Subsidiary Mathematics

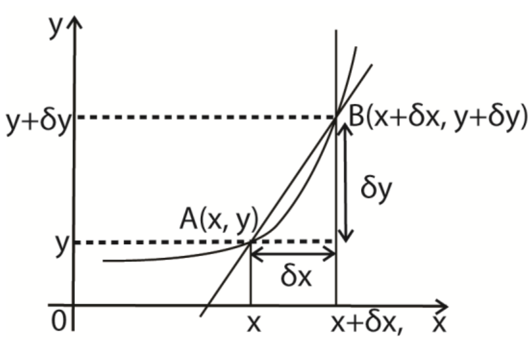
SENIOR Six term 1

TOPIC 2/2: Differentiation

Competency: The learner applies differentiation techniques in optimisation and analysing rates of change to interpret their significance in real life.

Sub math - Differentiation

Consider point A(x, y) lying on a curve drawn below, if another point B(x + δx, y + δy) lies in the same curve, where δx and δy are small increments in x and y respectively, the straight line AB, drawn through the curve is called a chord of the curve.



As the distance δx becomes smaller and smaller, point B moves close to A and the chord AB approaches the position of the target at A

Now, Gradient, $M_{AB} = \frac{(y+\delta y-y}{x+\delta x-x}$

$$M_{AB} = \frac{\delta y}{\delta x}$$

As δx tends to zero, i.e.δx →0.

$\frac{\delta y}{\delta x}$ approaches the value of the gradient of the target line at A. This value is called limiting value of $\frac{\delta y}{\delta x}$ and is written as $\lim_{\delta x\to 0}\frac{\delta y}{\delta x}$.

The limiting value of $\frac{\delta y}{\delta x}$ is called a differential coefficient or first derivative of y with respect to x which is denoted by $\frac{dy}{dx}$.

Note: the process of finding this limiting value is called differentiation.

Direct differentiation of explicit functions

Explicit functions are functions where one variable is expressed in terms of the other variable. Examples $y = x^2$, $y = x^4 + 2x$ etc.

Given the function $x = x^n$, the derivative of y with respect to x, denoted by either y' or $\frac{dy}{dx}$ is given by $y' = \frac{dy}{dx} = nx^{n-1}$.

This result applies for all rational values of n. This means that multiply the term given by the give power index and then reduce the power by one.

Example 1

Find the derivatives of the following with respect to x

(a) $y = x^3$
solution
$\frac{dy}{dx} = 3x^{3-2} = 3x^2$

(b) $y = 2x^2 + 3$
Solution
$y = 2x^2 + 3x^0$

$\frac{dy}{dx} = \frac{d}{dx}(2x^2) + \frac{d}{dx}(3x^0)$
$= 2(\,2x^{2-1}) + 0(3x^{0-1})$
$= 4x + 0 = 4x$

(c) $y = \frac{1}{x}$
Solution
$y = x^{-1}$
$\frac{dy}{dx} = -1x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$

(d) $y = \sqrt{x}$
Solution
$y = x^{\frac{1}{2}}$
$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

(e) $y = \frac{-2}{x}$
Solution
$y = -2x^{-1}$
$\frac{dy}{dx} = -2(-1x^{-1-1}) = 2x^{-2} = \frac{2}{x^2}$

(f) $y = x^4 + 3x^2 + 2$
Solution
$y = x^4 + 3x^2 + 2x^0$
$\frac{dy}{dx} = 4x^{4-1} + 2(3x^{2-1}) + 0(2x^{0-1})$
$= 4x^3 + 6x + 0$
$=4x^3 + 6x$

(g) $y = \frac{3}{\sqrt{x}} - 2\sqrt{x}$

Solution
$y \;=\; 3x^{-\frac{1}{2}} - 2x^{\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{1}{2}\left(3x^{-\frac{1}{2}-1} - \frac{1}{2}\left(2x^{\frac{1}{2}-1}\right)\right)$

$-\frac{3}{2}x^{\frac{-3}{2}} - x^{-\frac{1}{2}} = -\frac{3}{2x^{\frac{2}{3}}} + \frac{1}{x^{\frac{1}{2}}}$

(h) $y = x4(x + 1)$
solution
$y= x^5 + x^4$
$\frac{dy}{dx} = 5x^{5-1} + 4x^{4-1} = 5x^4 + 4x^3$

(i) $y =6\sqrt{x}(x^2 - 2x)$
Solution
$y = 6x^{\frac{5}{2}} - 12x^{\frac{3}{2}}$
$\frac{dy}{dx} = \frac{5}{2}\left(6x^{\frac{5}{2}-1}\right) - \frac{3}{2}\left(12x^{\frac{3}{2}-1}\right)$
$15x^{\frac{3}{2}} - 18x^{\frac{1}{2}}$

Revision exercise 1

(Answers are given in square brackets)

Find the derivatives of the following with respect to x

(a) $y = 3x^2$ [6x]
(b) $y = 2x^4 + 2$ [$8x^3$]
(c) y = b [0]
(d) $y = \frac{9}{2x^3}$ $\left[-\frac{27}{2x^4}\right]$
(e) $y = 2x^{-2}$ [$-4x^{-3}$]
(f) $y = \frac{-3}{4x^4}$ $\left[\frac{3}{x^5}\right]$
(g) $y = \sqrt[1]{x}$ $\left[\frac{1}{4x^{\frac{3}{4}}}\right]$
(h) $y = \frac{4}{5\sqrt{x}}$ $\left[\frac{2}{5x^{\frac{3}{2}}}\right]$
(i) $y = \frac{-6}{\sqrt[3]{x}}$ $\left[\frac{2}{x^{\frac{4}{3}}}\right]$
(j) $6\sqrt{x}(x^3 - 2x + 1)$
$\left[21x^{\frac{5}{2}} - 18x^{\frac{1}{2}} + \frac{3}{x^{\frac{1}{2}}}\right]$

Second derivatives

Suppose y has been given as a function of x, then the first derivative of y with respect to x is denoted by $\frac{dy}{dx}$.

The result of differentiating $\frac{dy}{dx}$ with respect to x is the second derivatives, $\frac{d^2y}{dx^2}$

Example 2

Find $\frac{d^2y}{dx^2}$ or f''(x) for each of the following

(a) $y = x^2$

Solution

$\frac{dy}{dx} = 2x$

$\frac{d^2y}{dx^2} = 2$

(b) $y = x^2(1 + x)$

Solution

$y = x^2(1 + x)$

$= x^2 + x^3$

$\frac{dy}{dx} = 2x + 3x^2$

$\frac{d^2y}{dx^2} = 2 + 6x$

Revision exercise 2

(Answers are given in square brackets)

Find $\frac{d^2y}{dx^2}$ or f''(x) for each of the following

(a) $y = x^3(4 - x^2)$ [$24x - 20x^2$]

(b) $y = 2(x - 3)^2$ [4]

(c) $y = 2x^2(x - 1)^2$ [$24x^2 - 24x + 4$]

(d) $f(x) = \frac{3x^3+5}{x^2}$ $\left[\frac{30}{x^4}\right]$

(e) $f(x) = \frac{(2x-5)(x-4)}{x^3}$ $\left(\frac{4}{x^3} + \frac{78}{x^4} + \frac{240}{x^5}\right)$

Differentiation of Natural Log, Inx

$$\frac{d}{dx}(Inx) = \frac{1}{x}$$

Application of differentiations

A Finding the gradient or a slope to the curve at a point

The gradient to the curve is given by $\frac{dy}{dx}$

Example 3

(a) Find the gradient of the curve $y = 4x^2(3x + 2)$ at the point (1, 20)

Solution

$y = 4x^2(3x + 2)$

$= 12x^3 + 8x^2$

$\frac{dy}{dx} = 36x^2 + 16x$

At x = 1

$\frac{dy}{dx} = 36 + 16 = 52$

Hence gradient = 52

(b) Determine the equation of the tangent to the curve $y = 2x^3 + 3x$ at a point x = 2.

Solution

Gradient = $\frac{dy}{dx}(2x^3 + 3x)$

$= 6x^2 + 3$

Substitution for x = 2

Gradient = $6 \times 2^2 + 3$

$= 6 \times 4 + 3$

$= 24 + 3$

$= 27$

Revision exercise 3

Find the gradients of the following curves at the given points.

1. $y = (x + 4)^2$ at x = 0 [8]
2. $y = \frac{3x^3+5}{x^2}$ at x = 1 [-7]
3. $y = \frac{(3x-1)^2}{2x}$ at x = 2 $\left[\frac{35}{8}\right]$
4. y =
5. $y = 5 + 4x - 2x^2$ at x = 2 [0]

B Determining the turning points of a curve and curve sketching

Turning point of the curve occur when

$\frac{dy}{dx}$ or f'(x) = 0

The turning point is either

- Maxima/ maximum when $\frac{d^2y}{dx^2}$ or f"(x) < 0
- Minima/ minimum when $\frac{d^2y}{dx^2}$ or f"(x) > 0
- Point of inflexion when $\frac{d^2y}{dx^2}$ or f"(x) = 0

Example 4

The equation of a curve is $y = 3 + 2x - x^2$.

(a) Determine the;

(i) coordinates and nature of the turning points of the curve.

Turning points occur when $\frac{dy}{dx} = 0$

$\frac{dy}{dx}(3 + 2x - x^2) = 2 - 2x$

At turning point 2 – 2x =0

x = 1

when x = 1, y = 3 + 2 -1 = 4

turning point is (1, 4)

Nature of turning point

$\frac{d^2y}{dx^2} = -2$,

since $\frac{d^2y}{dx^2} < 0$ the turning point is a maxima

(ii) y – and x – intercept of the curve

y intercept when x = 0, i.e. y = 3 or (0, 3)

x intercept when y = 0

$3 + 2x - x^2 = 0$

Or

$x^2 - 2x - 3 = 0$

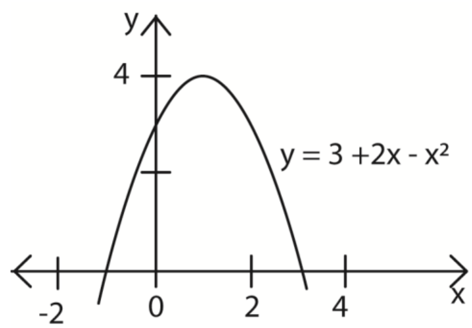
(x – 3)(x + 1) = 0

Either x – 3 = 0 and x = 3

Or x +1 = 0 and x = -1

Hence x intercepts are (-1, 0)and (3 , 0)

(b) (i) sketch the curve (02marks)



(ii) find the area enclosed by the curve and the x – axis.

Area = $\int_{-1}^{3}(3 + 2x - x^3)$

= $\left[3x + x^2 - \frac{x^3}{3}\right]_{-1}^{3}$

= $(9 + 9 - 9) - (-3 + 1 + \frac{1}{3})$

= $10\frac{2}{3}$ unit^2

Example 5

Determine the coordinates and nature of the stationary point of the curve

$y = \frac{1}{4}x^2 - 2x - 5$

At the stationary point occur when $\frac{dy}{dx} = 0$

$\frac{d}{dx}\left(\frac{1}{4}x^2 - 2x - 5\right) = \frac{1}{2}x - 2$

At stationary points

$\frac{1}{2}x - 2 = 0$

x = 4

substituting for x = 4 in the equation

y = $\frac{1}{4}(2)^2 - 2(2) - 5$ = -9

Hence stationary point is (4, -9)

$\frac{d^2y}{dx^2} = \frac{1}{2}$,

since $\frac{d^2y}{dx^2} > 0$ the turning point is a minima.

Example 6

Given the curve y = $3x^3 - 4x^2 - x$

(a) find the turning points of the curve

Turning points when $\frac{dy}{dx} = 0$

=> $9x^2 - 8x - 1 = 0$

(9x + 1)(x – 1) = 0

Either 9x +1 = 0; x = $-\frac{1}{9}$

Or (x – 1) = 0; x = 1

When x = $-\frac{1}{9}$

y = $3\left(-\frac{1}{9}\right)^3 - 4\left(-\frac{1}{9}\right)^2 - \left(-\frac{1}{9}\right) = \frac{14}{243}$

Turning point = $\left(-\frac{1}{9}, \frac{14}{243}\right)$

When x = 1

y = $3(1)^3 - 4(1)^2 - (-1) = -2$

Turning point (1, -2)

Hence the turning points (x, y) are $\left(-\frac{1}{9}, \frac{14}{243}\right)$ $and\ (1, -2)$

(b) distinguish between the nature of the turning points.

$\frac{d^2}{dx^2}(9x^2 - 8x - 1)$ = 18x – 8

When x = 1

$\frac{d^2y}{dx^2}$ = 18- 8 = 10

Since $\frac{d^2y}{dx^2} > 0$; the turning point (1, -2) is a minimum

When x = $-\frac{1}{9}$

$\frac{d^2y}{dx^2} = \frac{-18}{9} - 8$ = - 10

Since $\frac{d^2y}{dx^2} < 0$; the turning point $\left(-\frac{1}{9}, \frac{14}{243}\right)$ is a maximum

Revision exercise 4

Find and determine the nature of point of the following curves

(a) y = x3 + 3x2 + 1
[(0, 1) min, (-2, 5) max]

(b) y = x3 –x2 -5x +6
$\left[(-1, 9)\ max, \left(\frac{5}{3}, \frac{13}{27}\right) min\right]$

(c) $y = x^2 + \frac{16}{x}$ [(2,12) min]

C. Displacement, velocity and acceleration

Displacement

Displacement is the distance covered by a particle/body in a specified direction.

The displacement (r) of a particle is said to be maximum or minimum when $\frac{d}{dt}(r) = 0$ this enables us to obtain the time when r is maximum or minimum. Hence

$r_{\max\ or\ r_{min}}$ is the value $|r|$

Velocity

This is the rate of change of displacement or $v = \frac{d}{dt}(r)$ where r is displacement.

The velocity of a particle is maximum or minimum when $\frac{d}{dt}(v) = 0$, this enables us to obtain the time when v is maximum or minimum. Hence

$v_{\max\ or\ v_{min}}$ is the value $|v|$

Acceleration, a

This is the rate of change of velocity or $a = \frac{dv}{dt}$. The acceleration of a particle is minimum or maximum when $\frac{d}{dt}(a) = 0$

Example 7

(a) The distance, s meters of a particle from a fixed point is given by $s\ =\ t^2(t^2 + 6) - 4t(t - 1)(t + 1)$, where t is the time in seconds.

Find the velocity and acceleration of the particle when t = 1s.

Solution

$s\ =\ t^2(t^2 + 6) - 4t(t - 1)(t + 1)$

$= t^4 + 6t^2 - 4t(t^2 - 1)$

$= t^4 + 6t^2 - 4t^3 + 4t$

Velocity = $\frac{ds}{tt} = 4t^3 + 12t - 12t^2 + 4$

Whet t = 1

$v =\ 4 + 12 - 12 + 4 = 8ms^{-1}$

Acceleration = $\frac{dv}{dt} = 12t^2 + 12 - 24t$

When t = 1

$a = 12 + 12 - 24 = 0ms^{-2}$

(b) A particle moves along a straight line OX so that its displacement x meters from the origin, O at time t second is given by $x = 4t^3 - 18t^2 + 24t$

Find

(i) when and where the velocity of the particle is zero

$x\ = 4t^3 - 18t^2 + 24t$

$v\ = \frac{dx}{dt} = 12t^2 - 36t + 24$

For v = 0

$12t^2 - 36t + 24 = 0$

$t^2 - 3t + 2 = 0$

$(t - 1)(t - 2) = 0$

Either t = 1 or t= 2

$\therefore velocity = 0\ when$

$t = 1s\ or\ t\ = 2s$

When t = 1s

$x = 4(1)^3 - 18(1)^2 + 24(1)$

$x =\ 4 - 18 + 24\ =\ 10m$

When t= 2

$x = 4(2)^3 - 18(12)^2 + 24(2)$

$x =\ 32 - 72 + 48\ =\ 8m$

(ii) its acceleration at these instants

$a = \frac{dv}{dt} = \frac{d}{dt}(12t^2 - 36t + 24)$

$= 24t - 36$

When $t\ =\ 1s$,

$a\ =\ 24 - 36 = -12ms^2$

When $t\ =\ 2s$,

$a\ =\ 48 - 36 = 12ms^2$

(iii) its velocity when its acceleration is zero.

Acceleration is zero when $\frac{dv}{dt} = 0$

$24t - 36 = 0$

$t = \frac{36}{24} = \frac{3}{2}s$

Velocity v

$= 12\left(\frac{3}{2}\right)^2 - 36\left(\frac{3}{2}\right) = 24$

$= -3ms^{-1}$

i.e. the particle is moving in opposite direction.

(c) The acceleration of a car t s after starting from rest is $\frac{75-10t-t^2}{20}ms^{-2}$ until the instant when this expression vanishes. After this instant, the speed of this car remains constant. Find the maximum acceleration.

Solution

A is maximum when $\frac{d(a)}{dt} = 0$

$\frac{d}{dt}\left(\frac{75+10t-t^2}{20}\right) = \frac{10-2t}{20}$

A is maximum when $\frac{10-2t}{20} = 0$

t = 5s

$a_{max} = \frac{75+10(5)-(5)^2}{20} = \frac{100}{20} = 5ms^{-2}$

(d) The distance s m of a particle from a fixed point is given by
$s = t^2(t^2 + 6))$ where t is the time. Find the velocity and acceleration of the particles when t = 1s

Solution

$s = t^2(t^2 + 6)$

$= t^4 + 6t^2$

$v = \frac{d(s)}{dt} = \frac{d}{dt}(t^4 + 6t^2)$

$= 4t^3 + 12t$

$at\ t = 1s$

$v = 4(1)^3 + 12(1) = 16ms^{-1}$

$a = \frac{d(v)}{dt} = \frac{d}{dt}(4t^3 + 12t\ )$

$= 12t^2 + 12$

$at\ t = 1s$

$a = 12(1)^2 + 12 = 24ms^{-2}$

Revision exercise 1

1. A ball is thrown vertically upwards and its height after t seconds is h m where
$h = 25.2t - 4.9t^2$
Find
(a) its height and velocity after 3s
(b) when it is momentarily at rest
(c) the greatest height reached
(d) the distance moved in the 3rd second
(e) the acceleration when $t = 2\frac{4}{7}$

$\left[\begin{matrix}(a)\quad 31.5m, -4.2ms^{-1}; \\ (b)\ t = 2\frac{4}{7}; (c)32.4m; (d)2.5m; \\ (e) - 9.8ms^2(constant)\end{matrix}\right]$

2. A particle moves along a straight line in such a way that its distance s m from the origin after t s is given by $s = 7t + 12t^2$.
(a) What does it travel in the 9th second?
(b) What are its velocity and acceleration at the end of 9th second?
$[(a)211s; (b)223cms^{-1}\ (c)24ms^{-2}]$

3. A point moves along a straight line OX so that its distance x from the point O at t s is given by $s = t^3 - 6t^2 + 9t$.Find
(a) at what times and in what position the point will have zero velocity.
(b) its acceleration at those instants
(c) its velocity when its acceleration is zero.
$\left[\begin{matrix}(a)1s, 3s, 4cm, 0; (b) - 6, 6cms^{-2}; \\ (c) - 3cms^{-1}\end{matrix}\right]$

4. A particle moves in a straight line so that after t s it is 5m from a fixed point O on the line where$s\ = t^4 + 3t^2$. Find
(a) The acceleration when t = 1, t = 2 and t = 3s.
(b) The average acceleration between t = 1 and t =3s
$[(a)18, 54, 114ms^{-1}; (b)58ms^{-2}]$

5. A particle moves a long a straight line so that after t s, its distance from a fixed point O on the line is 5m where $s = t^3 - 3t^2 + 2t$
(a) When is the particle is at O?
(b) What is the velocity and acceleration at these times?
(c) What is the average acceleration between t = 0 and t = 2s.
$\left[\begin{matrix}(a)after\ 0, 1, 2s; (b)2, -1, 2ms^{-1}; \\ -6, 0, 6ms^{-2}; (c)0ms^{-1}; (d)0ms^{-1}\end{matrix}\right]$

Thank You

Dr. Bbosa Science