Subsidiary Mathematics

SENIOR FIVE term 3

TOPIC 3/3: Trigonometry

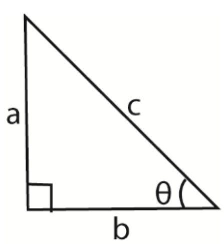
Competency: The learner analyses trigonometric concepts to solve real-world problems in various fields.

Trigonometry

The word 'trigonometry' suggests 'tri'-three, 'gono'-angle, 'metry'-measurement. As such, trigonometry is basically about triangles, most especially right-angled triangles.

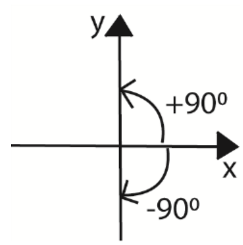
Important to note

(a) For a right angled triangle below

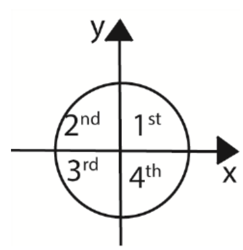


- $\sin\theta = \frac{a}{c}$ $\quad$ $\text{cosec}\theta = \frac{1}{\sin\theta} = \frac{c}{a}$
- $\cos\theta = \frac{b}{c}$ $\quad$ $\sec\theta = \frac{1}{\cos\theta} = \frac{c}{b}$
- $\tan\theta = \frac{\sin\theta}{\cos\theta} \frac{a}{b}$ $\quad$ $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{b}{a}$

(b) All positive angles are measured anticlockwise from positive x-axis

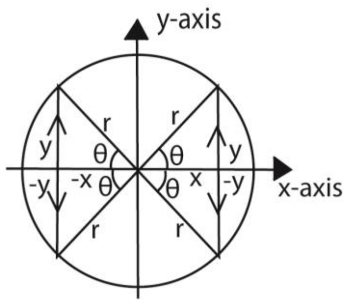


(c) A circle drawn with the centre O, divides the co-ordinate axis into four equal parts called quadrants



The quadrants are also labelled anti-clockwise from the positive x – axis.

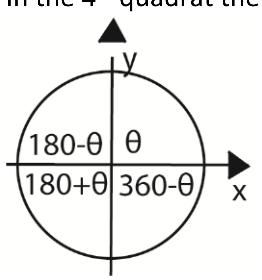
The signs the trigonometric ratios in the quadrants are given below



Ratio	Quadrant			
	1^{st}	2^{nd}	3^{rd}	4^{th}
cosθ	+	-	-	+
sinθ	+	+	-	-
tanθ	+	-	+	-
secθ	+	-	-	+
cosecθ	+	+	-	-
cotθ	+	-	+	-

Note

- If θ is the angle in the 1^{st} quadrat
- In the 2^{nd} quadrat the angle is (180 – θ)
- In the 3^{rd} quadrat the angle is (180 + θ)
- In the 4^{th} quadrat the angle is (360 –θ)



Solving equations

We make use of the quadrants to find the ranges of values within which the angle follows

Example 1

Solve the following equations for $0^0 \leq θ \leq 360^0$

(i) 3cosθ + 2 = 0

Solution

cosθ = $-\frac{2}{3}$

But cos is negative in the 2^{nd} and 3^{rd} quadrants.

Ignoring the negative sign, the angle obtained is referred to as the key or principle angle, i.e. key angle = $\cos^{-1}\frac{2}{3}$ = 48.2^0 (1d.p)

In the 2^{nd} quadrant, θ = 180 – 48.2 = 131.8^0

In the 3^{rd} quadrant, θ = 180 + 48.2 = 228.2^0

∴{θ: θ= 131.8^0, 228.2^0}

Note that: the key angle s not part of the solution but only a guide.

(ii) $4\cos^2θ - 1 = 0$

Solution

cosθ = $\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

Key angle, θ = $\cos^{-1}\frac{1}{2}$ = 60^0

When cosθ = ½ (positive is 1^{st} and 4^{th} quadrants)

1^{st} quadrant θ = 60^0

4^{th} quadrant θ = 360 – 60 = 300^0

When cosθ = -½ (positive is 2^{nd} and 3^{rd} quadrants)

3^{rd} quadrant θ = 180 – 60 = 120^0

4^{th} quadrant θ = 180 + 60 = 240^0

∴{θ: θ= 60^0, 120^0, 240^0, 300^0}

(iii) cosecθ + 2 = 0

Solution

cosecθ = -2 => sinθ = - ½ (taking reciprocal)

Key angle = $\sin^{-1}\frac{1}{2}$ = 30^0

In the 3^{rd} quadrant θ = 180 + 30 = 210^0

In the 4^{th} quadrant θ = 360 – 30 = 330^0

∴{θ: θ= 210^0, 330^0}

(iv) $3\sec^2\theta - 4 = 0$

Solution

$\sec\theta = \pm\frac{2}{\sqrt{3}}$ => $\cos\theta = \pm\frac{\sqrt{3}}{2}$

Key angle = $\cos^{-1}\frac{\sqrt{3}}{2} = 30^0$

For $\cos\theta = \frac{\sqrt{3}}{2}$; $\theta = 30^0, 330^0$

For $\cos\theta = -\frac{\sqrt{3}}{2}$; $\theta = 120^0, 210^0$

∴{θ: θ= $30^0, 120^0, 210^0, 330^0$}

(d) Definitions of angle

(i) **Acute angle** is an angle between 0^0 and 90^0. It lies in the 1^{st} quadrant

(ii) **Right angle** is an angle = 90^0

(iii) **Obtuse angle** is an angle between 90^0 and 180^0. It lies in the 2^{nd} quadrant

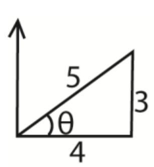
(iv) **Reflex angle** is an angle between 180^0 and 360^0. It lies in the 3^{rd} and 4^{th} quadrant

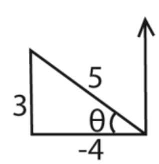
Example 2

(a) If $\sin\theta = \frac{3}{5}$ and $0^0 \leq \theta \leq 360^0$. Find the possible values of $3\tan\theta - \cot\theta$

Solution

If $\sin\theta = \frac{3}{5}$; θ lies in 1^{st} or 2^{nd} quadrants





In 1^{st} quadrant

$3\tan\theta - \cot\theta = 3\left(\frac{3}{4}\right) - \left(\frac{4}{3}\right) = \frac{11}{12}$

In 2^{nd} quadrant

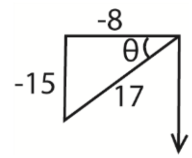
$3\tan\theta - \cot\theta = 3\left(-\frac{3}{4}\right) - \left(-\frac{4}{3}\right) = -\frac{11}{12}$

∴ the possible values are $\pm\frac{11}{12}$

(b) If $\cos\theta = -\frac{8}{17}$ and θ is reflex, find the value of $4\sec^2\theta + \tan\theta$

Solution

If $\cos\theta = -\frac{8}{17}$ and θ is reflex, θ lies in the 3^{rd} quadrant



$4\sec^2\theta + \tan\theta = 4\left(-\frac{17}{8}\right)^2 + \frac{15}{8} = \frac{319}{16}$

Example 3

Solve for θ, where $0^0 \leq \theta \leq 360^0$

(i) $3\tan^2 3\theta = 1$

Solution

$\tan 3\theta = \pm\frac{1}{\sqrt{3}}$

taking $\tan 3\theta = \frac{1}{\sqrt{3}}$

⇨ $3\theta = 30^0, 210^0, 390^0, 570^0, 750^0, 930^0$
$\theta = 10^0, 70^0, 130^0, 190^0, 250^0, 310^0$

taking $\tan 3\theta = -\frac{1}{\sqrt{3}}$

⇨ $3\theta = 150^0, 330^0, 510^0, 690^0, 870^0, 1050^0$
$\theta = 50^0, 110^0, 170^0, 230^0, 290^0, 350^0$

∴{θ: θ=$10^0, 50^0, 70^0, 110^0, 130^0, 170^0, 190^0, 230^0, 250^0, 290^0, 310^0, 350^0$}

Note

- If $0^0 \leq \theta \leq 360^0$ then $0^0 \leq 3\theta \leq 1080^0$ [multiply the interval through by 3]

(ii) $2\cos 2\theta + \sqrt{3} = 0$

Solution

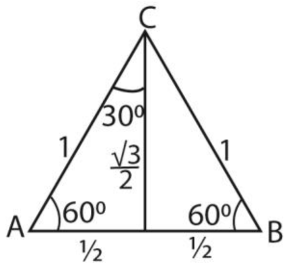
$\cos 2\theta = -\frac{\sqrt{3}}{2}$ and $0^0 \leq 2\theta \leq 720^0$

$2\theta = 150^0, 210^0, 510^0, 570^0$

∴{θ: θ = 75^0, 105^0, 255^0, 285^0}

Set square angles: 30^0, 45^0, and 60^0

(i) From equilateral triangle ABC with side equal to 1 unit



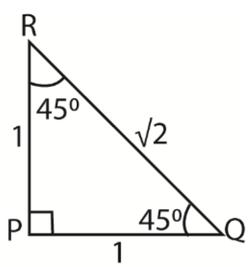
$\cos 60^0 = \sin 30^0 = \frac{1}{2}$

$\cos 30^0 = \sin 60^0 = \frac{\sqrt{3}}{2}$

$\tan 30^0 = \cot 60^0 = \frac{1}{\sqrt{3}}$

$\tan 60^0 = \cot 30^0 = \sqrt{3}$

(ii) From the right angled triangle PQR below

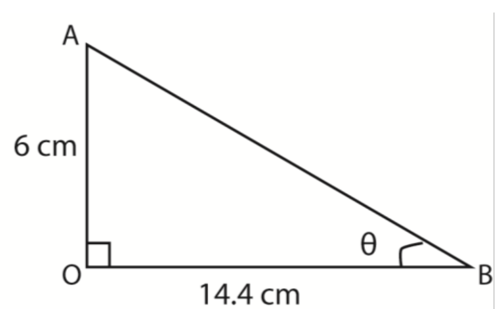


$\cos 45^0 = \sin 45^0 = \frac{1}{\sqrt{2}}$

$\tan 45^0 = 1$

Example 4

Triangle OAB is such that Angle AOB =90^0, angle ABO = θ, $\overline{OB}$ = 14.4cm and $\overline{OA}$ = 6cm.



Find sinθ + cotθ

Solution

$\tan\theta = \frac{6}{14.4} \Rightarrow \theta = \tan^{-1}\frac{6}{14.4} = 22.62^0$

sinθ + cotθ = sin 22.62 + cot 22.62

= 0.3846 + 2.4000

= 3.7846

Example 5

Without using tables or calculators find the value of

(i) $\cos 240^0$

Solution

$\cos 240^0 = -\cos(240 - 180)^0 = -\cos 60^0 = -\frac{1}{2}$

(ii) $\tan 3990^0$

Solution

$\tan 3990^0 = \tan [(360 \times 11) + 30]^0 = \tan 30^0 = \frac{1}{\sqrt{3}}$

(iii) $\sin 570^0$

Solution

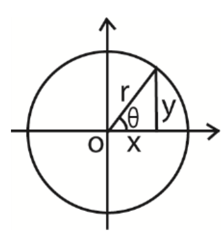
$\sin 570^0 = \sin \{(360 \times 1) + 210\}^0 = -\sin 30 = -\frac{1}{2}$

(iv) $\sec 225^0$

Solution

$\sec 225^0 = \sec (225 - 180)0 = \sec 45^0 = -\sqrt{2}$

The Pythagoras theorem



For any acute angle θ

x = rcosθ and y = rsinθ

By Pythagoras theorem

$x^2 + y^2 = r^2$

Substituting for x and y

$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$

$r^2(\cos^2\theta + \sin^2\theta) = r^2$

$\therefore \cos^2\theta + \sin^2\theta = 1$

Now $\tan\theta = \frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta}$

$\therefore \frac{\sin\theta}{\cos\theta} = \tan\theta$

Identities

$\cos^2\theta + \sin^2\theta = 1$ …………………(i)

Identity (i) ÷ $\cos^2\theta$

$1 + \tan^2\theta = \sec^2\theta$ ……………… (ii)

Identiy (i) ÷ $\sin^2\theta$

$1 + \cot^2\theta = \text{cosec}^2\theta$ …………(iii)

Example 6

Show that

(a) **$\sec^2\theta + \text{cosec}^2\theta = \sec^2\theta\,\text{cosec}^2\theta$**

Solution

$\sec^2\theta + \text{cosec}^2\theta$ $= \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta\sin^2\theta}$$

$$= \frac{1}{\cos^2\theta\sin^2\theta}$$

$$= \sec^2\theta\,\text{cosec}^2\theta$$

(b) $\frac{1-\cos^2\theta}{\sec^2-1} = \cos^2\theta$

Solution

$$\frac{1-\cos^2\theta}{\sec^2-1} = \frac{\sin^2\theta}{\tan^2\theta}$$

$$= \sin^2\theta \; x \; \frac{\cos^2\theta}{\sin^2\theta}$$

$= \cos^2\theta$ **as required**

(i) $\sin^2\theta + (1 + \cos\theta)^2 = 2(1 + \cos\theta)$

Solution

$\sin^2\theta + (1 + \cos\theta)^2$

$= \sin^2\theta + 1 + 2\cos\theta + \cos^2\theta$

$= \sin^2\theta + \cos^2\theta + 1 + 2\cos\theta$

$= 1 + 1 + 2\cos\theta$ (Recall that $\sin^2\theta + \cos^2\theta = 1$) $= 2 + 2\cos\theta = 2(1 + \cos\theta)$

$\therefore \sin^2\theta + (1 + \cos\theta)^2 = 2(1 + \cos\theta)$

(ii) $\frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\csc\theta} = \tan\theta$

Solution

$$\frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\text{cosec}\theta} = \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\frac{1}{\cos\theta}}{1+\frac{1}{\sin\theta}}$$

$$= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\frac{\cos\theta+1}{\cos\theta}}{\frac{\sin\theta+1}{\sin\theta}}$$

$$= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\cos\theta+1}{\cos\theta} \div \frac{\sin\theta+1}{\sin\theta}$$

$$= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\cos\theta+1}{\cos\theta} \, x \, \frac{\sin\theta}{\sin\theta+1}$$

$$= \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$\therefore \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\csc\theta} = \tan\theta$

(iii) $(\tan\theta + \sec\theta)^2 = \frac{1+\sin\theta}{1-\sin\theta}$

Solution

$(\tan\theta + \sec\theta)^2 = \left(\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)^2 = \left(\frac{\sin\theta+1}{\cos\theta}\right)^2$

$$= \frac{(1+\sin\theta)^2}{\cos^2\theta} = \frac{(1+\sin\theta)^2}{1-\sin^2\theta}$$

$$= \frac{(1+\sin\theta)(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} = \frac{1+\sin\theta}{1-\sin\theta}$$

$\therefore (\tan\theta + \sec\theta)^2 = \frac{1+\sin\theta}{1-\sin\theta}$

Example 7

Solve the following equations for - $180^0 \le x \le 360^0$

(a) **$\sec^2\theta - \tan\theta = 1$**

Solution

$\sec^2\theta - \tan\theta = 1$

$1 + \tan^2\theta - \tan\theta = 1$

$\tan^2\theta - \tan\theta = 0$

$\tan\theta(\tan\theta - 1) = 0$

Either $\tan\theta = 0$; $\theta = \tan^{-1} 0 = 0^0$

Or $\tan\theta = 1$, $\theta = \tan^{-1} 1 = 45^0, 225^0$

Hence $\theta = (0^0, 45^0, 225^0)$

Note that in the 1st and 3rd quadrants tanθ is is positive.

(b) **$3\sin^2\theta + \cos\theta + 1 = 0$**

Solution

$3(1 - \cos^2\theta) + \cos\theta + 1 = 0$

$3\cos^2\theta - \cos\theta - 4 = 0$

$(\cos\theta + 1)(3\cos\theta - 4) = 0$

Either $\theta = \cos^{-1} -1 = 180^0$

Or $\theta = \cos^{-1}\frac{4}{3}$ = no value

Hence = 180^0

(c) **$2\cos^2x = \sin x + 1$**

Solution

$2(1 - \sin^2x) = \sin x + 1$

$2\sin^2x - \sin x - 1 = 0$

$(\sin x + 1)(2\sin x - 1) = 0$

Either $\sin x + 1 = 0$

$\sin x = -1$; $x = 270^0$

Or

$2\sin x - 1 = 0$

$\sin x = \frac{1}{2}$; $x = 30^0, 150^0$

$x = (30^0, 150^0, 270^0)$

Note that in the 1st and 2nd quadrants sinθ is positive while in the 3rd and 4th quadrants sinθ is negative

(d) $2\cos^2\theta + \sin\theta - 1 = 0$

Solution

$2(1 - \sin^2\theta) + \sin\theta - 1 = 0$

$2\sin 2\theta - \sin\theta - 1 = 0$

$(\sin\theta - 1)(2\sin\theta + 1) = 0$

Either $\sin\theta = 1$ or $\sin\theta = -\frac{1}{2}$

When $\sin\theta = 1$; $\theta = 90^0$

When $\sin\theta = -\frac{1}{2}$; $\theta = -150^0, -30^0, 210^0, 330^0$

[θ: $\theta = -150^0, -30^0, 90^0$ for given range]

(e) $\cos\theta + \sqrt{3}\sin\theta = 1$

Solution

1st approach

$\sqrt{3}\sin\theta = 1 - \cos\theta$

Squaring both sides

$3\sin^2\theta = 1 - 2\cos\theta + \cos^2\theta$

$3(1 - \cos^2\theta) = 1 - 2\cos\theta + \cos^2\theta$

$4\cos^2\theta - 2\cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

$\cos\theta = -\frac{1}{2}$	$\cos\theta = 1$
$\theta = \pm 120^0$	$\theta = 0^0$

∴[θ: $\theta = 0^0, \pm 120^0$]

2nd approach

$\sqrt{3}\sin\theta = 1 - \cos\theta$

Dividing through by cosθ

$\sqrt{3}\tan\theta = \sec\theta - 1$

Squaring both sides

$3\tan^2\theta = \sec^2\theta - 2\sec\theta + 1$

$3\tan^2\theta = \sec^2\theta - 2\sec\theta + 1$

$3[\sec^2\theta - 1] = \sec^2\theta - 2\sec\theta + 1$

$2\sec^2\theta + 2\sec\theta - 4 = 0$

$\sec^2\theta + \sec\theta - 2 = 0$

$(\sec\theta + 2)(\sec\theta - 1) = 0$

$\sec\theta = -2$ or $\sec\theta = 1$

$\cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$

$\therefore[\theta: \theta = 0^0, \pm 120^0]$

3rd approach

$\sqrt{3}\sin\theta = 1 - \cos\theta$

Dividing through by sinθ

$\sqrt{3} = \csc\theta - \cot\theta$

Rearranging

$\sqrt{3} + \cot\theta = \csc\theta$

Squaring both sides

$3 + 2\sqrt{3}\cot\theta + \cot^2\theta = \csc^2\theta$

$3 + 2\sqrt{3}\cot\theta + \cot^2\theta = 1 + \cot^2\theta$

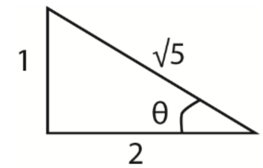
$\cot\theta = -\frac{1}{\sqrt{3}}$; => $\tan\theta = -\sqrt{3}$

$\therefore[\theta: \theta = -60^0, 120^0]$

Example 8

(a) **Given that tanθ = ½**
Evaluate $\csc^2\theta - \sec^2\theta$ without using mathematical tables or calculator.

Solution



$\csc\theta = \frac{\sqrt{5}}{1} = \sqrt{5}$; $\csc^2\theta = 5$

$\sec\theta = \frac{\sqrt{5}}{2}$; $\sec^2\theta = \frac{5}{4}$

⇨ **$\csc^2\theta - \sec^2\theta = 5 - \frac{5}{4} = \frac{15}{4} = 3\frac{3}{4}$**

(b) **Prove that** $\frac{1+\cos 2x}{2\sin 2x} = \frac{1}{2}\cot x$

Solution

$\frac{1+\cos 2x}{2\sin 2x} = \frac{1+\cos^2 x - \sin^2 x}{4\cos x\sin x}$

$= \frac{2\cos^2 x}{4\cos x\sin x}$

$= \frac{\cos x}{2\sin x}$

$= \frac{1}{2}\cot x$ **as required**

Hence solve the equation $\frac{1+\cos 2x}{2\sin 2x} = 1$ **for $0^0 \le x \le 180^0$.**

Solution

Hence $\frac{1}{2}\cot x = 1$

Cot x = 2

$x = 26.6^0$

(c) **Show that** $\frac{1-\cos^2\theta}{\sec^2 - 1} = \cos^2\theta$

$\frac{1-\cos^2\theta}{\sec^2 - 1} = \frac{\sin^2\theta}{\tan^2\theta} = \sin^2\theta \; x \frac{\cos^2\theta}{\sin^2\theta} = \cos^2\theta$ **as required**

Hence, solve the equation $\frac{1-\cos^2\theta}{\sec^2 - 1} = \frac{3}{4}$ **for $0^0 \le \theta \le 90^0$.**

$\cos^2\theta = \frac{3}{4}$

$\cos\theta = \sqrt{3/4} = \pm 0.866$

For $\cos\theta = 0.866$; $\cos^{-1}(0.866) = 30^0, 330^0$

For $\cos\theta = -0.866$; $\cos^{-1}(-0.866) = 150^0, 210^0$

Hence $\theta = 30^0$ for $0^0 \le \theta \le 90^0$

Example 9

(a) Given that 7 tanθ + cotθ = 5secθ, derive a quadratic equation for sinθ. Hence or otherwise, find all values of θ in the interval $0^0 \leq \theta \leq 180^0$ which satisfy the equation, giving your answer to the nearest 0.10 where necessary

Solution

7 tanθ + cotθ = 5secθ

$7\frac{sin\theta}{cos\theta} + \frac{cos\theta}{sin\theta} = \frac{5}{cos\theta}$

$7\frac{sin^2\theta + cos^2\theta}{cos\theta sin\theta} = \frac{5}{cos\theta}$

$7\sin^2\theta + \cos^2\theta = 5\sin\theta$

$7\sin^2\theta + (1 - \sin^2\theta) = 5\sin\theta$

$6\sin^2\theta - 5\sin\theta + 1 = 0$

$(3\sin\theta - 1)(2\sin\theta - 1) = 0$

$\sin\theta = \frac{1}{3}$	$\sin\theta = \frac{1}{2}$
$\theta = 19.5^0, 160.5^0$	$\theta = 30^0, 150^0$

∴[θ: θ= 19.5^0, 30^0, 150^0, 160.5^0]

Example 10

Find the solution of 3cotθ + cosecθ = 2 for $0^0 \leq \theta \leq 180^0$.

Solution

3cotθ + cosecθ = 2

$3\frac{cos\theta}{sin\theta} + \frac{1}{sin\theta} = 2$

$(3cos\theta + 1)^2 = (2sin\theta)^2$

$9cos^2\theta + 6cos\theta + 1 = 4sin^2\theta$

$9cos^2\theta + 6cos\theta + 1 = 4(1 - cos^2\theta)$

$13cos^2\theta + 6\cos\theta - 3 = 0$

$cos\theta = \frac{-6 \pm \sqrt{6^2 + 4x3x13}}{2x13}$

cosθ = 0.3021	cosθ = 0.7637
θ = 72.40	θ = 40.2

∴[θ: θ = 72.4^0, 40.2^0]

Elimination of trigonometric parameter

This involves the use of identities to eliminate the trigonometric values in equation

Example 11

(a) If x = tanθ + secθ and y = tanθ – secθ; show that xy + 1 = 0

Solution

x + y = tanθ

x – y = 2secθ

secθ = $\frac{1}{2}(x - y)$

Using identity: $1 + \tan^2\theta = \sec^2\theta$

$1 + (x + y)\ 2 = \left[\frac{1}{2}(x - y)\right]^2$

$4 + x^2 + 2xy + y^2 = x^2 - 2xy + y^2$

4xy + 4 = 0

xy + 1 = 0 as required

(b) x = 2 + 3sinθ and y = 3 + 2cosθ show that $4(x - 2)^2 + (y - 3)^2 = 36$

Solution

x = 2 + 3sinθ => sinθ = $\frac{x-2}{3}$

y = 3 + 2cosθ => cosθ = $\frac{y-3}{2}$

Using identity $\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{x-2}{3}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$

$4(x - 2)^2 + (y - 3)^2 = 36$ as reqyured

(c) x = 2sinθ and y = tanθ, prove that

$x = \pm\frac{2y}{\sqrt{(1 + y^2)}}$

Solution

x = 2sinθ; => cosecθ = $\frac{2}{x}$

y = tanθ; => cotθ = $\frac{1}{y}$

Using identity: $1 + \cot^2\theta = \text{cosec}^2\theta$

$1 + \left(\frac{1}{y}\right)^2 = \left(\frac{2}{x}\right)^2$

$x = \pm\frac{2y}{\sqrt{(1 + y^2)}}$

Revision exercise 1

1. Solve for θ, where $\theta^0 \le \theta \le 360^0$
(a) secθcosecθ + 2secθ – 2cosecθ – 4 = 0 [θ: θ = 60^0, 210^0, 300^0, 330^0]
(b) $\tan^2\theta - (\sqrt{3}+1)\tan\theta + \sqrt{3} = 0$
[θ: θ = 45^0, 60^0, 225^0, 240^0]
2. Show that
(a) $\frac{1-cos\theta+sin\theta}{1-cos\theta} = \frac{1+cos\theta+sin\theta}{sin\theta}$
(b) tanθ + cotθ = secθcosecθ
(c) $\cos^4\theta - \sin^4\theta + 1 = 2\cos^2\theta$
(d) $\frac{sin\theta}{1+cos\theta} + \frac{1+cos\theta}{sin\theta} = 2cosec\theta$
(e) $\sqrt{\frac{1+sin\theta}{1-sin\theta}}$ = secθ + tanθ
3. Solve the following equations for $-180^0 \le x \le 180^0$
(i) $2\cos^2\theta$ + sinθ – 1 = 0 [θ: θ = -150^0, -30^0, 90^0]
(ii) sin2θ + 5cos2θ = 3
[θ: θ = $\pm 45^0$, $\pm 135^0$]
(iii) $4\cot^2\theta$ + 24cosecθ + 39 = 0
[θ: θ = 16.6^0, 23.6^0, 156.4^0, 163.4^0]

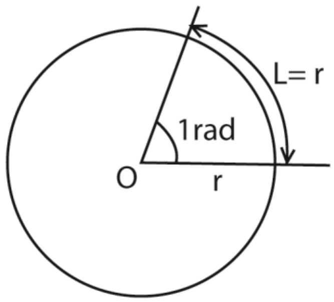
4. Solve each of the following equations in the stated range
(a) $4\cos^2\theta$ + 2sin θ = 4 $0^0 \le \theta \le 360^0$
[θ: θ = 0^0, 48.6^0, 131.4^0, 180^0, 360^0]
(b) $2\sec^2\theta$ – 4tanθ – 2 = $-180^0 \le \theta \le 360^0$
[θ: θ = -135^0, -161.6^0, 18.4^0, 45^0]
(c) $5\cos^2 3\theta$ = 3(1 + sin3θ), $0^0 \le \theta \le 360^0$
[θ: θ = 7.9^0, 52.1^0, 90^0, 127.9^0, 172.1^0]
5. Solve for θ; 00 ≤ θ ≤ 3600
(a) tanθ + 3cotθ = 4
[θ: θ = 45^0, 71.6^0, 225^0, 251.6^0]
(b) 4cosθ – 3sinθ = 2
[θ: θ = 29.50, 256.70]
6. Solve
(a) cosθ + √3sinθ = 2 0 ≤ θ ≤ π
$\left[\theta = \frac{\pi}{3}\right]$
(b) 2cosθ – cosecθ = 0 $0^0 \le \theta \le 270^0$
[θ: θ = 45^0, 225^0]
(c) $2\sin^2\theta$ + 3cosθ = 0 $0^0 \le \theta \le 360^0$
[θ: θ = 240^0, 120^0]
(d) 3sinθ + 4cosθ = 2 $-180^0 \le \theta \le 180^0$
[θ: θ = -29.55^0, 103.29^0]
(e) $3\tan^2\theta + 2\sec^2\theta$ = 2(5 – 3tanθ) for $0^0 < \theta < 180^0$
[θ: θ = 38.66^0, 116.57^0]

7. Without using a tables or calculator, show that tan 15^0 = 2 - √3
8. Solve equation
$8\cos^4\theta - 10\cos^2\theta + 3$ for $0^0 \le \theta \le 180^0$
[θ: θ = 30^0, 45^0, 135^0, 150^0]
9. Eliminate θ from the following equation
(a) x = asecθ and y = b + ccosθ [ac = x(y – b)]
(b) x = secθ +tanθ and y = secθ – tanθ
[xy = 1]
10. Solve the simultaneous equation
Cos x + 4siny = 1
4secx – 3cosecy = 5 for values of x and y between 0^0 and 360^0
[x = 78.8^0, 281.5^0; y = 11.5^0, 168.5^0]

11. Prove each of the following identities
(a) Sinxtanx + cosx =secx
(b) Cosecx + tanxsecx = $cosecx\sec^2x$
(c) Cosecx – sinx = cotxcosx
(d) $(\sin x + \cos x)^2 - 1$ = 2sinxcosx
12. Eliminate θ from each of the following pairs of relationships
(a) x =3sinθ, y = cosecθ [xy = 3]
(b) 5x = sinθ, y = 2cosθ [$100x^2 + y^2 - 4 = 0$]
(c) x = 3 + sinθ, y = cosθ [$(x-3)^2 + y^2 = 1$]
(d) x = 2 + sinθ, cosθ = 1+y [$(x-2)^2 + (y+1)^2 = 1$]

Measuring angles in radians

A radian is defined as an angle subtended at the centre of a circle by an arc that is equal to the radius of the circle. One radian is represented by π, where $\pi = \frac{22}{7}$



How to convert between degrees and radians

1 revolution = circumference of a circle

But circumference of a circle subtends an angle 2π at the centre.

⇨ 1 revolution = 2π = 360^0
π = 180^0
$1^0 = \frac{\pi}{180}$ radians
$x^0 = \frac{\pi}{180}x$ radians

Example 12

Convert the following angles to radians

(a) 330^0
(b) 90^0
(c) 30^0

Solution

(a) $330^0 = \frac{\pi}{180}x\ 330 = \frac{11\pi}{6}$ radians
(b) $90^0 = \frac{\pi}{180}x\ 90 = \frac{\pi}{2}$ radians
(c) $30^0 = \frac{\pi}{180}x\ 30 = \frac{\pi}{6}$ radians

Converting radians to degrees

2π radians = 360^0

1 radian = $\frac{180^0}{\pi}$

x radians $= \frac{180^0}{\pi}x$

Example 13

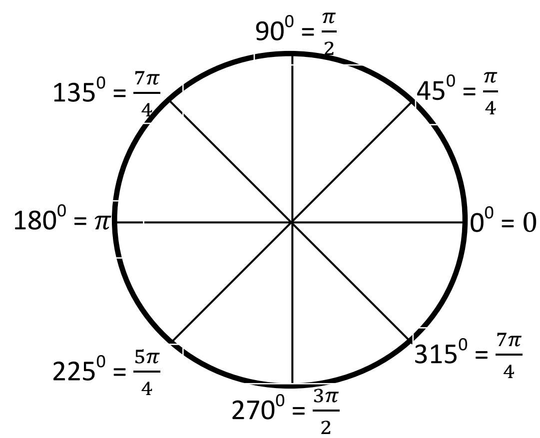
Convert each of the following radians to degrees

(i) $\frac{\pi}{3}$ radians
(ii) $\frac{2\pi}{5}$ radians
(iii) π radians

Solution

(i) $\frac{\pi}{3}$ radians = $\frac{180^0}{\pi}\ x\frac{\pi}{3}$ = 60^0
(ii) $\frac{2\pi}{5}$ radians= $\frac{180^0}{\pi}\ x\frac{2\pi}{5}$ = 72^0
(iii) π radians = $\frac{180^0}{\pi}\ x\ \pi$ = 180^0

Some equivalent angles in degrees and radians



Example 14

Find each of the following values

(a) $\sin\left(\frac{2\pi}{3}\right)$
(b) $\cos\left(\frac{4\pi}{3}\right)$
(c) $\tan\left(\frac{7\pi}{4}\right)$

Solution

Convert the angles from radian to degrees

(a) $\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{2\ x\ 180}{3}\right) = \sin 120^0 = \frac{\sqrt{3}}{2}$
(b) $\cos\left(\frac{4\pi}{3}\right) = \cos\left(\frac{4\ x\ 180}{3}\right) = \cos 240^0 = -\frac{1}{2}$
(d) $tan\left(\frac{7\pi}{4}\right) = \tan\left(\frac{7x\ 180}{4}\right) = \tan 60^0 = \sqrt{3}$

Compound angles

1. cos (α + β) = cosαcosβ – sinαsinβ
   ⇨ $\cos 2\theta = \cos^2\theta - \sin^2\theta$
2. cos (α – β) = cosαcosβ + sinαsinβ
   ⇨ $\cos 2\theta = \cos^2\theta + \sin^2\theta$
3. sin (α + β) = sinαcosβ + sinβcosα
   ⇨ sin2θ = 2cosθsinθ
4. sin (α - β) = sinαcosβ – sinβcosα
5. tan (α + β) = $\frac{tan\alpha+tan\beta}{1-tan\alpha tan\beta}$
   ⇨ $tan2\theta = \frac{2tan\theta}{1-tan^2\theta}$
6. tan (α - β) = $\frac{tan\alpha-tan\beta}{1+tan\alpha tan\beta}$

Example 15

Solve the equation cosθ = sin2θ for values of θ from $0^0$to 360^0. (05 marks)

cosθ = 2cosθsinθ

cos(1-2sinθ) = 0

Either cosθ = 0 or θ= 90^0, 270^0

Or

1-2sinθ = 0

sinθ = ½ Or θ = 30^0, 150^0

hence θ = 30^0, 90^0, 150^0, 270^0

Example 16

Show that

cosec2θ + cot2θ = cotθ

Solution

$$cosec2\theta + cot2\theta = \frac{1}{sin2\theta} + \frac{cos2\theta}{sin2\theta}$$
$$= \frac{1+cos2\theta}{sin2\theta}$$
$$= \frac{1+2cos^2\theta-1}{2sin\theta cos\theta}$$
$$= \frac{2cos^2\theta}{2sin\theta cos\theta}$$
$$= cot\theta$$

Example 17

If tanα = $\frac{3}{4}$ and α is acute, without using tables or calculator work out the value of

(a) tan2α

$$tan2\alpha = \frac{2tan\alpha}{1-tan^2\alpha}$$
$$= \frac{2x\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}$$
$$= \frac{\frac{2}{2}}{\frac{3}{4}}$$
$$= \frac{24}{7}$$

(b) tan$\frac{\alpha}{2}$

$$tan\alpha = \frac{2tan\frac{\alpha}{2}}{1-tan^2\frac{\alpha}{2}} = \frac{3}{4}$$

⇨ $3tan^2\frac{\alpha}{2} + 8tan\frac{\alpha}{2} - 3 = 0$

$(3tan\frac{\alpha}{2} - 1)(tan\frac{\alpha}{2} + 3)$ =0

$tan\frac{\alpha}{2} = \frac{1}{2}$ $or\ tan\frac{\alpha}{2} = -3$

Since α is acute, tanα cannot be negative

$\therefore tan\frac{\alpha}{2} = \frac{1}{3}$

Example 18

Prove that cot 2θ = $\frac{cot^2\theta-1}{2cot\theta}$. Hence solve the equation cot2θ + 2cotθ = 2 for $0^0 \le \theta \le 360^0$

Solution

cot 2θ = $\frac{cos2\theta}{sin2\theta} = \frac{cos^2\theta-sin^2\theta}{2sin\theta cos\theta}$

dividing through by $\sin^2\theta$

cot 2θ = $\frac{cot^2\theta-1}{2cot\theta}$

Hence, cot2θ + 2cotθ =0

$$\frac{cot^2\theta-1}{2cot\theta} + 2cot\theta = 0$$

$5\cot^2\theta$ -4cotθ – 1 = 0

(5cotθ + 1)(cotθ -1) = 0

cotθ = $-\frac{1}{5}$ or cotθ = 0

⇨ tanθ = -5 ot tanθ = 1

When tanθ= -5; θ= 101.3^0, 281.3^0

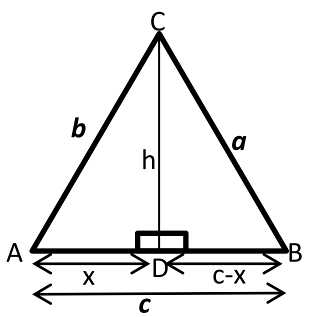
When tanθ = 1, θ = 45^0, 225^0

∴{θ: θ= 45^0, 101.3^0, 225^0, 281.3^0}

Triangles

1. The cosine rule

(a) Given an acute angle A



From triangle

ACD: $x^2 + h^2 = \boldsymbol{b}^2$ ....(i)

BCD: $(\boldsymbol{c} - x)^2 + h^2 = \boldsymbol{a}^2$

$\boldsymbol{c}^2 - 2\boldsymbol{c}x + x^2 + h^2 = \boldsymbol{a}^2$ .... (ii)

Substituting eqn. (i) into eqn. (ii)

$\boldsymbol{c}^2 - 2\boldsymbol{c}x + \boldsymbol{b}^2 = \boldsymbol{a}^2$

But

x = $\boldsymbol{b}$cosA

⇨ $\boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A = a^2$

$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A$ (1)

Similarly;

$\boldsymbol{b}^2 = \boldsymbol{a}^2 + \boldsymbol{c}^2 - 2\boldsymbol{ac}\cos B$ (2)

$\boldsymbol{c}^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 - 2\boldsymbol{ab}\cos C$ (3)

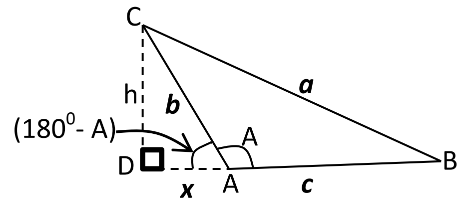
It follows that

$$cosA = \frac{b^2+c^2-a^2}{2bc}$$

$$cosB = \frac{a^2+c^2-b^2}{2ac}$$

$$cosC = \frac{a^2+b^2-c^2}{2ab}$$

(b) Given an obtuse angle A



In triangle ABC, A is an obtuse angle and CD is the altitude.

From triangle

ACD: $x^2 + h^2 = \boldsymbol{b}^2$ ....(i)

BCD: $(\boldsymbol{c} - x)^2 + h^2 = \boldsymbol{a}^2$

$\boldsymbol{c}^2 - 2\boldsymbol{c}x + x^2 + h^2 = \boldsymbol{a}^2$ .... (ii)

Substituting eqn. (i) into eqn. (ii)

$\boldsymbol{c}^2 - 2\boldsymbol{c}x + \boldsymbol{b}^2 = \boldsymbol{a}^2$

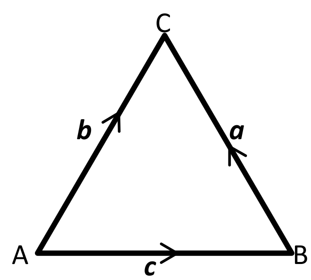
But

x = $\boldsymbol{b}\cos(180^0 - A) = -\boldsymbol{bc}\cos A$

From triangle ACD

$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A$ as before

The cosine rule can be derived using the vector approach.



Given a triangle above with BC = $\boldsymbol{a}$, AC = $\boldsymbol{c}$ and AB = $\boldsymbol{b}$

BC = BA + AC = AC – AB

$\boldsymbol{a} = \boldsymbol{b} - \boldsymbol{c}$

⇨ $\boldsymbol{a}.\boldsymbol{a} = (\boldsymbol{b} - \boldsymbol{c})(\boldsymbol{b} - \boldsymbol{c})$

$= \boldsymbol{b}.\boldsymbol{b} - 2\boldsymbol{b}.\boldsymbol{c} + \boldsymbol{c}.\boldsymbol{c}$

$= \boldsymbol{b}.\boldsymbol{b} + \boldsymbol{c}.\boldsymbol{c} - 2\boldsymbol{b}.\boldsymbol{c}$

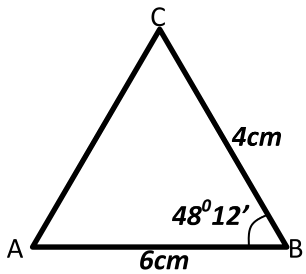
$\therefore \boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A$

since $\boldsymbol{b}.\boldsymbol{c} = |bc|\cos A$

Example 19

Solve the triangle in which AB = 6cm, BC = 4cm and angle ACB = $48^0 12'$

Solution



Using: $b^2 = a^2 + c^2 - 2ac\cos B$

$= 6^2 + 4^2 - 2(6)(4)\cos 48.2^0$

1^0 (degree) = 60' (minutes)

b = 4.47cm

Using: $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{20.0+36-16}{2(4.47)(6)}$

A $=41.8^0$

But A + B + C = 180^0

$41.8^0 + 48.2^0 + C = 180^0$

C = 90^0

∴ AC = 4.47cm, angles BAC = 41.8^0 and ACB = 90^0

Example 20

In a triangle ABC, prove that

(a) $a^2 = (b-c)^2 + 4bc\sin^2\left(\frac{A}{2}\right)$ hence that a = (b – c)secα where $\tan\alpha = \frac{\sqrt{bc}\sin\left(\frac{A}{2}\right)}{b-c}$

From $\cos A = 1 - 2\sin^2\left(\frac{A}{2}\right)$

Substituting for cosA into the cosine formula $a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = b^2 + c^2 - 2bc[1 - 2\sin^2\left(\frac{A}{2}\right)]$

$a^2 = b^2 + c^2 - 2bc + 4\sin^2\left(\frac{A}{2}\right)$

$a^2 = (b-c)^2 + 4bc\sin^2\left(\frac{A}{2}\right)$

Hence, substituting for $\sin^2\left(\frac{A}{2}\right)$ into tanα expression we get

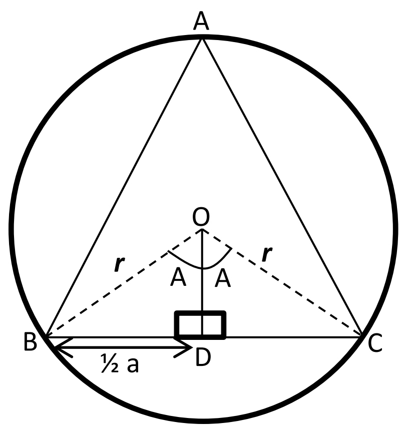
$a^2 = (b-c)^2 + (b-c)^2\tan^2\alpha$

$a^2 = (b-c)^2(1 + \tan^2\alpha)$

$a^2 = (b-c)^2\sec^2\alpha$

$a = (b-c)\sec\alpha$

2. The Sine Rule



The figure shows a circle with centre O and radius r circumscribing triangle ABC

Angle BOC = 2A [angle subtended by the same arc at the centre of the circle is twice the angle formed at any point on the circumference]

Triangle BOC is isosceles

OD bisects angle BOC and side BC

∴BD = ½a

From triangle BOD

$\sin A = \frac{a}{2r}$ i.e. $\frac{a}{\sin A} = 2r$

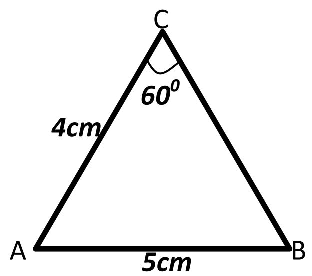
if instead we consider triangles AOC and AOB, we obtain $\frac{b}{\sin B} = 2r$ and $\frac{c}{\sin C} = 2r$

In general: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Example 21

Solve the triangle in which AB = 5cm, AC = 4cm and angle ACB = 60^0

Solution



Using sine rule

$\frac{b}{sinB} = \frac{c}{sinC}$ => B = $\sin^{-1}\left(\frac{bsinC}{c}\right)$

B = $\sin^{-1}\left(\frac{4}{5}sin60^0\right)$ = 43.9^0

From A + B + C+ = 180^0

A = $(180 - 60 - 43.9)^0$ = 76.1^0

Similarly a = $\frac{bsinA}{sinB} = \frac{4sin76.1^0}{sin43.9^0}$ = 5.6cm

$\therefore \overline{AB} = 5.6cm, B\hat{A}C = 76.1^0, A\hat{B}C = 43.9^0$

Example 23

Prove that in any triangle

$\frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$

Solution

From sine rule formula;
a = 2rsinA, b = 2rsinB, c = 2rsinC

By substitution

$\frac{a^2-b^2}{c^2} = \frac{(2rsinA)^2-(2rsinB)^2}{(2rsinC)^2} = \frac{sin^2A-sin^2B}{sin^2C}$

But A + B + C = 180^0

C = 180^0 – (A + B)

sinC = sin[180^0 – (A + B)] = sin (A + B)

By substitution

$\frac{a^2-b^2}{c^2} = \frac{sin^2A-sin^2B}{\sin(A+B)} = = \frac{(sinA+\sin B)(sinA-sinB}{\sin(A+B)}$

$= \frac{2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B).2cos\frac{1}{2}(A+B)sin\frac{1}{2}(A-B)}{\sin(A+B)}$

$= \frac{2sin\frac{1}{2}(A-B)cos\frac{1}{2}(A-B)}{} = \frac{\sin(A-B)}{\sin(A+B)}$

Hence $\frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$

Example 24

Prove that in any triangle ABC,

$sin\frac{1}{2}(B-C) = \frac{b-c}{a}cos\frac{1}{2}A$

Solution

From sine rule formula;
a = 2rsinA, b = 2rsinB, c = 2rsinC

By substitution

$\frac{b-c}{a} = \frac{2rsinB-2rsinC}{2rsinA} = \frac{SinB-sinC}{sinA}$

But A + B + C = 180^0

A = 180^0 – (B + C)

sinA = sin[180^0 – (B + C)] = sin(B + C)

By substitution

$\frac{b-c}{a} = \frac{SinB-sinC}{sinA} = \frac{SinB-sinC}{\sin(B+C)}$

$= \frac{2cos\frac{1}{2}(B+C)sin\frac{1}{2}(B+C)}{2cos\frac{1}{2}(B+C)cos\frac{1}{2}(B+C)}$

$= \frac{sin\frac{1}{2}(B+C)}{cos\frac{1}{2}(B+C)}$

From A + B + C = 180^0

B + C = 180^0 – A

$\frac{1}{2}(B+C) = \left(90^0 - \frac{1}{2}A\right)$

$sin\frac{1}{2}(B+C) = sin\left(90^0 - \frac{1}{2}A\right) = cos\frac{1}{2}A$

By substitution

$\frac{b-c}{a} = \frac{sin\frac{1}{2}(B+C)}{cos\frac{1}{2}A}$

$\therefore sin\frac{1}{2}(B-C) = \frac{b-c}{a}cos\frac{1}{2}A$

Thank You

Dr. Bbosa Science