Subsidiary Mathematics

SENIOR FIVE term 1

TOPIC 2/3: Quadratic equations

Competency: The learner applies concepts of quadratic equations to solve real-life problems in different contexts.

Quadratic equations

These are equations expressed in the form $ax^2 + bx + c = 0$ where a, b and c are constants and $a \neq 0$

They have at most two roots which may be real or complex.

Example of quadratic equations

$2y^2 + 3y + 5 = 0$; a = 2, b = 3 and c = 5

$x^2 + 4x – 10 = 0$; a = 1, b = 4, c = -10

Forming quadratic equations

Suppose that the roots of a quadratic equation are α and β,

then x – α = 0 and x – β = 0

When forming a quadratic equation

(x – α)(x – β) = 0

$x^2 – (\alpha + \beta)x + \alpha\beta = 0$

This means that if the roots of a quadratic equation are given, its equation in terms of x is x^2-(sum of roots)x + product of roots = 0

Example 1

Form quadratic equation I terms of x with roots

(a) (2, 3)

Solution

Let x = 2 or x = 3

x – 2 = 0 x – 3 = 0

⇨ (x – 2)(x – 3) = 0

$x^2 – 5x – 6 = 0$

(b) (p, q)

Solution

Let x = p or x = q

x – p = 0 x – q = 0

⇨ (x – p)(x – q) = 0

$x^2 – (p + q)x – pq = 0$

(c) (-3, -2)

Solution

Let x = -3 or x = -2

x + 3 = 0 x + 2 = 0

⇨ (x + 3)(x + 2) = 0

$x^2 + 5x + 6 = 0$

Example 2

The root of the equation $4x^2 + 9x - k = 0$ are α and 2. Find the values of α and k. (05marks)

$2 + \alpha = -\frac{9}{4}$

$\alpha = -\frac{9}{4} - 2 = -\frac{17}{4} = -4\frac{1}{4}$

$2\alpha = -\frac{k}{2}$

$2x - \frac{17}{4} = -\frac{k}{2}$

$-\frac{k}{2} = -\frac{34}{4}$

$k = 34$

Example 3

The roots of the equation $2x^2 + 4x - 1 = 0$ are α and β. Find the value of $\alpha^2 + \beta^2$. (05marks)

$2x^2 + 4x - 1 = 0$

$x^2 + 2x - \frac{1}{2} = 0$

$\alpha + \beta = -2$

$\alpha\beta = -\frac{1}{2}$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= 4 - (-1) = 5$

Example 4

The roots of the equation $2x^2 - 6x + 7 = 0$ are α and β. Determine the

(a) values of $(\alpha - \beta)^2$ and $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2}$ (12 marks)

Solution

$\alpha + \beta = 3$

$\alpha\beta = \frac{7}{2}$

$(\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$

But $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

=> $(\alpha - \beta)^2 = (\alpha+\beta)^2 - 2\alpha\beta - 2\alpha\beta$

$= (\alpha+\beta)^2 - 4\alpha\beta$

$= 3^2 - 4 \times \frac{7}{2}$

$= -5$

$\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2} = \frac{\alpha+\beta}{\alpha^2\beta^2} = \frac{\alpha+\beta}{(\alpha\beta)^2} = \frac{2}{\left(\frac{7}{2}\right)^2} = \frac{3x4}{49} = \frac{12}{49}$

(b) Quadratic equation with integral coefficients whose roots $(\alpha - \beta)^2$ and $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2}$(03marks)

Solution

Sum of root = $-5 + \frac{12}{49} = -\frac{233}{49}$

Product of roots = $-5 \times \frac{12}{49} = -\frac{60}{49}$

Equation

$x^2 - \left(-\frac{233}{49}\right)x - \frac{60}{49} = 0$

Hence equation is $x^2 - \frac{233}{49}x - \frac{60}{49} = 0$

Example 5

Determine the possible values of α for which the equation

$2x^2 + (\alpha + 2)x + (\alpha + 2) = 0$ has equal roots (05marks)

Solution

For equal roots $b^2 = 4ac$

$(\alpha + 2)^2 = 4(2)(\alpha + 2)$

$\alpha^2 + 4\alpha + 4 = 8\alpha + 16$

$\alpha^2 - 4\alpha - 12 = 0$

$(\alpha - 6)(\alpha + 2) = 0$

Either $\alpha - 6 = 0$, $\alpha = 6$

Or $\alpha + 2 = 0$, $\alpha = -2$

Example 6

1. (a) Given that α and β are the root of the quadratic equation $x^2 - 3mx + n^2 = 0$, show that $\alpha + \beta = 3m$ and $\alpha\beta = n^2$.

Solution

$(x - \alpha)(x - \beta) = 0$

$x^2 - \alpha x - \beta x + \alpha\beta = 0$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

Comparing

$\alpha + \beta = 3m$ and $\alpha\beta = n^2$

(b) If α and β are the root of the equation $x^2 - 9x + 4 = 0$, find the;

(i) value of $\alpha^2 + \beta^2$. (03marks)

Solution

From $(\alpha+\beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= (9)^2 - 2\times4$

$= 81 - 8$

$= 73$

(ii) value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$.

Solution

$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{\alpha^2\beta^2} = \frac{\alpha^2+\beta^2}{(\alpha\beta)^2} = \frac{73}{4^2} = \frac{73}{16} = 4.5625$

(iii) quadratic equation whose roots are $\frac{1}{\alpha^2}$ *and* $\frac{1}{\beta^2}$.

Solution

Sum of the root, $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{73}{16}$

Product of the roots, $\frac{1}{\alpha^2} \cdot \frac{1}{\beta^2} = \frac{1}{(\alpha\beta)^2} = \frac{1}{16}$

Hence quadratic equation

$x^2 - \frac{73}{16}x + \frac{1}{16} = 0$

Exercise 1

(Answers are given in square brackets, [ ] at the end of each questions

1. Given that the root of the equation $x^2 + px + q = 0$ are α and β, Form quadratic equations with roots $\frac{\alpha}{\beta}$ *and* $\frac{\beta}{\alpha}$

   [$x^2 - \left(\frac{p^2-2q}{q}\right)x + 1 = 0$

   Or $x^2 - (p^2 - 2q)x + q = 0$]

2. Given the equation $x^3 + x - 10 = 0$.

(a) Show that x = 2 is a root of the equation

Let $f(x) = x^3 + x - 10$

Substituting for x = 2

$f(2) = 2^3 + x - 10$

$= 8 + 2 - 10$

$= 10 - 10 = 0$

Hence x = 2 is a root of $x^3 + x - 10 = 0$

(b) Deduce the values of α + β and αβ where α and β are roots of the equation.

Hence form a quadratic equation whose roots are α^2 and β^2.

⇨ $x^3 + x - 10 = (x-2)(x^2 + 2x + 5)$

Either $x - 2 = 0$

Or $(x^2 + 2x + 5) = 0$

$\alpha + \beta = 2$

$\alpha\beta = 5$

Sum of roots = $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (-2)^2 - 2(5) = 4 - 10 = -6$

Product= $\alpha^2\beta^2 = (\alpha\beta)^2 = 5^2 = 25$

The equation become

$x^2 - (-6)x + 25 = 0$

$x^2 + 6x + 25 = 0$

Solving quadratic equations

Quadratic equations may be solved by

(a) Factorization method
(b) Completing square method

(a) Factorization method

It is used for quadratic equations that are easy to factorise

Example 7

(a) $4x^2 + 7x + 3 = 0$

Solution	side work
$4x(x + 1) + 3(x + 1) = 0$	Product = 4 x 3 = 12
$(x + 1)(4x + 3) = 0$	Sum = 7
Either $x + 1 = 0$; $x = -1$	Factor = (4, 3)
Or $4x + 3 = 0$; $x = -\frac{3}{4}$	

(b) $2x^2 + 5x + 3 = 0$

Solution	side work
$2x(x + 1) + 3(x + 1) = 0$	Product = 3 x 2 = 6
$(x + 1)(2x + 3) = 0$	Sum = 5
Either $x + 1 = 0$; $x = -1$	Factors 3, 2
Or $2x + 3 = 0$; $x = -\frac{3}{2}$	

(c) $x^2 + x - 20 = 0$

Solution	Side work
$x(x - 4) + 5(x - 4) = 0$	Product = -20
$(x - 4)(x + 5) = 0$	Sum = 1
Either $x - 4 = 0$; $x = 4$	Factors (5, - 4)
Or $x + 5 = 0$; $x = -5$	

(d) $10x^2 + x - 3 = 0$

Solution	Side work
$5x(2x - 1) + 3(2x - 1) = 0$	product 10 x – 3 = -30
$(5x + 3)(2x - 1) = 0$	sum = 1
Either $5x + 3 = 0$; $x = -\frac{3}{5}$	Factors (6, -5)
Or $2x - 1 = 0$; $x = \frac{1}{2}$	

(b) Method II: Completing squares approach

The idea is to create a perfect square on one side of the equation:

Given the equation $ax^2 + bx + c = 0$

- Dividing the equation by and transposing the constant term to the RHS
  $x^2 + \frac{b}{a}x = -\frac{c}{a}$
- Marking the LHS a perfect square, add a half the coefficient of x squared on both sides

  $$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

- Factorise the terms on the LHS
- $\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2-4ac}{4a^2}$
- Taking square root on both sides of the equation
- $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$
- Solving $x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$
- $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

This is the general quadratic equation formula for finding the square root of any quadratic equation. This formula is locally known as ***bull dozer formula***

Example 8

Solve the following equations by completing squares

(a) $2x^2 - x - 3 = 0$

Solution

$2x^2 - x - 3 = 0$

$x^2 - \frac{1}{2}x = \frac{3}{2}$

$x^2 - \frac{1}{2}x + \left(-\frac{1}{4}\right)^2 = \frac{3}{2} + \left(-\frac{1}{4}\right)^2$

$\left(x - \frac{1}{4}\right)^2 = \frac{25}{16}$

$x - \frac{1}{4} = \sqrt{\frac{25}{16}} = \pm\frac{5}{4}$

Either $x = \frac{5}{4} + \frac{1}{4} = \frac{6}{4} = \frac{3}{2}$

Or $x = -\frac{5}{4} + \frac{1}{4} = \frac{-4}{4} = -1$

Hence x = -1 and x = $\frac{3}{2}$

(b) $18x^2 + 7x - 1 = 0$

Solution

$18x^2 + 7x - 1 = 0$

$x^2 + \frac{7}{18}x = \frac{1}{18}$

$x^2 + \frac{7}{18}x + \left(\frac{7}{36}\right)^2 = \frac{1}{18} + \left(-\frac{7}{36}\right)^2$

$\left(x + \frac{7}{36}\right)^2 = \frac{1}{18} + \frac{49}{1296} = \frac{121}{1296}$

$x + \frac{7}{36} = \sqrt{\frac{121}{1296}} = \pm\frac{11}{36}$

Either $x = \frac{11}{36} - \frac{7}{36} = \frac{4}{36} = \frac{1}{9}$

Or $x = -\frac{11}{36} - \frac{7}{36} = -\frac{18}{36} = -\frac{1}{2}$

Hence x = $\frac{1}{9}$ or x = $-\frac{1}{2}$

(c) $3x^2 + 7x + 2 = 0$

Solution

$3x^2 + 7x + 2 = 0$

$x^2 + \frac{7}{3}x = -\frac{2}{3}$

$x^2 + \frac{7}{3}x + \left(\frac{7}{6}\right)^2 = -\frac{2}{3} + \left(\frac{7}{6}\right)^2$

$\left(x + \frac{7}{6}\right)^2 = -\frac{2}{3} + \frac{49}{36} = \frac{25}{36}$

$x + \frac{7}{6} = \sqrt{\frac{25}{36}} = \pm\frac{5}{6}$

Either x = $\frac{5}{6} - \frac{7}{6} = -\frac{2}{6} = -\frac{1}{3}$

Or x = $-\frac{5}{6} - \frac{7}{6} = -\frac{12}{6} = -2$

Hence x = -2 and x = $-\frac{1}{3}$

Example 8

Solve the following equations by using the quadratic formula

(a) $7x^2 - 5x - 2 = 0$

Solution

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{5\pm\sqrt{-5^2-4\ x\ 7\ x-2}}{2\ x\ 7} = \frac{5\pm 9}{14}$

Either $x = \frac{14}{14} = 1$ or $x = \frac{-4}{14} = -\frac{2}{7}$

Hence $x = 1\ and\ x = -\frac{2}{7}$

(b) $3x^2 - 7x - 6 = 0$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{7\pm\sqrt{-7^2-4\ x\ 3\ x-6}}{2\ x\ 3} = \frac{7\pm 11}{6}$

Either $x = \frac{18}{6} = 3$ or $x = \frac{-4}{6} = -\frac{2}{3}$

Hence $x = 3\ and\ x = -\frac{2}{3}$

Revision exercise 2

[Answers are given in square brackets]

1. Given that the roots of the equation $x^2 + px + q = 0$ are α and β, find the values of:
   (a) $\alpha^2 + \beta^2 [p^2 - 2q]$
   (b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} \left[\frac{p^2-2q}{q}\right]$
   (c) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} \left[\frac{p}{q}(3q - p^2)\right]$
   (d) $\alpha - \beta \left[\pm\sqrt{p^2 - 4q}\right]$
   (e) $\alpha^2 - \beta^2 \left[-p\sqrt{(p^2 - 4q)}\right]$
   (f) $\alpha^3 + \beta^3 [p(3q - p^2)]$
2. If the roots of the equation $x^2 + 2x + 3 = 0$ are α and β, form an equation whose roots are $\alpha^2 - \beta$ and $\beta^2 - \alpha$. $[x^2 + 2 = 0]$

3. Given that the roots of the equation $x^2 - 2x + 10 = 0$ are α and β, form an equation whose roots are

$\frac{1}{(2+\alpha)^2}$ and $\frac{1}{(2+\beta)^2}$ $[324x^2 + 1 = 0]$

4. If α and β are roots of the equation $x^2 - px + q = 0$, find the equation whose roots are $\frac{\alpha^3-1}{\alpha}$ *and* $\frac{\beta^3-1}{\beta}$
   $[qx^2 - (p^2q - 2q^2 - p)x + (q^3 - p^3 + 3pq) + 1 = 0]$
5. The roots of the equation $3x^2 - ax + 6b = 0$ are α and β. Find the condition for one root to be
   (a) Twice the other $[81b = a^2)$
   (b) The cube of the other
   $[a^4 - 648(b - 1)b^2 - 18(4a^2 + 9)b = 0]$
6. By Factorization method solve the following quadratic equations
   (a) $x^2 + 9x + 14$ [x = -7, x= -2]
   (b) $x^2 + 2x - 8 = 0$ [x = -4, x = 2)
   (c) $2x^2 + x - 10 = 0$ [x= 2, x = $-\frac{5}{2}$]
   (d) $6x^2 - 19x + 10$ $\left[x = \frac{5}{2},\ x = \frac{2}{3}\right]$
7. Solve the following equations by completing squares
   (a) $2x^2 + 5x + 3 = 0$ $\left[x = -1 \text{ and } x = -\frac{3}{2}\right]$
   (b) $x^2 + 9x + 20 = 0$ [x = -5 and x = -4]
   (c) $x^2 + x - 20 = 0$ [x = -5 and x = 4]
   (d) $x^2 - x - 20 = 0$ [x = 4 and x = 5]
8. Solve the following equations using the quadratic formula
   (a) 2x2 + 5x + 3 = 0 $\left[\frac{-3}{2}, -1\right]$
   (b) $x^2 + 9x + 20$ [-5, -4]
   (c) $x^2 + x - 20$ [-5, 4]

The roots of the equation $2x^2 + 4x - 1 = 0$ are α and β. Find the value of $\alpha^2 + \beta^2$. (05marks)

$2x^2 + 4x - 1 = 0$

$x^2 + 2x - \frac{1}{2} = 0$

$\alpha + \beta = -2$

$\alpha\beta = -\frac{1}{2}$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= 4 - (-1) = 5$

The roots of the equation $2x^2 - 6x + 7 = 0$ are α and β. Determine the

(a) values of $(\alpha - \beta)^2$ and $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2}$ (12 marks)

Solution

$\alpha + \beta = 3$

$\alpha\beta = \frac{7}{2}$

$(\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$

But $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

=> $(\alpha - \beta)^2 = (\alpha+\beta)^2 - 2\alpha\beta - 2\alpha\beta$

$= (\alpha+\beta)^2 - 4\alpha\beta$

$= 3^2 - 4 \times \frac{7}{2}$

$= -5$

$$\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2} = \frac{\alpha+\beta}{\alpha^2\beta^2} = \frac{\alpha+\beta}{(\alpha\beta)^2} = \frac{2}{\left(\frac{7}{2}\right)^2} = \frac{3x4}{49} = \frac{12}{49}$$

(b) Quadratic equation with integral coefficients whose roots $(\alpha - \beta)^2$ and $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2}$(03marks)

Sum of root = $-5 + \frac{12}{49} = -\frac{233}{49}$

Product of roots $= -5 \times \frac{12}{49} = -\frac{60}{49}$

Equation

$$x^2 - \left(-\frac{233}{49}\right)x - \frac{60}{49} = 0$$

Hence equation is $x^2 - \frac{233}{49}x - \frac{60}{49} = 0$

2. (a) Given that α and β are the root of the quadratic equation $x^2 - 3mx + n^2 = 0$, show that $\alpha + \beta = 3m$ and $\alpha\beta = n^2$. (06 marks)

Solution

$(x - \alpha)(x - \beta) = 0$

$x^2 - \alpha x - \beta x + \alpha\beta = 0$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

Comparing

$\alpha + \beta = 3m$ and $\alpha\beta = n^2$

(b) If α and β are the root of the equation $x^2 - 9x + 4 = 0$, find the;

(i) value of $\alpha^2 + \beta^2$. (03marks)

Solution

From $(\alpha+\beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= (9)^2 - 2x4$

$= 81 - 8$

$= 73$

(ii) value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$. (03marks)

Solution

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{\alpha^2\beta^2} = \frac{\alpha^2+\beta^2}{(\alpha\beta)^2} = \frac{73}{4^2} = \frac{73}{16} = 4.5625$$

(iii) quadratic equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$. (03marks)

Solution

Sum of the root, $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{73}{16}$

Product of the roots, $\frac{1}{\alpha^2} . \frac{1}{\beta^2} = \frac{1}{(\alpha\beta)^2} = \frac{1}{16}$

Hence quadratic equation

$x^2 - \frac{73}{16}x + \frac{1}{16} = 0$

Thank You

Dr. Bbosa Science