | |
|---|---|---|---|---|---|---|
| X | -3 | -2 | -1 | 0 | 1 | 2 |
| y | 6 | -1 | -4 | -3 | 2 | 11 |



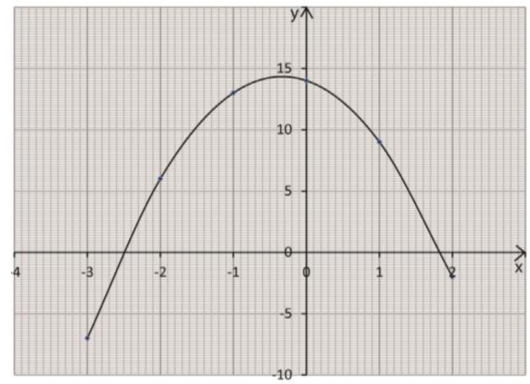
$y_{min}$ = - 4.1 at -0.75

(b) $14 - 2x - 3x^2$

Ley $y = 14 - 2x - 3x^2$ ; $-3 \le x \le 2$

Table of results

| | | | | | | |
|---|---|---|---|---|---|---|
| x | -3 | -2 | -1 | 0 | 1 | 2 |
| y | -7 | 6 | 13 | 14 | 9 | -2 |



$y_{max}$ = 14.3 at x =-0.3

## Revision exercise on quadratic equation

1. Given that the roots of the equation $x^2 + px + q = 0$ are α and β, find the values of:
   (a) $\alpha^2 + \beta^2 [p^2 - 2q]$
   (b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} \left[\frac{p^2-2q}{q}\right]$
   (c) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} \left[\frac{p}{q}(3q - p^2)\right]$
   (d) $\alpha - \beta \left[\pm\sqrt{p^2 - 4q}\right]$
   (e) $\alpha^2 - \beta^2 \left[-p\sqrt{(p^2 - 4q)}\right]$
   (f) $\alpha^3 + \beta^3 [p(3q - p^2)]$

2. If the roots of the equation $x^2 + 2x + 3 = 0$ are α and β, form an equation whose roots are $\alpha^2 - \beta$ and $\beta^2 - \alpha$. $[x^2 + 2 = 0]$

3. Given that the roots of the equation $x^2 - 2x + 10 = 0$ are α and β, form an equation whose roots are

   $\frac{1}{(2+\alpha)^2}$ and $\frac{1}{(2+\beta)^2}$ $[324x^2 + 1 = 0]$

4. If α and β are roots of the equation $x^2 - px + q = 0$, find the equation whose roots are $\frac{\alpha^3-1}{\alpha}$ $and$ $\frac{\beta^3-1}{\beta}$

[$qx^2 – (p^2q – 2q^2–p)x + (q^3 – p^3 + 3pq)+1=0$]

5. The roots of the equation $3x^2 – ax + 6b = 0$ are α and β. Find the condition for one root to be
   (a) Twice the other [$81b = a^2$)
   (b) The cube of the other
   [$a^4 – 648(b – 1)b^2 – 18(4a^2 + 9)b = 0$]
6. By Factorization method solve the following quadratic equations
   (a) $x^2 + 9x + 14$ [x = -7, x= -2]
   (b) $x^2 + 2x – 8 = 0$ [ x = -4, x = 2)
   (c) $2x^2 + x – 10 = 0$ [x= 2, x = $-\frac{5}{2}$]
   (d) $6x^2 – 19x + 10$ $\left[x = \frac{5}{2},\ x = \frac{2}{3}\right]$
7. Solve the following equations by completing squares

(a) $2x^2 + 5x + 3 = 0$ $\left[x = -1 \text{ and } x = -\frac{3}{2}\right]$
   (b) $x^2 + 9x + 20 = 0$ [x = -5 and x = -4]
   (c) $x^2 + x – 20 = 0$ [x = -5 and x = 4]
   (d) $x^2 – x – 20 = 0$ [x = 4 and x = 5]
8. Solve the following equations using the quadratic formula
   (a) 2x2 + 5x + 3 = 0 $\left[\frac{-3}{2}, -1\right]$
   (b) $x^2 + 9x + 20$ [-5, -4]
   (c) $x^2 + x - 20$ [-5, 4]

9. Determine the maximum or minimum values of the following expression
   (a) $3x^2 – 2x + 1$ $\left[y_{min} = \frac{2}{3} at\ x = \frac{1}{3}\right]$
   (b) $4 – x – x^2$ $\left[y_{max} = \frac{33}{8} at\ x = -\frac{1}{4}\right]$
10. Determine the maximum or minimum values of the following expression using graphical method
   (a) $14 – 2x – 2x^2$

## Application of quadratic equation

### 1. Projectile Motion

**Application**: Predicting the path of a thrown object (e.g., ball, arrow).

**Example**: If a ball is thrown upward, its height h at time t can be modeled by:

$h(t)=-4.9t^2+20t+1$

This equation helps determine how high the ball goes and when it hits the ground.

### 2. Area Optimization

**Application**: Maximizing or minimizing area with fixed perimeter or constraints.

**Example**: A farmer wants to fence a rectangular field next to a river (no fence needed on one side). With 100 meters of fencing, the area A can be modeled by:

$A(x) = x(100-2x)$

Solving this quadratic helps find the dimensions that give the largest area.

### 3. Business and Economics

**Application**: Maximizing profit or minimizing cost.

**Example**: A company finds that profit P from selling x items is:

$P(x) =-5x^2+150x-100$o

Solving this helps determine how many items to sell for maximum profit.

### 4. Engineering and Design

**Application**: Designing structures like bridges or roller coasters.

**Example**: The arch of a bridge might follow a quadratic curve:

$Y =-0.01x^2+x$

Engineers use this to calculate height and span.

# Simultaneous equation

## Simultaneous equations in two unknowns

These are equations containing two unknowns. The equations may all be linear (equations of straight lines) or one of them may be linear and the other non-linear.

Linear simultaneous equations

Simultaneous equations may solved by any of the three methods

- Elimination method
- Substitution method
- Graphical method
- Solving simultaneous equations using matrixes

## Solving simultaneous equations using elimination method

This involves elimination of one of the unknown variables so as to be in position to find the other.

Example 1

(a) $5x + 3y = 7$
$2x - 4y = 3$

Solution

$5x + 3y = 7$ ………… (i)

$2x - 4y = 3$ ………….(ii)

To eliminate y the equation are multiplied by relevant factors to make the coefficients of y in both equations equal. Thus eqn. (i) is multiplied by 4 and eqn. (ii) by 3

i.e. 4 x eqn. (i) + 3 x eqn. (ii)

$20x + 12y = 28$

$+ \ 6x - 12y = 9$

$26x = 37$

$x = \frac{37}{26}$

Substituting x in eqn.(i)

$5x + 3y = 7$

$3y = 7 - 5 \times \frac{37}{26}$

$Y = \frac{-1}{26}$

(b) $3x + 2y = 8$

$3y + 4x = 11$

Solution

Rearrange the eqns.

$3x + 2y = 8$ ………… (i)

$4x + 3y = 11$ ………. (ii)

Eliminate x as follows

4 x eqn. (i) – 3 x eqn. (ii)

$12x + 8y = 32$

$- \ 12x + 9y = 33$

$y = 1$

Substituting y in eqn. (i)

$3x + 2y = 8$

$3x + 2 \times 1 = 8$

$3x = 6$

$x = 2$

## Solving simultaneous equations using substitution method

This involves the expression of one of the unknown variable in terms of the other.

Example 2

Solve the following pairs of simultaneous equation for x and y by substitution method

(a) $5x + 3y = 7$

$2x - 4y = 3$

Solution

$5x + 3y = 7$

$5x = 7 - 3y$

$x = \frac{7-3y}{5}$ ……….. (i)

$2x - 4y = 3$ ……… (ii)

Substituting x in eqn. (ii)

$2\left(\frac{7-3y}{5}\right) - 4y = 3$

Multiply 5 through

$2(7 - 3y) - 20y = 15$

$14 - 6y - 20y = 15$

$-26y = 1$

$y = \frac{-1}{26}$

Substituting y into eqn. (i)

$x = \frac{7-3y}{5} = \frac{7-3\left(\frac{-1}{26}\right)}{5} = \frac{26 \ x\ 7+-3\ x-1}{5\ x\ 26} = \frac{37}{26}$

$\therefore x = \frac{37}{26}$ and $y = \frac{-1}{26}$

(b) $3x + 2y = 8$

$y = \frac{8-3x}{2}$………………… (i)

$3y + 4x = 11$ …………. (ii)

Substituting y in equation (ii)

$3\left(\frac{8-3x}{2}\right) + 4x = 11$

Multiplying 2 through

$3(8 - 3x) + 8x = 22$

$24 - 9x + 8x = 22$

$-1x = = -2$

$x = 2$

substituting x into equation (i)

$y = \frac{8-3x}{2} = \frac{8-3\ x\ 2}{2} = \frac{2}{2} = 1$

$\therefore x = 2$ and $y = 1$

## Solving simultaneous equations using graphical method

This method involve drawing graphs of the two linear equations and finding the coordinates of their points of intersection

Establish at least two possible points with known coordinates satisfying the equations. The coordinates of the point of intersection of the lines drawn are the solutions to the equations

***Example 3***

Solve the following pairs of simultaneous equations for x and y using graphical method

(a) $3x + 2y = 8$

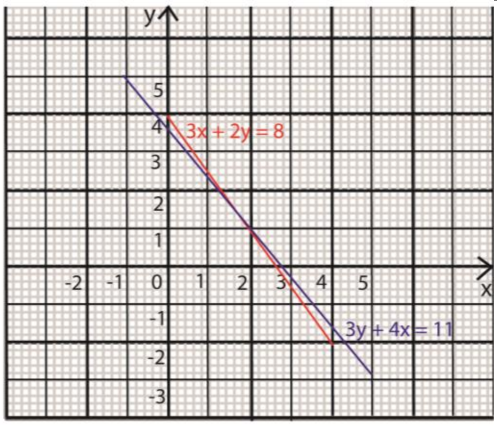
$3y + 4x = 11$

For $3x + 2y = 8$

| | | |
|---|---|---|
| x | 0 | 4 |
| y | 4 | -2 |

For $3y + 4x = 11$

| | | |
|---|---|---|
| x | -1 | 2 |
| y | 5 | 1 |




From the graph the point of intersection is (2, 1) Hence $x = 2$ and $y = 1$

(b) $x + y = 4$

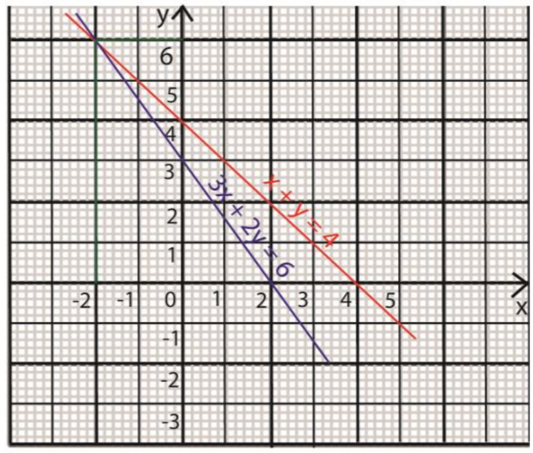
$3x + 2y = 6$

For $x + y = 4$

| | | |
|---|---|---|
| x | 0 | 4 |
| y | 4 | 0 |

For $3x + 2y = 6$

| | | |
|---|---|---|
| x | 0 | 0 |
| y | 3 | 2 |




From the graph, the point of intersection is (-2, 6). Hence x = -2, y = 6

## Solving simultaneous equations using matrixes

**A. Determinant method**

Consider the simultaneous equations

$a_1x + b_1y = c_1$

$a_2x + b_1y = d_1$

$\Rightarrow \; A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ d_1 \end{pmatrix}$ …………… (i)

$B = \begin{pmatrix} c_1 & b_1 \\ d_2 & b_2 \end{pmatrix}$

Matrix B is obtained by interchanging the coefficients of x in eqn. (i) with the column matrix on the right hand side

$$C = \begin{pmatrix} a_1 & c_1 \\ a_2 & d_1 \end{pmatrix}$$

Matric C is obtained by interchanging the coefficients of y in equation (i) with the column matrix on the right and side of eqn.(i)

For the determinant method

$x = \frac{|B|}{|A|}$ and y = $\frac{|C|}{|A|}$

***Example 4***

(a) x + 3y – 15 = 0
3x = 17 – 2y
Re-arranging the equation
x + 3y = 15
3x + 2y = 17
Expressing the equation in matrix form

$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 17 \end{pmatrix}$

$x = \frac{\begin{pmatrix} 15 & 3 \\ 17 & 2 \end{pmatrix}}{\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}}$

$x = \frac{(15\,x\,2) - (17\,x\,3)}{(1\,x\,2) - (3\,x\,3)} = \frac{-21}{-7} = 3$

$y = \frac{\begin{pmatrix} 1 & 15 \\ 3 & 17 \end{pmatrix}}{\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}}$

$y = \frac{(1\,x\,17) - (3\,x\,15)}{(1\,x\,2) - (3\,x\,3)} = \frac{-28}{-7} = 4$

Hence x = 3 and y = 4

(b) 5x + 3y = 7

2x -4y = 3

Expressing the equation in matrix form

$\begin{pmatrix} 5 & 3 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

$x = \frac{\begin{pmatrix} 7 & 3 \\ 3 & -4 \end{pmatrix}}{\begin{pmatrix} 5 & 3 \\ 2 & -3 \end{pmatrix}}$

$x = \frac{(7\,x - 4) - (3\,x\,3)}{(5\,x - 4) - (2\,x\,3)} = \frac{-37}{-26} = \frac{37}{26}$

$y = \frac{\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}}{\begin{pmatrix} 5 & 3 \\ 2 & 3 \end{pmatrix}}$

$$y = x = \frac{(5\,x\,3) - (2\,x\,7)}{(5\,x - 4) - (2\,x - 3)} = \frac{-1}{26}$$

(c) 34 + 3y = 3x
3x – 4y- 16 = 0

Rearranging the equations

-3x + 3y = -34

3x – 4y = 16

Expressing the equations in matrix form

$\begin{pmatrix} -3 & 3 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -34 \\ 16 \end{pmatrix}$

$x = \frac{\left|\begin{pmatrix} -34 & 3 \\ 16 & -4 \end{pmatrix}\right|}{\left|\begin{pmatrix} -3 & 3 \\ 3 & -4 \end{pmatrix}\right|}$

$x = \frac{(-34\,x - 4) - (16\,x\,3)}{(-3\,x - 4) - (3\,x\,3)}$

$x = \frac{136 - 48}{12 - 9} = \frac{88}{3}$

$y = \frac{\left|\begin{pmatrix}-3 & -34\\ 3 & 16\end{pmatrix}\right|}{\left|\begin{pmatrix}-3 & 3\\ 3 & -4\end{pmatrix}\right|}$

$x = \frac{(-3\,x16)-(3\,x-34)}{(-3\,x-4)-(3\,x\,3)}$

$x = \frac{-48+102}{12-9} = \frac{54}{3} = 18$

Hence x = $\frac{88}{3}$ and y = 18

**B. Adjunct method**

Consider the simultaneous equations

$a_1x + b_1y = c_1$

$a_2x + b_1y = d_1$

Arranging in matrix form

$\begin{pmatrix}a_1 & b_1\\ a_2 & b_2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}c_1\\ d_1\end{pmatrix}$ …………… (i)

If A= $\begin{pmatrix}a_1 & b_1\\ a_2 & b_2\end{pmatrix}$

Adjunct A = $\begin{pmatrix}b_2 & -b_1\\ -a_2 & a_1\end{pmatrix}$

To obtain the values of x and y, the adjunct is pre-multiplied on the both sides, i.e.

$\begin{pmatrix}b_2 & -b_1\\ -a_2 & a_1\end{pmatrix}\begin{pmatrix}a_1 & b_1\\ a_2 & b_2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}b_2 & -b_1\\ -a_2 & a_1\end{pmatrix}\begin{pmatrix}c_1\\ d_1\end{pmatrix}$

***Example 5***

Solve the following equations using matrix method

(a) 4x – 3y = 2
    x + 2y = 1
(b) 4x + 3y = 17
    5x – 2y = 4
(c) 7x – y = -1
    3x – 2y = - 24

**Solution**

(a) 4x – 3y = 2
    x + 2y = 1

Express the equation in matrix form

$\begin{pmatrix}4 & -3\\ 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}2\\ 1\end{pmatrix}$ ………………… (i)

Let $A = \begin{pmatrix}4 & -3\\ 1 & 2\end{pmatrix}$

Adjunct $A = \begin{pmatrix}2 & 3\\ -1 & 4\end{pmatrix}$

Pre- multiply both sides of eqn. (i) the adjunct matrix A

$\begin{pmatrix}2 & 3\\ -1 & 4\end{pmatrix}\begin{pmatrix}4 & -3\\ 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}2 & 3\\ -1 & 4\end{pmatrix}\begin{pmatrix}2\\ 1\end{pmatrix}$

$\begin{pmatrix}2x4 + 3x1 & 2x - 3 + 3x2\\ -1x4 + 1x4 & -1x - 3 + 4x2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}2x2 + 3x1\\ -1x2 + 4x1\end{pmatrix}$

$\begin{pmatrix}11 & 0\\ 0 & 11\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}7\\ 2\end{pmatrix}$

11x = 7

$x = \frac{7}{11}$

11y =2

$y = \frac{2}{11}$

Hence $= \frac{7}{11}, y = \frac{2}{11}$

(b) 4x + 3y = 17
    5x – 2y = 4

Express in matrix form

$\begin{pmatrix}4 & 3\\ 5 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}17\\ 4\end{pmatrix}$ …………. (i)

Let $A = \begin{pmatrix}4 & 3\\ 5 & -2\end{pmatrix}$

Adjunct A$\begin{pmatrix}-2 & -3\\ -5 & 4\end{pmatrix}$

Pre- multiply both sides of eqn. (i) the adjunct matrix A

$\begin{pmatrix}-2 & -3\\ -5 & 4\end{pmatrix}\begin{pmatrix}4 & 3\\ 5 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-2 & -3\\ -5 & 4\end{pmatrix}\begin{pmatrix}17\\ 4\end{pmatrix}$

$\begin{pmatrix}-2\,x4 + -3x5 & -2\,x\,3 + -3x - 2\\ -5x4 + 4x5 & -5x3 + 4x - 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$
$= \begin{pmatrix}-2\,x\,17 + -3x4\\ -5x17 + 4x4\end{pmatrix}$

$\begin{pmatrix}-23 & 0\\ 0 & -23\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-46\\ -69\end{pmatrix}$

-23x = -46

$x = \frac{-46}{-23} = 2$

-23y = -69

$y = \frac{-69}{-23} = 3$

Hence x = 2, y = 3

(c) 7x – y = -1
 3x – 2y = - 24

Arrange in matrix form

$\begin{pmatrix} 7 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -24 \end{pmatrix}$ ……………..(i)

Let $A = \begin{pmatrix} 7 & -1 \\ 3 & -2 \end{pmatrix}$

Adjunct A= $\begin{pmatrix} -2 & 1 \\ -3 & 7 \end{pmatrix}$

Pre- multiply both sides of eqn. (i) the adjunct matrix A

$$\begin{pmatrix} -2 & 1 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 7 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} -1 \\ -24 \end{pmatrix}$$

$$\begin{pmatrix} -2\,x\,7 + 1\,x\,3 & -2\,x - 1 + 1\,x - 2 \\ -3\,x\,7 + 7\,x\,3 & -3\;\,x - 1 + 7\,x - 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2\,x - 1 + 1\,x - 24 \\ -3\,x - 1 + 7x - 24 \end{pmatrix}$$

$$\begin{pmatrix} -11 & 0 \\ 0 & -11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -22 \\ -165 \end{pmatrix}$$

-11x = -22

$x = \frac{-22}{-11} = 2$

-23y = -69

$y = \frac{-165}{-11} = 15$

Hence x = 2 and y = 15

### C. Inverse method

***Example 6***

(a) 4x – 3y = 2
 x + 2y = 1

(b) 4x + 3y = 17
 5x – 2y = 4

(c) 7x – y = -1
 3x – 2y = - 24

**Solution**

(a) 4x – 3y = 2
 x + 2y = 1

Arrange n matrix form

$\begin{pmatrix} 4 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Let $A = \begin{pmatrix} 4 & -3 \\ 1 & 2 \end{pmatrix}$

det A = (4 x 2) – (1 x -3) = 11

Inverse, $A^{-1} = \frac{1}{11}\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} \frac{2}{11} & \frac{3}{11} \\ \frac{-1}{11} & \frac{4}{11} \end{pmatrix}$

Pre-multiply both sides of eqn. (i) by the inverse matrix A

$$\begin{pmatrix} \frac{2}{11} & \frac{3}{11} \\ \frac{-1}{11} & \frac{4}{11} \end{pmatrix}\begin{pmatrix} 4 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{11} & \frac{3}{11} \\ \frac{-1}{11} & \frac{4}{11} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{2}{11}x4 + \frac{3}{11}x1 & \frac{2}{11}x - 3 + \frac{3}{11}x2 \\ \frac{-1}{11}x4 + \frac{4}{11}x1 & \frac{-1}{11}x - 3 + \frac{4}{11}x2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{11}x2 + \frac{3}{11}x1 \\ \frac{-1}{11}x2 + \frac{4}{11}x1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{7}{11} \\ \frac{2}{11} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{7}{11} \\ \frac{2}{11} \end{pmatrix}$$

Hence x = $\frac{7}{11}$, y=$\frac{2}{11}$

Note: when we pre-multiply a matrix by its inverse, we obtain an identity matrix, i.e. $AA^{-1} = 1$

$$\begin{pmatrix} \frac{2}{11}x4 + \frac{3}{11}x1 & \frac{2}{11}x - 3 + \frac{3}{11}x2 \\ \frac{-1}{11}x4 + \frac{4}{11}x1 & \frac{-1}{11}x - 3 + \frac{4}{11}x2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{11}x2 + \frac{3}{11}x1 \\ \frac{-1}{11}x2 + \frac{4}{11}x1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{11}x2 + \frac{3}{11}x1 \\ \frac{-1}{11}x2 + \frac{4}{11}x1 \end{pmatrix} = \begin{pmatrix} \frac{7}{11} \\ \frac{2}{11} \end{pmatrix}$$

Hence x = $\frac{7}{11}$, y=$\frac{2}{11}$

(b) 4x + 3y = 17

5x – 2y = 4

Arrange in matrix form

$\begin{pmatrix}4 & 3\\5 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}17\\4\end{pmatrix}$ ………………….(i)

Let $A = \begin{pmatrix}4 & 3\\5 & -2\end{pmatrix}$

det A = 4 x -2 – (5 x 3) = -23

Adjunct A = $\begin{pmatrix}-2 & -3\\-5 & 4\end{pmatrix}$

Inverse A ($A^{-1}$)= $\frac{-1}{23}\begin{pmatrix}-2 & -3\\-5 & 4\end{pmatrix} = \begin{pmatrix}\frac{2}{23} & \frac{3}{23}\\\frac{5}{23} & \frac{-4}{23}\end{pmatrix}$

Pre-multiply both sides of eqn. (i) by the inverse matrix A

$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{2}{23} & \frac{3}{23}\\\frac{5}{23} & \frac{-4}{23}\end{pmatrix}\begin{pmatrix}17\\4\end{pmatrix}$

$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{2}{23}x17 + & \frac{3}{23}x4\\\frac{5}{23}x17 + & \frac{-4}{23}x4\end{pmatrix} = \begin{pmatrix}\frac{46}{23}\\\frac{69}{23}\end{pmatrix}$

$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$

Hence x = 2, y = 3

(c) 7x – y = -1

3x – 2y = - 24

Express in matrix form

$\begin{pmatrix}7 & -1\\3 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-1\\-24\end{pmatrix}$

Let $A = \begin{pmatrix}7 & -1\\3 & -2\end{pmatrix}$

det A = (7 x -2)- (3 x -1) = -11

Adjunct A= $\begin{pmatrix}-2 & 1\\-3 & 7\end{pmatrix}$

Inverse of matrix A ($A^{-1}$) =$\frac{-1}{11}\begin{pmatrix}-2 & 1\\-3 & 7\end{pmatrix}$

=$\begin{pmatrix}\frac{2}{11} & \frac{-1}{11}\\\frac{3}{11} & \frac{-7}{11}\end{pmatrix}$

Pre-multiply both sides of eqn. (i) by the inverse matrix A

$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{2}{11} & \frac{-1}{11}\\\frac{3}{11} & \frac{-7}{11}\end{pmatrix}\begin{pmatrix}-1\\-24\end{pmatrix}$

$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{2}{11}x-1+ & \frac{-1}{11}x-24\\\frac{3}{11}x-1 & \frac{-7}{11}x-24\end{pmatrix} = \begin{pmatrix}\frac{22}{11}\\\frac{165}{11}\end{pmatrix}$

$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\15\end{pmatrix}$

Hence x = 2 and y 15

## Solving Non-linear simultaneous equation

These are solved basically by using substitution method

***Example 7***

(a) $x^2 + 2x - y = 14$

$2x^2 - 3y = 47$

(b) $2x^2 - xy + y^2 = 32$

$y = -\frac{5}{x}$

***Solution***

(a) $x^2 + 2x - y = 14$

$y = x^2 + 2x - 14$……………… (i)

$2x^2 - 3y = 47$ …………………. (ii)

Substituting eqn. (i) into eqn. (ii)

$2x^2 - 3(x^2 + 2x - 14) = 47$

$2x^2 - 3x^2 - 6x + 42 = 47$

$-x^2 - 6x - 5 = 0$

$x^2 + 6x + 5 = 0$

(x+1)(x+5)= 0

x = -1 or x = -5

Substituting x into eqn. (i)

When x = -1, y = $(-1)^2 + 2(-1) - 14 = -15$

When x = -5, y = $(-5)^2 + 2(-5) - 14 = 1$

Hence (x, y) = (-1, -15) and (-5, -1)

(b) $2x^2 - xy + y^2 = 32$ ................... (i)

$y = -\frac{5}{x}$ .................................. (ii)

Substituting equation (ii) into eqn. (i)

$2x^2 - x\left(-\frac{5}{x}\right) + \left(-\frac{5}{x}\right)^2 = 32$

$2x^2 + 5 + \frac{25}{x^2} = 32$

$2x^2 + \frac{25}{x^2} - 27 = 0$

$2x^4 - 27x^2 + 25 = 0$

$(x^2 - 1)(2x^2 - 25) = 0$

$x^2 = 1$

$x = \pm 1$

$2x^2 - 25 = 0$

$2x^2 = 25$

$x \pm \frac{5}{\sqrt{2}}$

Substituting for x into eqn. (i)

$y = -\frac{5}{x}$

When x = 1, $y = -\frac{5}{1} = -5$

When x = -1, $y = -\frac{5}{-1} = 5$

When x = $\frac{5}{\sqrt{2}}$, $y = -\frac{5}{\frac{5}{\sqrt{2}}} = -\frac{5\sqrt{2}}{5} = -\sqrt{2}$

When x = $-\frac{5}{\sqrt{2}}$, $y = -\frac{5}{-\frac{5}{\sqrt{2}}} = \frac{5\sqrt{2}}{5} = \sqrt{2}$

Hence the solution to simultaneous equations are (x, y)= (1, -5), (-1, 5), $\left(\frac{5}{\sqrt{2}}, -\sqrt{2}\right)$, $\left(\frac{5}{\sqrt{2}}, \sqrt{2}\right)$,

(c) $(x - 4y)^2 = 1$

3x = 8y = 11 (06marks)

Solving equations

(x – 4y) = 1 ....................... (i)

3x = 8y = 11....................... (ii)

Eqn. (ii) - 3Eqn. (i)

20y = 8

$y = \frac{8}{20} = \frac{2}{5}$

From eqn. (i)

$x = 1 + 4\left(\frac{2}{5}\right) = \frac{13}{5}$

And

(x – 4y) = -1 ....................... (i)

3x = 8y = 11....................... (ii)

2(eqn (i)) + eqn. (ii)

5x = 9

$x = \frac{9}{5}$

From equation (i)

$4y = \frac{9}{5} + 1$

$y = \frac{7}{10}$

$\therefore$ (x,y) = $\left(\frac{13}{5}, \frac{2}{5}\right), \left(\frac{9}{5}, \frac{7}{10}\right)$

## Solving three linear simultaneous equations in three unknown

When solving for three unknowns, there must be three equation that will be solved simultaneously. The methods that will be used to such equation are

- Elimination and substitution
- Row reduction to echelon

**Elimination and substitution**

This involves elimination of one unknown variable so as to remain two unknowns which can easily be solved

***Example 7***

(a) Solve the simultaneous equations

x – 2y - 2z = 0

2x + 3y + z = 1

3x –y – 3z = 3

X – 2y -2z = 0 .......................... (i)

2x + 3y + z = 1...................... (ii)

3x –y – 3z = 3 ..........................(iii)

Eqn. (i) + 2eqn. (ii)

4x + 4y = 2 ...............................(iv)

3Eqn. (ii) + eqn. (iii)

9x + 8y = 6 ............................. (v)

2eqn. (iv) – eqn. (v)

x = -2

Substituting x = -2 into eqn. (iv)

5(-2) + 4y = 2

y = 3

Substituting x = -2 and y = 3 into eqn. (ii)

2(-2) + 3(3) + z =2

z = -4

∴ x = -2, y = 3 and z = -4

(b) 2x = 3y = 4z

$x^2 - 9y^2 - 4z + 8 = 0$

2x = 3y = 4z, substituting 4z = 2x and y = $\frac{2x}{3}$ into the equation $x^2 - 9y^2 - 4z + 8 = 0$

$x^2 - (2x)^2 - 2x + 8 = 0$

$-3x^2 - 2x + 8 = 0$

x = $\frac{2\pm\sqrt{(-2)^2-4(-3)(8)}}{2(-3)}$; x = -2 or x = $\frac{4}{3}$

When x = -2; y = $\frac{2\,x\,(-2)}{3} = \frac{-4}{3}$; z= $\frac{2\,x\,(-2)}{4}$ = -1

(x, y, z) = ( -2, $\frac{-4}{3}, -1$)

When x = $\frac{4}{3}$; y = $\frac{2\,x\,(\frac{4}{3})}{3} = \frac{8}{9}$; z= $\frac{2\,x\,(\frac{4}{3})}{4} = \frac{2}{3}$

(x, y, z) = ($\frac{4}{3}, \frac{8}{9}, \frac{2}{3}$)

Alternatively

2x = 3y = 4z, substituting 4z = 3y and x = $\frac{3y}{2}$ into the equation $x^2 - 9y^2 - 4z + 8 = 0$

$\left(\frac{3}{2}y\right)^2 - 9y^2 - 3y + 8 = 0$

$9y^2 - 36y^2 - 12y + 32 = 0$

$-27y^2 - 12y + 32 = 0$

y = $\frac{12\pm\sqrt{(-12)^2-4(-27)(32)}}{2(-27)}$; y = $\frac{-4}{3}$ or x = $\frac{8}{9}$

When y = $\frac{-4}{3}$; x = $\frac{3}{2}\,x\,\frac{-4}{3} = -2$; z= $\frac{3}{4}\,x\,\frac{-4}{3}$ = -1

(x, y, z) = ( -2, $\frac{-4}{3}, -1$)

When y =$\frac{8}{9}$; x = $\frac{3}{2}\,x\,\frac{8}{9} = \frac{4}{3}$; z= $\frac{3}{4}\,x\,\frac{8}{9} = \frac{2}{3}$

(x, y, z) = ($\frac{4}{3}, \frac{8}{9}, \frac{2}{3}$)

**Row reduction to Echelon matrix form**

This method involves the expression of the three equations into matrix form known as augmented matrix and thereafter transforming the augmented matrix to a unity triangular matrix (a matrix whose elements in the major diagonal are unity and zero below)

Example

Solve the simultaneous equations

(a) x – 2y - 2z = 0
2x + 3y + z = 1
3x –y – 3z = 3

**Solution**

Expressing the equation in matrix form

$$\begin{pmatrix}1 & -2 & -2\\ 2 & 3 & 1\\ 3 & -1 & -3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 1\\ 3\end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix}1 & -2 & -2:0\\ 2 & 3 & 1\ :1\\ 3 & -1 & -3:3\end{pmatrix}$$

Transforming augmented matrix to a unity triangular matrix

$$\begin{matrix}R_1\\ R_2\\ R_3\end{matrix}\begin{pmatrix}1 & -2 & -2:0\\ 2 & 3 & 1\ :1\\ 3 & -1 & -3:3\end{pmatrix}\begin{matrix}\to \quad R_1 = R_1\\ \to 2R_1 - R_2 = R_2\\ \to 3R_1 - R_3 = R_3\end{matrix}\begin{pmatrix}1 & -2 & -2:\ \ 0\\ 0 & -7 & -5:-1\\ 0 & -5 & -3:-3\end{pmatrix}$$

$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\begin{pmatrix}1 & -2 & -2: & 0\\0 & -7 & -5: & -1\\0 & -5 & -3: & -3\end{pmatrix}\begin{matrix}\rightarrow\\\rightarrow\\\rightarrow\end{matrix}\begin{matrix}R_1 = R_1\\2R_1 - R_2 = R_2\\\frac{3R_1-R_3}{-4} = R_3\end{matrix}\begin{pmatrix}1 & -2 & -2: & 0\\0 & 1 & \frac{5}{7}: & {}^1/_7\\0 & 0 & 1: & -4\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1 & -2 & -2\\0 & 1 & \frac{5}{7}\\0 & 0 & -4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\{}^1/_7\\16\end{pmatrix}$$

z = -4

y + $\frac{5}{7}z$ = ${}^1/_7$
y + $\frac{5}{7}x - 4$ = ${}^1/_7$
y = 3

x – 2y – 2z = 0
x -2(3) – 2(-4) = 0
x = -2
∴ x = -2, y = 3 and z = -4

(b) 3x – y – 2z = 0
x +3y – z = 5
2x – y + 4z = 26

Expressing the equation into matrix form

$$\begin{pmatrix}3 & -1 & -2\\1 & 3 & -1\\2 & -1 & 4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\5\\26\end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix}3 & -1 & -2: & 0\\1 & 3 & -1: & 5\\2 & -1 & 4: & 26\end{pmatrix}$$

Transforming the augmented matrix into unity

$$\begin{pmatrix}3 & -1 & -2: & 0\\1 & 3 & -1: & 5\\2 & -1 & 4: & 26\end{pmatrix}\begin{matrix}\rightarrow\\\rightarrow\\\rightarrow\end{matrix}\begin{matrix}R_1 = R_1\\R_1 - 3R_2 = R_2\\2R_1 - 3R_3 = R_3\end{matrix}\begin{pmatrix}3 & -1 & -2: & 0\\0 & -10 & 1: & -15\\0 & 1 & -16: & -78\end{pmatrix}$$

$$\begin{pmatrix}3 & -1 & -2: & 0\\0 & -10 & 1: & -15\\0 & 1 & -16: & -78\end{pmatrix}\begin{matrix}\rightarrow\\\rightarrow\\\rightarrow\end{matrix}\begin{matrix}R_1 \div 3 = R_1\\10R_1 \div -10 = R_2\\\frac{R_2+10R_3}{159}\end{matrix}\begin{pmatrix}1 & \frac{-1}{3} & \frac{-2}{3} & : 0\\0 & 1 - \frac{1}{10} & & : \frac{15}{10}\\0 & 0 & 1 & : 5\end{pmatrix}$$

Z = 5

y - $\frac{1}{10}$ $x5$ =$\frac{15}{10}$
10y - 5 =15
y = 2

x – $\frac{1}{3}$y – $\frac{2}{3}$z = 0
x -$\frac{1}{3}$ (2) – $\frac{2}{3}$ (5) = 0
x = 4
∴ x = 4, y = 2 and z = 5

(c) 3x – 2y – z = 5
x +3y – z = 4
2x – y + 4z = 13 [3, 1, 2]

Expressing the equation into matrix form

$$\begin{pmatrix}3 & -2 & -1\\ 1 & 3 & -1\\ 2 & -1 & 4\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}5\\ 4\\ 13\end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix}3 & -2 & -1 & : 5\\ 1 & 3 & -1 & : 4\\ 2 & -1 & 4 & :13\end{pmatrix}$$

Transforming the augmented matrix into unity

$$\begin{pmatrix}3 & -2 & -1 & : 5\\ 1 & 3 & -1 & : 4\\ 2 & -1 & 4 & :13\end{pmatrix}\begin{matrix}\to R_1 = R_1\\ \to R_1 - 3R_2 = R_2\\ \to 2R_1 - 3R_3 = R_3\end{matrix}\begin{pmatrix}3 & -2 & -1 & : 5\\ 0 & -11 & 2 & : -7\\ 0 & -1 & -14 & : -29\end{pmatrix}$$

$$\begin{pmatrix}3 & -2 & -1 & : 5\\ 0 & -11 & 2 & : -7\\ 0 & -1 & -14 & : -29\end{pmatrix}\begin{matrix}\to R_1 \div 3\\ \to R_2 \div -11\\ \to \frac{R_2 - 11R_3}{156}\end{matrix}\begin{pmatrix}1 & \frac{-2}{3} & \frac{-1}{3} & : \frac{5}{3}\\ 0 & 1 & \frac{-2}{11} & : \frac{7}{11}\\ 0 & 0 & 1 & : 2\end{pmatrix}$$

Z = 2

$y - \frac{2}{11}\, x\, z = \frac{7}{11}$

11y - 4 =7

y = 1

$x - \frac{2}{3}y - \frac{1}{3}z = \frac{5}{3}$

$x - \frac{2}{3}(1) - \frac{1}{3}(2) = \frac{5}{3}$

x = 3

∴ x = 3, y = 1 and z = 2

## Applications of simultaneous equations

### 1. Business and Finance

**Application**: Calculating costs, revenues, and profits when multiple variables are involved.

**Example**: A company sells two products. If 3 units of Product A and 2 units of Product B cost sh. 80, and 2 units of Product A and 4 units of Product B cost sh.100, simultaneous equations help determine the individual prices of each product.

### 2. Mixture Problems

**Application**: Determining concentrations or proportions in chemistry or cooking.

**Example**: Mixing two solutions with different concentrations to get a desired concentration. If Solution A is 30% acid and Solution B is 50% acid, simultaneous equations help find how much of each to mix for a 40% solution.

### 3. Travel and Motion

**Application**: Solving problems involving speed, time, and distance.

**Example**: Two cyclists start from different points and travel toward each other. One travels 5 km/h faster than the other. Simultaneous equations help determine when and where they meet.

### 4. Economics

**Application**: Market equilibrium analysis.

**Example**: Supply and demand equations intersect to find the equilibrium price and quantity in a market.

### 5. Engineering

**Application**: Circuit analysis and structural design.

**Example**: In electrical engineering, Kirchhoff's laws generate simultaneous equations to calculate current and voltage in circuits.

## Revision exercise for simultaneous equations

1. Using elimination method solve the following pairs of simultaneous equation
   (a) -3x + 2y =-16
   x + 5y = 11 [x = 6, y = 1]
   (b) 3y – 2x = -18
   2y + 3x = -6 [ x = 0, y = -3]
   (c) 2x - 3y = 7
   X + 4y = -2 [x = 2, y = -1]
   (d) 5x + 3y = 8
   3x + 2y = 6 [x = -2, y = 6)
2. Using substitution method solve the following pairs of simultaneous equation
   (a) -3x + 2y = 16
   5x + 3y = 33 [x = 6, y = 1]
   (b) y – 2x = -3
   7y + 3x = -21 [x = 0, y = -3]
   (c) 2x + 5y = 26
   3x + 2y = 6 [ x = -2, y = 6]
3. Solve the following pairs of simultaneous equation using the matrix method

(a) 2x – 3y = 7
2x + 3y = 1 [x =2, y = -1]

(b) 2x – 7y = 1
3x + 3y = 15

(c) 3x – 4y = 5
6x – 3y = 0 [x = -1, y =-2]

(d) 3x + 2y = 3
x -6y = 1[x =1, y = 0]

(e) 2x + 3y= 1
3x + y = 5 [x =2, y = -1]

4. Solve the following simultaneous equations elimination and substitution method
   (a) 3x – 2y – 2z = -2
   x + 3y – 3z = -5
   2x – y + 4z = 26 [(x, y, z)=(4, 2, 5)]
   (b) 2x + 2y – 3z = 1
   3x + 3y – z = 5
   4x – 2y + 2z = 4 [(x, y, z)=(1, 1, 1)]
   (c) 4x –y + 2z = 7
   x + y + 6z = 2
   8x + 3y – 10z = -3 [(x, y, z)=(1, -2, $\frac{1}{2}$)]
5. By row reducing the appropriate matric to echelon form solve the systems of equations below

(a) x +2y – 2z =0
2x + y – 4z = -1
4x – 3y + z = 11 [(x, y, z), (3, 1, 2]

(b) x – 2y + 3z = 6
3x + 4y – z = 3
4x + 6y – 5z = 0 [(x, y, z)=(2, $\frac{-1}{2}$, 1)]

(c) 2x – y + 3z = 10
x +2y -5z = 9
5x + y + 4z [ x =2, y = -2, z= 1]

(d) p + 2q – r = -1
3p –q + 2r = 16
2p + 3q + r =3 [p= 4, q = -2, r = 1]

# Inequalities

An inequality is a logical statement that states relationship between two mathematical expressions.

The basic inequalities commonly used are

- Less than (<)
- More than (>)
- Less than or equal (≤)
- Greater than or equal (≥)

When solving for equations, the solutions or answers are individual values but when solving inequalities, the solutions are a range of possible real values.

**Linear inequalities**

## Solving Linear inequalities in one unknown given one in one equation

Solving linear inequalities in one unknown given one in one equation is done in the same way as solving for linear equation except

- The inequality symbols must be maintained
- The inequality symbol changes when dividing both sides of inequality equation by a negative number.

***Example 1***

Solve the following inequalities

(a) 4x – 2 > x + 7
Solution
4x – x > 7 + 2
3x > 9
x > 3

(b) 3(2 – x) > 5(3 + 2x)
Solution
6 – 3x > 15 + 10x
-9 > 13x
x < $\frac{-9}{13}$

(c) $\frac{x-2}{4} < \frac{2x-3}{3}$
Solution
Multiply both sides by 12
$12\left(\frac{x-2}{4}\right) < 12\left(\frac{2x-3}{3}\right)$
3(x – 2) < 4(2x – 3)
3x – 6 < 8x – 12
-5x < -6
x > $\frac{6}{5}$

(d) $\frac{1}{2}(x-1) + \frac{1}{3}(x-2) \leq \frac{1}{4}(x-3)$
Multiply both sides by 12
6(x – 1) + 4(x – 2) ≤ 3(x – 3)
6x – 6 + 4x – 8 ≤ 3x – 9
7x ≤ 5
x ≤ $\frac{5}{7}$

## Solving linear inequalities involving indices

When solving inequalities involving indices such as $a^x > b$, where a and b are positive integers, introduce natural logarithms to both sides of the inequality. i.e.

$\text{In}a^x > b$

xlna > lnb

x > $\frac{lna}{lnb}$

***Example 2***

Solve the following inequalities correct to 3 decimal places

(a) $5^{2x} > 8$
Solution
$\text{In}5^{2x} > \text{In}8$
2xln5 > ln8
$x > \frac{ln8}{2ln5} = 0.646$
x > 0.646

(b) $20^{-3x} < 15$

Solution

$\ln 20^{-3x} > \ln 15$

$-3x\ln 20 > \ln 15$

$x > -\frac{\ln 15}{3\ln 20} = -0.301$

x > -0.301

(c) $(0.8)^{-3x} > 2.4$

Solution

$\ln(0.8)^{-3x} > \ln 2.4$

-3xln0.8- > ln2.4

$x > \frac{\ln 2.4}{-3\ln 0.8} = 1.308$

x > 1.308

(d) $(0.8)^{3x} > 2.4$

Solution

$\ln(0.8)^{3x} > \ln 2.4$

$3x\ln 0.8 > \ln 2.4$

Note that logarithm of any number between 0 and 1 is negative; so ln0.8 is negative

$x < \frac{\ln 2.4}{3\ln 0.8} = -1.308$

x < −1.308

## Solving linear inequalities in one unknown given two inequalities equations

The solution to two linear inequalities can be best handled by use of a number line. When finding a set of integers that satisfy the equations, we only take on integral (discrete) values.

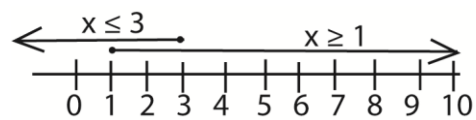
***Example 3***

Find the set of integers which satisfy simultaneously both of the following equations

(a) $4x + 3 \geq 2x + 5$;  $x + 4 \leq 7$

Solution

| | |
|---|---|
| $4x + 3 \geq 2x + 5$; | $x + 4 \leq 7$ |
| $2x \geq 2$ | $x \leq 3$ |
| $x \geq 1$ | |

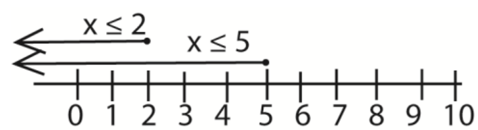



The number line show that the set of integers that satisfies the two equations are {1, 2, 3}

(b) $5 - 2x \geq 3 - x$;  $1 - 2x \leq 11 - 4x$

Solution

| | |
|---|---|
| $5 - 2x \geq 3 - x$; | $1 - 2x \leq 11 - 4x$ |
| $-x \geq -2$ | $2x \leq 10$ |
| $x \leq 2$ | $x \leq 5$ |

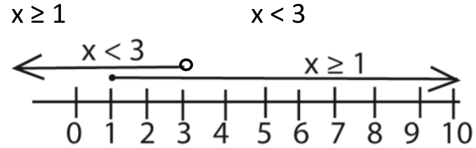



The number line show that the set of integers that satisfies the two equations are {x:x ≤ 2}

(c) $5x - 4 \geq 4x - 3$,  $\frac{1}{3}x < 1$

Solution

| | |
|---|---|
| $5x - 4 \geq 4x - 3$, | $\frac{1}{3}x < 1$ |
| $x \geq 1$ | $x < 3$ |




The number line show that the set of integers that satisfies the two equations are {1, 2} (3 is not included

***Example 4***

Show that there is just one integer which simultaneously satisfies the three inequalities and find that number

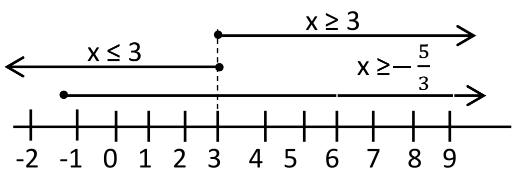
$\frac{1}{2}(x-1) \geq 1 \quad 2 - 3x \leq 7 \quad \frac{1}{3}x \leq 1$

Solution

$\frac{1}{2}(x-1) \geq 1$ $\quad 2-3x \leq 7$ $\quad \frac{1}{3}x \leq 1$

x – 1 ≥ 2 -3x ≤ 5 x ≤ 3

x ≥ 3 x ≥ - $\frac{5}{3}$ x ≤ 3




From the number line, there is only one point of intersection of the three inequalities, which is 3

Hence the set of integers that satisfy the three inequalities is {3}

## Solving Non-linear inequalities in one unknown

The following methods are employed

- Sign change
- Graphical method

When using graphical method, the set of values above the axis are positive and those below are negative

When using sign change method, a table describing specific regions of inequalities is used and the necessary tests are performed

If the inequality symbol is ≥ or ≤, care must be taken, because the critical values of the function and the numerator in case of fractions will always satisfy the inequalities

Before solving inequality, all terms must be taken to one side preferably the LHS

Method I: Graphical method

***Example 5***

Solve the following inequalities

(a) $2x^2 - 3x + 1 \leq 0$

Solution

Let y = $2x^2 - 3x + 1$

The curve cuts the x – axis when y = 0

⇨ $2x^2 - 3x + 1 = 0$
(x – 1)( 2x – 1) = 0

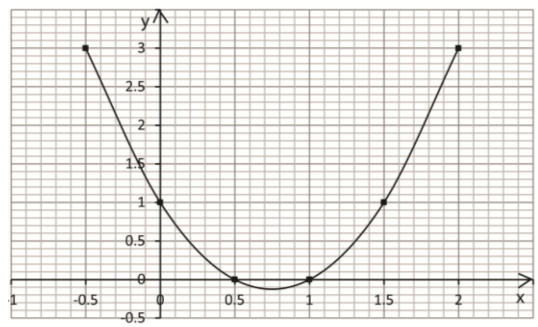
Either x – 1 = 0 or 2x – 1 = 0;

x = 1 or x = $\frac{1}{2}$

the curve cuts the y – axis when x = 0

⇨ y = 1

Since the coefficient of $x^2$ is positive, that the curve is U shaped




The solution set is 0.5 ≤ x ≤ 1

(b) $7x^2 > 1 - 6x$

Solution

$7x^2 > 1 - 6x$

$7x^2 + 6x - 1 > 0$

Let y = $7x^2 + 6x - 1$

The curve cuts the x – axis when y = 0

⇨ $7x^2 + 6x - 1 = 0$

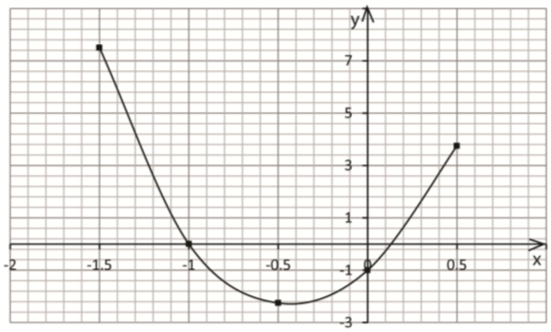
(x + 1)(7x – 1) = 0

Either x + 1 = 0 or 7x – 1 = 0

x = -1 or x = $\frac{1}{7}$

the curve cuts the y – axis when x = 0

⇨ $y = -1$




From the graph, the solution is $x < -1$ and $x > \frac{1}{7}$

(c) $2x^3 + 3x^2 \geq 2x$

Solution

$2x^3 + 3x^2 - 2x \geq 0$

Let $y = 2x^3 + 3x^2 - 2x$

The curve cuts the x – axis when y = 0

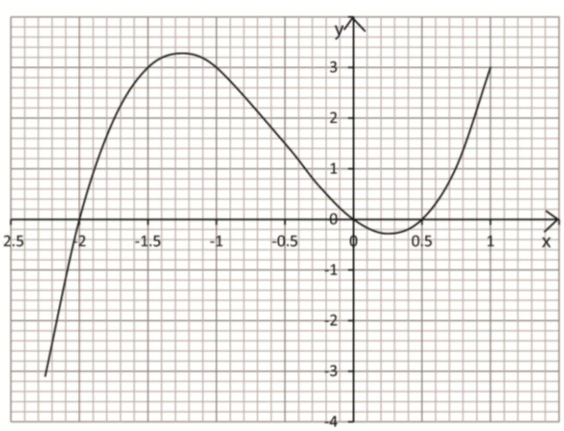
⇨ $2x^3 + 3x^2 - 2x = 0$
$x(2x - 1)(x + 2) = 0$

Either $x = 0$, $2x - 1 = 0$ or $x + 2 = 0$

$x = 0$, $x = \frac{1}{2}$ or $x = -2$

the curve cuts y – axis when x = 0

⇨ $y = 0$




From the graph the solution $-2 \leq x \leq 0$ and $x \geq 0.5$

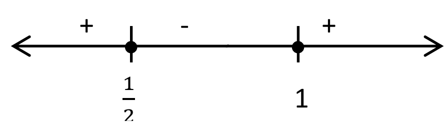
(d) Solve the inequality

$4x^2 + 2x < 3x + 6$ (06marks)

**Method II: sign change**

***Example 6***

(a) $2x^2 - 3x + 1 \leq 0$
$(2x - 1)(x - 1) \leq 0$
The critical values of $(2x - 1)(x - 1) = 0$ are $x = 1$ and $x = \frac{1}{2}$ respectively
The above illustration shows that the numbers $\frac{1}{2}$ and 1 subdivide the number line into three regions namely
$x \leq \frac{1}{2}$, $\frac{1}{2} \leq x \leq 1$, $x \geq 1$
The corresponding sign in the respective regions can be analysed by choosing any random value in each region, substitute it in the equation $(2x - 1)(x - 1)$ and put the sign of the answer on the following number line




Note that the solution for $(2x - 1)(x - 1) \leq 0$ is equal or less than zero or negative.
Closed circles indicate that the critical values are part of the solution.

Hence the solution set for $(x - 1)(2x - 1) \leq 0$

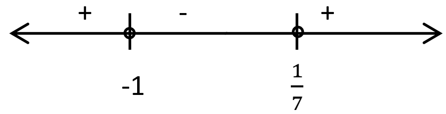
is $\frac{1}{2} \leq x \leq 1$

(b) $7x^2 + 6x - 1 > 0$

$7x^2 + 7x - x - 1 = 0$

$(x + 1)(7x - 1) = 0$

The critical values x = -1 and $\frac{1}{7}$




The solution for (x + 1)(7x -1) > 0 is positive . Open circles indicate that the critical values are not part of the solution

Hence the solution set for (x + 1)(7x -1) > 0 is

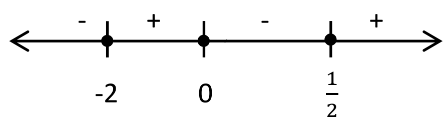
x < -1 and x > $\frac{1}{7}$

(c) $2x^3 + 3x^2 - 2x \geq 0$

$x(2x^2 + 3x - 2) \geq 0$

$x(x + 2)(2x - 1) \geq 0$

Critical values x = 0, x =-2, and x = $\frac{1}{2}$




The solution for x(x + 2)(2x – 1) ≥ 0 is positive and the critical values are part of the solution

Hence the solution for x(x + 2)(2x – 1) ≥ 0

-2 ≤ x ≤ 0 and x ≥ $\frac{1}{2}$

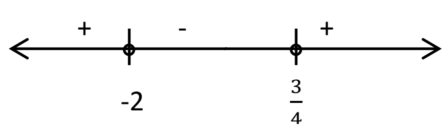
(d) $4x^2 + 5x - 6 < 0$

$4x^2 + 5x - 6 = 0$

Critical values

$$x = \frac{-5 \pm \sqrt{5^2 - 4(4)(-6)}}{2(4)} = \frac{-5 \pm \sqrt{121}}{8} = \frac{-5 \pm 11}{8}$$

x = -2, $\frac{3}{4}$




The solution $4x^2 + 5x - 6 < 0$ is negative and the critical values are not part of the solution

∴ the solution is -2 < x < $\frac{3}{4}$

(e) $\frac{3x^2-1}{x+2} \geq 2$

$\frac{3x^2-1}{x+2} - 2 \geq 0$

$\frac{3x^2-1-2(x+2)}{x+2} \geq 0$

$\frac{3x^2-1-2x-4}{x+2} \geq 0$

$\frac{(3x-5)(x+1)}{x+2} \geq 0$

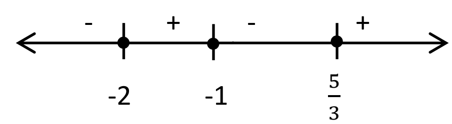
Finding critical values

3x – 5 = 0; x = $\frac{5}{3}$

(x + 1) = 0; x = -1

(x + 2) = 0, x = -2

Testing for correct region




The solution for $3x^2 - 2x - 5 \geq 0$ is positive and the critical values are part of the solution

Hence the solution for $3x^2 - 2x - 5 \geq 0$ is -2 ≤ x ≤ -1 and x ≥ $\frac{5}{3}$

The modulus of inequalities

The modulus of a number is the magnitude of that number (absolute value) which is always positive, e.g. $|1| = |-1|$ = 1

When finding modulus of an inequality, the following must be considered

- The modulus on one side of the linear inequality is removed by introducing a negative number of the given value on the other side
  i.e. if $|x| < 3$, then -3 < x < 3

- The modulus on both sides of the linear inequality is removed by squaring both sides
- When the terms under modulus are fractional, square both sides of the inequality

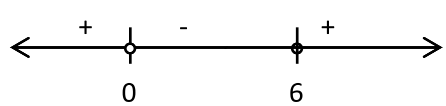
***Example 7***

Solve the following inequalities

(a) $|x - 6| < 4$
-4< x – 6 < 4
-4 + 6 < x 4 + 6
2 < x < 10

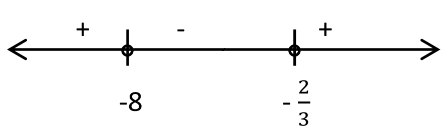
(b) $|3x + 4| < 6$
-6 < 3x + 4 < 6
-6 – 4 < 3x < 6 – 4
-10 < 3x < 2
$-\frac{10}{3} < x < \frac{2}{3}$

(c) $|2x - 3| > |x + 3|$
$(2x – 3)^2 > (x + 3)^2$
$4x^2 – 12x + 9 > x^2 + 6x + 9$
$3x^2 – 18x > 0$
3x(x – 6) = 0
Critical values are x = 0 and x = 6




The solution x < 0 and x > 6

(d) $|2x + 5| < |x - 3|$
$(2x + 5)^2 < (x – 3)^2$
$4x^2 + 20x + 25 < x^2 – 6x + 9$
$3x^2 + 26x + 16 < 0$
(3x + 2)(x + 8) < 0
Critical values are x = -8 and x = $-\frac{2}{3}$




The solution is -8 < x < $-\frac{2}{3}$

***Example 8***

Find the range of value of x can take for the following inequality to be true

$\left|\frac{x}{x-3}\right| < 2$

Solution

Squaring both sides

$\frac{x^2}{x^2-6x+9} < 4$

$x^2 < 4(x^2 – 6x + 9)$

$x^2 < 4x^2 – 24x + 36$

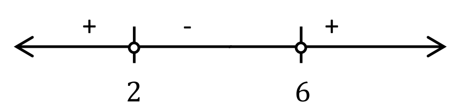
$0 < 3x^2 – 24x + 36$

Divide through by 3

$x^2 – 8x + 12 > 0$

(x – 2)(x – 6) > 0

Critical values are x = 2 and x = 6




Solution for (x – 2)(x – 6) > 0 (positive)

Hence the solution x < 2 and x > 6

## Limits of inequality

This refers to interval within which the inequalities lies or does not lie.

This is done by expressing the function given as quadratic equation in x.

For real values of x, $b^2 \geq 4ac$

***Example 9***

(a) Given the function y = $\frac{3x-6}{x^2+6x}$, find the range of values within which y does not lies

Solution

y = $\frac{3x-6}{x^2+6x}$

y($x^2$ + 6x) = 3x – 6

y$x^2$ + (6y – 3)x + 6 = 0

For real values of x, $b^2 \geq 4ac$

$(6y – 3)^2$ = 24y

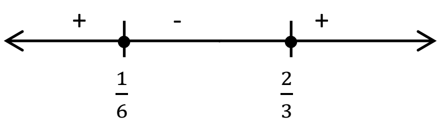
$36y^2 – 60y + 9 \geq 0$

Dividing through by 3

$12y^2 – 2y + 3 \geq 0$

$(6y – 1)(2y – 3) \geq 0$

The critical values are y = $\frac{1}{6}$ and y = $\frac{2}{3}$




Since the solution of the equation is positive; the required range $\frac{1}{6} < x < \frac{2}{3}$

(b) Find the range of values within which the function y =$\frac{3-2x}{4+x^2}$ lies

Solution

$y(4 + x^2) \geq 3 – 2x$

$yx^2 + 2x + 4y – 3 \geq 0$

For real values of x, $b^2 \geq 4ac$

$2^2 \geq 4y(4y – 3)$

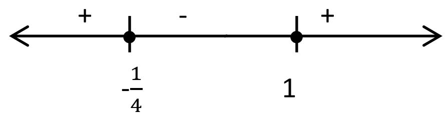
$1 \geq 4y^2 – 3y$

$0 \geq 4y^2 – 3y – 1$

$4y^2 – 3y – 1 \leq 0$

$(y – 1)(4y + 1) \leq 0$

Critical values y = 1 and y = $-\frac{1}{4}$




Solution for $(y – 1)(4y + 1) \leq 0$ is negative and critical values are part of the solution

Hence range of values is $-\frac{1}{4} \leq x \leq 1$

Simultaneous inequalities

Solving two simultaneous inequalities is best done by representing the inequalities on the graph. The unshaded (feasible) region represents the solution to the inequalities.

***Example 10***

Show by shading the unwanted regions; the region satisfying the inequalities y ≤ 2x +1 and y ≥ 3

Solution

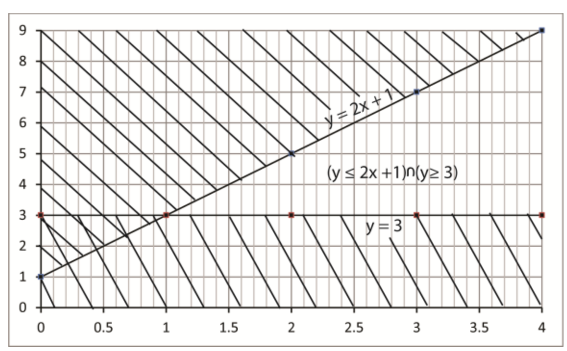
For y ≤ 2x +1, the boundary line is y = 2x +1

If x= 0, y= 1, (x, y) = (0, 1)

If x= 2, y= 5, (x, y) = (2, 5)

Testing for wanted region using point (0,0); 0 ≤ 1. Hence this point is in wanted region.

For y ≥ 3 boundary line is y = 3




Show by shading the unwanted regions; the region satisfying the inequalities x +2y ≥ 6, y > x, x< 5 and 3x + 5y ≤ 30

**Solution**

For x +2y ≥ 6 the boundary line is

x + 2y = 6

| | | |
|---|---|---|
| x | 0 | 6 |
| y | 3 | 0 |

For x> y

The boundary line is x = y

| | | |
|---|---|---|
| x | 0 | 5 |
| y | 0 | 5 |

For x < 5

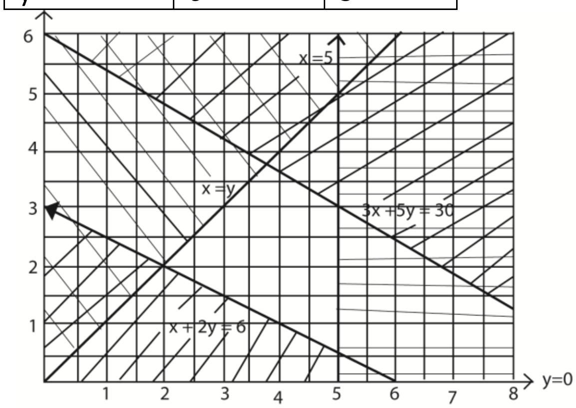
The boundary line is x = 5

For 3x + 5y ≤ 30

The boundary line

3x + 5y = 30

| | | |
|---|---|---|
| x | 0 | 5 |
| y | 6 | 3 |




## Applications of inequalities

### 1. Budgeting and Finance

**Application**: Setting spending limits or estimating costs.

**Example**: If you have **sh.** 500 to spend on groceries for the month, the inequality is:

Total cost≤500

This helps ensure you stay within budget.

### 2. Construction and Manufacturing

**Application**: Ensuring materials meet safety or size requirements.

**Example**: A beam must be at least 10 meters long but no more than 15 meters:

10≤Length≤15

### 3. Health and Medicine

**Application**: Monitoring safe dosage or health metrics.

**Example**: A patient's blood pressure should stay below 140 mmHg:

Blood pressure<140

### 4. Law and Regulation

**Application**: Enforcing limits like speed or age.

**Example**: Legal driving age is 18 or older:

Age≥18

### 5. Software and Staffing

**Application**: Managing resource allocation or scheduling.

**Example**: A server can handle up to 100 users:

Number of users≤100

### 6. Real Estate

**Application**: Filtering properties by price or size.

**Example**: Looking for homes under sh.300,000:

Price<300,000

### 7. Education

**Application**: Grading systems and eligibility.

**Example**: Students need at least 50% to pass:

Score≥50%

### 8. Transportation

**Application**: Fuel limits or travel time estimates.

**Example**: A car can travel no more than 600 km on a full tank:

Distance≤600km

### 9. Project Management

**Application**: Estimating time or cost ranges.

**Example**: A project must be completed in 30 days or less:

Time≤30

## Revision exercise on inequalities

1. Solve the following inequalities
   (a) 7x – 3 ≥ 2x – 1 $\left[x \geq \frac{2}{5}\right]$
   (b) 5(2 – x) – 2(3 – 6x) + 2(x – 1) > 0 $\left[x > -\frac{2}{9}\right]$
   (c) $\frac{1}{2}(x+3) \leq \frac{1}{3}(x-5)$ $[x \leq -19]$
   (d) $\frac{1}{3}(x- \ 3) + \frac{1}{2}(3x-1) > 2$ $\left[x \geq \frac{19}{11}\right]$
2. Solve the following inequalities. Correct 2 decimal places
   (a) (0.8)-3x > 4.0 [ x > 2.07 (2dp)]
   (b) (0.6)-2x< 3.6 [x< 1.25 (2dp)]
3. Find the integers which simultaneously satisfy the following inequalities
   (a) 3x + 2 ≥ 2x – 1, 7x + 3 < 5x + 2 {-3, -2, -1}
   (b) $\frac{1}{2}(x+1) > 1$, 5x + 1 < 4(x + 2) {2, 3, 4, 5, 6}
4. Find the set of values of x for which
   (a) $\frac{3x^2-1}{2+x} > 2$ $\left[x < -1, x > \frac{5}{3}\right]$
   (b) $\frac{3x^2-1}{1+x^2} > 1$[ x < -1, x > 1]
   (c) 2($x^2$ – 5) < $x^2$ + 6 [-4 < x < 4]
   (d) $x^2$ – x – 12 > 0 [ x < -3, x > 4]
   (e) 2x(x + 3) > (x + 2)(x – 3) [ x < -6, x > -1]
5. Solve the following inequalities
   (a) $\frac{x}{x+1} \leq \frac{x-2}{x+3}$ $\left[x \leq -3 \ and -1 \leq x \leq -\frac{1}{2}\right]$
   (b) $\frac{x+2}{x-3} < \frac{x+5}{x-5}$ $[1 < x < 3 \ and \ x > 5]$
   (c) $\frac{x-1}{2+2} > 2x$ $\left[x < -2 \ and -1 < x < -\frac{1}{2}\right]$
6. Solve the following inequalities
   (a) $|x-3| < |2x+3|$ $[x < -18 \ and \ x > 0]$
   (b) $\left|\frac{2x-4}{x+1}\right| < 4$ [x < -4 and x> 0]
   (c) $\left|\frac{x-4}{x+1}\right| > 3$ $\left[-\frac{7}{2} < x < \frac{1}{4}\right]$
   (d) $\left|\frac{x}{x-3}\right| < 2$ [x < 2, and x > 6]
7. If y = $\frac{x}{x^2+4}$, find the range of possible values of y for which x is real $\left[\frac{-1}{4} \leq x \leq \frac{1}{4}\right]$
8. Find the range of values of x can take for the following inequalities to be true
   (a) $\left|\frac{2x-4}{x+1}\right| < 4$ [ x < -4, x > 0]
   (b) $|x+1| > |x-3|$ [x > 1]

# Polynomials

A polynomial in x is a function in the form

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$ where a≠0

### (a) Quadratics

If α and β are the two roots, then, x = α and x =β

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

### (b) Cubic

If α, β and λ are the three roots, then,
x = α, x =β and x = λ

⇨ $(x - \alpha)(x - \beta)(x - \lambda) = 0$

$[x^2 - (\alpha + \beta)x + \alpha\beta](x - \lambda) = 0$

$x^3 - \lambda x^2 - (\alpha + \beta)x^2 + (\alpha + \beta)\lambda x + \alpha\beta x - \alpha\beta\lambda = 0$

$x^3 - (\alpha + \beta + \lambda)x^2 + (\alpha\beta + \alpha\lambda + \beta\lambda)x - \alpha\beta\lambda = 0$

### (c) Qurtic

If α, β, λ and ρ are the four roots, then

x = α, x =β, x = λ and x = ρ

⇨ $(x - \alpha)(x - \beta)(x - \lambda)(x - \rho) = 0$

$[x^3 - (\alpha + \beta + \lambda)x^2 + (\alpha\beta + \alpha\lambda + \beta\lambda)x - \alpha\beta\lambda](x - \rho) = 0$

$X^4 - (\alpha + \beta + \lambda + \rho)x^3 + (\alpha\beta + \alpha\lambda + \alpha\rho + \beta\lambda + \beta\rho + \lambda\rho)x^2 - (\alpha\beta\lambda + \alpha\beta\rho + \alpha\lambda\rho + \beta\lambda\rho)x + \alpha\beta\lambda\rho = 0$

The above illustration shows that:

(i) The signs of the terms alternate in the order: +, - , + ….. starting with the first term.

(ii) The coefficient of the second term is – (sum of the roots) and the last term with its appropriate sign is the product of the roots

***Example 1***

Find the sums and products of the roots of the following equations

(a) $5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 6 = 0$

Solution

Dividing through by 5

$x^5 + \frac{4}{5}x^4 - \frac{3}{5}x^3 + \frac{2}{5}x^2 - \frac{1}{5}x + \frac{6}{5} = 0$

sum of roots = $-\frac{4}{5}$

product of roots = $\frac{6}{5}$

(b) $3x^4 - 5x^3 + 2x^2 + 0x + 9 = 0$

Dividing through by 3

$x^4 - \frac{5}{3}x^3 + \frac{2}{3}x^2 + 0x + 3 = 0$

Sum of roots = $\frac{5}{3}$

Product of roots = 3

## Addition and subtraction of polynomial

Polynomial are added or subtracted if they are of the same degree. This is done by adding or subtracting the coefficients of the corresponding term

***Example 2***

(a) Given the polynomial

$f(x) = 5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 6$ and

$g(x) = 3x^4 - 5x^3 + 2x^2 + 9$

Find (i) f(x) + g (x) (ii) f(x) – g(x)

**Solution**

(i) f(x) + g(x)

$$\begin{array}{r} 5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 6 \\ +\quad 3x^4 - 5x^3 + 2x^2 + 0x + 9 \\ \hline = 5x^5 + 7x^4 - 8x^3 + 2x^2 - x + 15 \end{array}$$

f(x) - g(x)

$$\begin{array}{r} 5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 6 \\ -\quad 3x^4 - 5x^3 + 2x^2 + 0x + 9 \\ \hline = 5x^5 + x^4 + 2x^3 + 0x^2 - x - 3 \end{array}$$

(b) Given polynomials

$f(x) = 2x^3 + 4x^2 - 2x - 8$ and

$g(x) = x^3 - 2x^2 + 3x + 5$

Find (i) $f(x) + g(x)$ (ii) $f(x) - g(x)$

Solution

(i) $f(x) + g(x)$
$= (2 + 1)x^3 + (4 - 2)x^2 + (-2 + 3)x + (-8 + 5)$
$= 3x^3 + 2x^2 + x - 3$

(ii) $f(x) - g(x)$
$= (2 - 1)x^3 + (4 - -2)x^2 + (-2 - 3)x + (-8 - 5)$
$= x^3 + 6x^2 - 5x - 13$

## Multiplication of polynomials

When multiplying two functions together, the terms of the first function are multiplied by the terms of the second function

***Example 3***

(a) Given the polynomial
$f(x) = 5x^3 + 2x^2 + 9$ and
$g(x) = 4x^4 - 3x^3 + 2x^2 - x + 6$
Find $f(x) \times g(x)$
Solution
$= 5x^3 (4x^4 - 3x^3 + 2x^2 - x + 6)$
$+ 2x^2 (4x^4 - 3x^3 + 2x^2 - x + 6)$
$+ 9 (4x^4 - 3x^3 + 2x^2 - x + 6)$
$= (20x^7 - 15x^6 + 10x^5 - 5x^4 + 30x^3)$
$+ (8x^6 - 6x^5 + 4x^4 - 2x^3 + 12x^2)$
$+ (36x^4 - 27x^3 + 18x^2 - 9x + 54)$
$= 20x^7 - 7x^6 + 4x^5 + 35x^4 + x^3 + 30x^2 - 9x + 54$
Hence $f(x) . g(x) =$

$20x^7 - 7x^6 + 4x^5 + 35x^4 + x^3 + 30x^2 - 9x + 54$

***Example 4***

Find coefficient of $x^3$ in the expansion

$(2x^3 + x^2 - 5x + 6)(2x + 4)$

Solution

$(2x^3 + x^2 - 5x + 6)(2x + 4)$

$= 2x(2x^3 + x^2 - 5x + 6) + 4(2x^3 + x^2 - 5x + 6)$

$= (2x^4 + 2x^3 - 10x^2 + 12x) + 8x^3 + 4x^2 - 20x + 16$

$= 2x^4 + 10x^3 - 6x^2 - 8x + 16$

The coefficient of $x^3$ is 10

## Division of polynomials

The division of polynomial may be done by long division as follows.

***Example 5***

(a) Divide $x^3 - 7x - 6$ by $(x + 1)$
Solution

$x^2 - x - 6$
$(x + 1) \overline{)x^3 - 7x - 6}$
$- \; x^3 - x^2$
$x^2 - 7x - 6$
$- \; x^2 - x$
$-6x - 6$
$- \; -6x - 6$
$0 + 0$

***Example 6***

Find the remainder when $2x^3 + x^2 + 5x - 4$ is divided by $2x - 1$

$x^2 + x + 3$
$(2x - 1) \overline{)2x^3 + x^2 + 5x - 4}$
$- \; 2x^3 - x^2$
$2x^2 + 5x - 4$
$- \; 2x^2 - x$
$6x - 4$
$- \; 6x - 3$
$-1$

Hence the remainder is -1

***Example 7***

Show that $x = -2$ is a root of the equation $2x^3 - x2 - 8x + 4 = 0$. Hence find the other roots

Solution

If x =-2 is a root of the function, then its

Remainder must be equal to zero

Hence x = -2 is a root of $2x^3 – x^2 – 8x+ 4 = 0$

x = -2 => x + 2 =0

$$
\begin{array}{r}
2x^2 - 5x + 2 \\
(x-2)\overline{)2x^3 - x^2 - 8x + 4} \\
-\ \underline{2x^3 + 4x^2} \\
-5x^2 - 8x + 4 \\
-\ \underline{5x^2 - 10x} \\
2x + 4 \\
-\ \underline{2x + 4} \\
0
\end{array}
$$

Since a cubic equation has at most three roots; the remaining two roots are obtained by solving the quadratic equation by factorization

2x2 – 5x + 2 = 0

2x2 – 4x – 2x + 2 =0

2x(x – 2) – 1(x – 2) = 0

(2x – 1)(x – 2) = 0

2x – 2 = 0 or x – 2 = 0

$x = \frac{1}{2}$ or x = 2

∴ the other roots are $x = \frac{1}{2}$ and x = 2

## Revision exercise 1 on polynomials

1. Find the degree of each of the following polynomials
   (a) $x^6 + 4x^4 – 2$ [6]
   (b) $2x^3 – x^2 – 8x+ 4$ [3]
   (c) $5x^3 + 2x^2 + 9$ [3]
   (d) $2x^4 + 10x^3 – 6x^2 – 8x + 16$ [4]
2. Given that
   (a) $f(x) = x^3 + 2x^2 – 3x + 2$ and $g(x) = 2x^3 – x^2 + 5x – 4$, find f(x) – g(x) $[-x^3 + 3x^2 – 8x + 6]$
   (b) $f(x) = x^3 + 2x^2 – 3x + 2$ and $g(x) = 2x^3 – x^2 + 5x – 4$, find g(x) – f(x) $[x^3 - 3x^2 + 8x - 6]$
   (c) $f(x) = 2x^3 – 5x^2 + 6x$ and $g(x) = x^3 – 6x^2 + 5x + 1$, find f(x) – g(x) $[x^3 + x^2 + x – 1]$
   (d) $f(x) = 2x^3 – 5x^2 + 6x$ and $g(x) = x^3 – 6x^2 + 5x + 1$, find 2f(x) + g(x) $[5x^3 – 16x^2 + 7x + 1]$
3. Given that
   (a) $f(x) = x^2 – 2x + 5$ and $g(x) = x^3 + 6x – 4$, find xf(x) + 3g(x) $[4x^3 – 2x^2 + 23x – 12]$
   (b) $f(x) = x^3 + 6x – 4$ and $g(x) = x^2 – 2x + 5$, find 3f(x) + xg(x) $[2x^3 + 2x^2 + 13x – 12]$

### Other methods of finding the remainder

Apart from using long division, the remainder when a function is divided by a certain factor can be obtained by **remainder theorem** and **synthetic approach**.

## The remainder and factor theorems

When a number say 186 is divided by 4, this can be represented simply as follow

$$
\begin{array}{r}
46 \\
4\overline{)186} \\
-\ \underline{16} \\
26 \\
-\ \underline{24} \\
2
\end{array}
$$

∴ the quotient is 46 and the remainder is 2

The above alogarithm can be written as

$\frac{186}{4} = 46 + \frac{2}{4}$

Or simply 186 = 4Q + R, where the quotient Q = 46 and the remainder, R = 2. This is referred to as the remainder theorem

The **remainder theorem** states that, when a function f(x) is divided by by (x – a) and leaves a remainder, then the remainder of the function is f(a)

From f(x) = (x –a)Q(x) + R

When we substitute for x = a; f(a) = R

When R = 0, => f(a) = 0. This is referred to as the **factor theorem**,

***Example 8***

Find the remainder when

(a) $f(x) = x^3 + 3x^2 - 4x + 2$ is divided by x – 1

Solution

Let $x^3 + 3x^2 - 4x + 2 = (x - 1)Q(x) + R$

Putting x = 1: 1 + 3 – 4 + 2 = R

2 = R

(b) $f(x) = 3x^3 + 2x - 4$ is divided by x – 2

Solution

Let $3x^3 + 2x - 4 = (x - 2)Q(x) + R$

Putting 2: 24 + 4 – 4 = R

R = 24

(c) $f(x) = 2x^3 + 4x^2 - 6x + 5$ is divided by x – 1

Let $2x^3 + 4x^2 - 6x + 5 = (x - 1)Q(x) + R$

Putting 1: 2 + 4 – 6 + 5 = R

R = 5

(d) $8x^3 + 4x + 3$ is divided by 2x -1

Let $8x^3 + 4x + 3 = (2x - 1)Q(x) + R$

Putting $\frac{1}{2}$: 1 + 2 + 3 = R

R = 6

## Synthetic approach for finding the remainder of polynomial

The synthetic approach can be illustrated as follows

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + .... + a_n$ is divided by x – a, then the following steps are taken using a table

(i) The factor in question must be linear

(ii) The 1st row of the table contains the coefficients of x in f(x) in descending order.

(iii) The at most left hand side of the 3rd row contains a value a from the division, x – a i.e. if x – a = 0, then x = a

(iv) The first digit in the second row comes under the second digit in the first row and this digit is equal to $a_0$ x a, where $a_0$ is the coefficient of $x^n$.

(v) The second digit in the 3rd row is the same as the sum of the digits in the 1st and 2nd rows

(vi) The next digit in the 2nd row is equal to $a_0$ x the 2nd digit in the 3rd row

(vii) The corresponding numbers in the 1st and 2nd row are added to give the digit in the 3rd row and the process continues

(viii) The last digit in the 3rd row is the remainder of the polynomial and the digit to the left of the remainder are coefficients of the **quotient** provided the divisor is in the form tx + b, where t = 1. If t ≠ 0, then we divide the digit to left of the remainder by t to obtain the coefficients of the quotient.

***Example 9***

Find the remainder and quotient when the function

(a) $f(x) = x^3 + 3x^2 – 4x + 2$ is divided by $x – 1$
from $f(x) = x^3 + 3x^2 – 4x + 2$, the coefficients of x in a descending order are 1, 3, -4 and 2
From $x – 1 = 0$, => $x = a = 1$

| | | | | | |
|---|---|---|---|---|---|
| $1^{st}$ row | 1 | 3 | -4 | 2 | +; a=1 |
| $2^{nd}$ row | | 1 | 4 | 0 | |
| $3^{rd}$ row | 1 | 4 | 0 | (2) | ←remainder |

The remainder is 2
Quotient $x^2 + 4x$

(b) $x^5 + x – 9$ is divided by $x + 1$

$x^5 + x – 9 \equiv x^5 + 0x^4 + 0x^3 +0x^2 + x - 9$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $1^{st}$ row | 1 | 0 | 0 | 0 | 1 | -9 | +; x = -1 |
| $2^{nd}$ row | | -1 | 1 | -1 | 1 | -2 | |
| $3^{rd}$ row | 1 | -1 | 1 | -1 | 2 | (-11) | ←remainder |

Remainder: -11

Quotient: $x^4 – x^3 + x^2 –x + 2$

(c) $4x^5 – 3x^3 + 2x + 7$is divided by $2x – 1$

$4x^5 – 3x^3 +2x+ 7 \equiv 4x^5+ 0x^4– 3x^3 +0x^2 +2x+7$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $1^{st}$ row | 4 | 0 | -3 | 0 | 2 | 7 | +; $x = \frac{1}{2}$ |
| $2^{nd}$ row | | 2 | 1 | -1 | $-\frac{1}{2}$ | $\frac{3}{4}$ | |
| $3^{rd}$ row | 4 | 2 | -2 | -1 | $\frac{3}{2}$ | $\left(\frac{31}{4}\right)$ | ←remainder |

The quotient is $\frac{1}{2}[4x^4 + 2x^3 -2x^2 – x + \frac{3}{2}] = 2x^4 + x^3 – x^2 -\frac{1}{2}x +\frac{3}{4}$ and remainder is $\frac{31}{4}$

***Example 10***

The polynomial $x^4 + px^3$ –x2 + qx – 12 has factors x + 1 and x + 2.

Find the values of p and q, hence factorize the polynomial completely.

Solution

Let $f(x) = x^4 + px^3 –x^2 + qx – 12$

By factor theorem

Putting x = -1

f(-1) = 1 – p - 1 – q – 12 =0

p + q = -12 ………………………… (i)

Putting x = -2

f(-2) = 16 – 8p – 4 – 2q – 12 = 0

4p + q = 0 …………………………(ii)

Eqn. (ii) – eqn. (i)

3p = 12

p = 4

substituting for p into eqn. (i)

4 + q = -12

q = -16

∴ $f(x) = x^4 + 4x^3 –x^2 + 16x – 12$

Now $(x + 1)(x + 2) = x^2 + 3x + 2$

Since (x + 1)(x + 2) is a factor, then $x^2 + 3x + 2$ is also a factor of f(x)

By long division to find other factors

$$
\begin{array}{r}
x^2 + x - 6 \\
x^2 + 3x + 2 \overline{) x^4 + 4x^3 - x^2 - 16x - 12} \\
-\underline{\; x^4 + 3x^3 + 2x^2 \;} \\
x^3 - 3x^2 - 16x \\
-\underline{\; x^3 + 3x^2 + 2x \;} \\
-6x^2 - 18x - 12 \\
-\underline{\; -6x^2 - 18x - 12 \;} \\
0 + 0 + 0
\end{array}
$$

$\therefore x^4 + 4x^3 - x^2 + 16x - 12$

$= (x^2 + 3x + 2)(x^2 + x - 6)$

Now $x^2 + x - 6 = (x + 3)(x - 2)$

Hence $x^4 + 4x^3 - x^2 + 16x - 12$

$= (x + 1)(x + 2)(x + 3)(x - 2)$

***Example 11***

When $f(x) = x^3 - ax + b$ is divided by x + 1, the remainder is 2 and x + 2 is a factor. Find a and b.

Solution

By substitution for x = -1 in the function

(-1)3 – a(-1) + b = 0

-1 + a + b = 2

a + b = 3 ……………… (i)

since x + 2 is a factor, substituting for x = -2 in the function gives zero

$(-2)^3 - a(-2) + b = 0$

-8 + 2a + b = 0

2a + b = 8 …………….(ii)

Eqn. (ii) – eqn. (i)

a = 5

substituting for a in eqn. (i)

5 + b = 3 => b = -2

***Example 12***

The function $f(x) = x^3 + px^2 - 5x + q$ has a factor x – 2 and has a value of 5 when x = -3. Find the values of p and q

Solution

By substitution for x = 2 in the function

$2^3 + p(2)^2 - 5(2) + q = 0$

4p + q = 2 …………(i)

By substitution for x = -3 in the function

$(-3)^3 + p(-3)^2 - 5(-3) + q = 5$

9p + q = 17 ………. (ii)

Eqn. (ii) – eqn.(i)

5p = 15

p = 3

From eqn. (i)

4 x 3 + q = 2

q = -10

Hence p = 3 and q = -10

***Example 13***

The polynomial $x^4 + px^3 - x^2 + qx - 12$ has a factor $x^2 + 3x + 2$.

Find the value of p and q and hence factorise completely

Solution

$x^2 + 3x + 2 = (x + 1)(x + 2)$

Let $f(x) = x^4 + px^3 - x^2 + qx - 12$

$f(-1) = (-1)^4 + p(-1)^3 - (-1)^2 + q(-1) - 12 = 0$

1 – p – 1 + q – 12 = 0

-p + q = 12…………….(i)

$f(-2) = (-2)^4 + p(-2)^3 - (-2)^2 + q(-2) - 12 = 0$

16 – 8p – 4 – 2q – 12 = 0

-8p – 2q = 0

4p – q = 0 …………….(ii)

Eqn. (i) + eqn. (ii)

3p = 12

p = 4

From eqn. (i)

$q = -4 - 12 = -16$

Hence $p = 4$ and $q = -16$

$\therefore x^4 + px^3 - x^2 + qx - 12 = x^4 + 4x^3 - x^2 - 16x - 12$

By long division to find other factors

$$
\begin{array}{r}
x^2 + x - 6 \\
x^2 + 3x + 2 \overline{)\, x^4 + 4x^3 - x^2 - 16x - 12} \\
-\ \underline{x^4 + 3x^3 + 2x^2} \\
x^3 - 3x^2 - 16x \\
-\ \underline{x^3 + 3x^2 + 2x} \\
-6x^2 - 18x - 12 \\
-\ \underline{-6x^2 - 18x - 12} \\
0 \quad + 0 \quad + 0
\end{array}
$$

∴

Now $x^2 + x - 6 = (x + 3)(x - 2)$

Hence $x^4 + 4x^3 - x^2 + 16x - 12$

$= (x + 1)(x + 2)(x + 3)(x - 2)$

Example 14

If $4x^3 + ax^2 + bx + 2$ is divisible by $x^2 + k^2$, show that $ab = 8$

Solution

By long division

$$
\begin{array}{r}
4x + a \\
(x^2 + k^2) \overline{)\, 4x^3 + ax^2 + bx + 2} \\
\underline{-4x^3 + 4k^2} \\
ax^2 + (b - 4k^2)x + 2 \\
-\ \underline{ax^2 + ak^2} \\
(b - 4k^2)x + 2 - ak^2
\end{array}
$$

Comparing coefficients of x:

$b - 4k^2 = 0$ ………………… (i)

$2 - ak^2 = 0$

$k^2 = \frac{2}{\mu}$……………………… (ii)

Eqn. (ii) into eqn. (i)

$b - 4\left(\frac{2}{a}\right) = 0$

⇨ $ab - 8 = 0$

$\therefore ab = 8$

## Revision exercise 2 on polynomials

1. Given that $f(x)\ 3x^3 - 4x^2 - 5x + 2$, factorize f(x) completely and hence solve the equation f(x) = 0 $\left[x = -1, 2\ or\ \frac{1}{3}\right]$
2. Prove that a – b is a factor of $a^2(b - c) + b^2(c - a) + c^2(a - b)$ and write down two other factors of the expression. Hence or otherwise factorize the expression completely
   [-(a – b)(b – c)(c – a) or (b – a) )(b – c)(c – a)]
3. The polynomial $ax^3 + bx^2 - cx - 2$ is divisible by x +2. When divided by x – 1 it leaves a remainder of 18 and when divided by x + 3, it leaves a remainder -50. Determine the values of a, b, and c. Hence factorize the polynomial completely
   [ a = 6, b = 13, c = -1; (x + 2)(2x +1)(3x-1)]
4. Factorize completely
   $f(xyz) = (x + y)^3(x - y) + (y + z)^3(y - z) + (z + x)^3(z - x)$
   [(x + y + z)(x – y)(y – z)(z – x)]
5. When the quadratic expression ax2 + bx + c is divided by x – 1, x – 2 and x + 1, the remainders are 1, 1, and 25 respectively, determine the factors of the expression [(2x – 3) and (2x – 3)]
6. The remainder when $px^3 + 2x^2 - 5x + 7$ is divided by x – 2 is equal to the remainder when the same expression is divided by x + 1. Find the value of p [p = 1]

7. Given that $x - 4$ is a factor of $2x^3 - 3x^2 - 7x + k$, where k is a constant, find the remainder when the expression is divided by $2x - 1$ [k = - 52, remainder = -56]
8. The expression $px^3 + qx^2 + 3x + 8$ leaves a remainder of -6 when divided by $x - 2$ and a remainder of -34 when divided by $x + 2$. Find the values of constants p and q. [p = 1, q = -7]
9. Show that $x - 2$ is a factor of $x^3 - 9x^2 + 26x - 24$. Find the set of values of x for which $x^3 - 9x^2 + 26x - 24 < 0$ [x < 2 and 3< x < 4]
10. The remainder when $x^3 - 2x^2 + kx + 5$ is divided by $x - 3$ is twice when the same expression is divided by $x + 1$. Find the value of the constant k [k = -2]

## Application of polynomials

### 1. Physics and Motion

**Application**: Modeling acceleration, velocity, and displacement.

**Example**: The position of an object under constant acceleration can be expressed as:

$s(t) = at^2+bt+c$

where s(t)s(t) is displacement, tt is time, and a,b,ca, b, c are constants.

### 2. Engineering and Design

**Application**: Designing curves and structures.

**Example**: The shape of a bridge arch or roller coaster track can be modeled using polynomial equations to ensure safety and aesthetics.

### 3. Economics and Business

**Application**: Forecasting trends and optimizing profits.

**Example**: A company's profit based on production volume might follow a polynomial:

$P(x) = -2x^3+15x^2-30x+100$

Solving this helps find the production level that maximizes profit.

### 4. Computer Graphics

**Application**: Rendering curves and animations.

**Example**: Bézier curves used in graphic design and animation are based on polynomial functions, enabling smooth transitions and shapes.

### 5. Signal Processing

**Application**: Filtering and analyzing signals.

**Example**: Polynomial functions help approximate and smooth out noisy data in audio and image processing.

### 6. Astronomy

**Application**: Predicting planetary motion and orbits.

**Example**: Polynomial approximations are used to model the trajectory of celestial bodies over time.

### 7. Medicine and Biology

**Application**: Modeling growth rates and drug absorption.

**Example**: Polynomial regression can estimate tumor growth or the concentration of a drug in the bloodstream over time.

**Thank you Dr. Bbosa Science**