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The Science Foundation College
Uganda East Africa
Senior one to sinior six
+256 778 633682 0753 143413
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Subsidiary Mathematics

SENIOR FIVE term 2

TOPIC 1/4: Descriptive Statistics

Competency: The learner applies indices and logarithms to solve societal problems involving exponential relationships and scaling in various fields.

Indices

There are five basic rules of indices

(a) $a^p \ x \ a^q = a^{p+q}$

(b) $\frac{a^p}{a^q} = a^{p-q}$

(c) $(a^p)^q = a^{pq}$

(d) $a^{\frac{1}{q}} = \sqrt[q]{a}$

(e) $a^{\frac{p}{q}} = \left(\sqrt[q]{a}\right)^p$

Example 1

Evaluate the following

(a) $2^2 \text{ x } 2^3$

Solution

$2^2 \text{ x } 2^3 = 2^{2+3}$

$= 2^5$

$= 32$

(b) $\frac{4^3}{4^2}$

Solution

$\frac{4^3}{4^2} = 4^{3-2}$

$= 4^1$

$= 4$

(c) $(3^2)^3$

Solution

$(3^2)^3 = 3^{2 \ x \ 3}$

$= 3^6$

$= 729$

(d) $2^{\frac{1}{2}} \ x \ 2^{\frac{1}{2}}$

Solution

$2^{\frac{1}{2}} \ x \ 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}}$

$= 2^1$

$= 2$

(e) $\sqrt[3]{27}$

Solution

$\sqrt[3]{27} = (3^3)^{\frac{1}{3}}$

$= 3^{3 \ x \frac{1}{3}}$

$= 3^1$

$= 3$

(f) $125^{\frac{2}{3}}$

Solution

$125^{\frac{2}{3}} = (5^3)^{\frac{2}{3}}$

$= 5^{3x\frac{2}{3}}$

$= 5^2$

$= 25$

Example 2

Evaluate the following

(a) $\left(\frac{125}{27}\right)^{\frac{4}{3}}$

Solution

$\left(\frac{125}{27}\right)^{\frac{4}{3}} = \left(\frac{(5^3)^{\frac{4}{3}}}{(3^3)^{\frac{4}{3}}}\right)$

$= \frac{5^4}{3^4}$

$= \frac{625}{81}$

(b) $81^{\frac{3}{4}}$

Solution

$81^{\frac{3}{4}} = (3^4)^{\frac{3}{4}}$

$= 3^{4x\frac{3}{4}}$

$= 3^3$

= 27

The zero index

From $\frac{a^p}{a^p} = a^{p-p} = a^0 = 1$

∴ Any number raised to power zero =1

i.e. $100^0 = 529^0 = 83^0 = 1$

Negative indices

It can be shown that

$\frac{1}{a} = \frac{a^0}{a^1} = a^{0-1} = a^{-1}$

Also

$\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$

Hence a negative index is the inverse of a given number

Example 3

Evaluate the following

(a) $16^{\frac{-3}{2}}$

Solution

$16^{\frac{-3}{2}} = (2^4)^{\frac{-3}{2}}$

$= 2^{4x\frac{-3}{2}}$

$= 2^{-6}$

$= \frac{1}{2^6}$

$= \frac{1}{64}$

(b) $\left(\frac{64}{27}\right)^{-\frac{2}{3}}$

Solution

(a) $\left(\frac{64}{27}\right)^{-\frac{2}{3}} = \left(\frac{27}{62}\right)^{\frac{2}{3}} = \left(\frac{\sqrt[3]{27}}{\sqrt[3]{64}}\right)^2 = \frac{9}{16}$

Example 4

Show that $\sqrt{\frac{25^3+5^6}{5^7-5^6}} = \frac{\sqrt{2}}{2}$

Solution

$\sqrt{\frac{25^3+5^6}{5^7-5^6}} = \sqrt{\frac{(5^2)^3+5^6}{5^7-5^6}}$

$= \sqrt{\frac{5^6+5^6}{5x5^6-5^6}}$

$= \sqrt{\frac{2x5^6}{4\,x5^6}}$

$= \sqrt{\frac{2}{4}}$

= $\frac{\sqrt{2}}{2}$ **(as required)**

Solving equations with unknown indices

It involves making appropriate substitution after expressing terms containing powers in simplified form

Example 5

Solve the equation

$2^{2x+1} - 7(2^x) + 6 = 0$

Solution

$2^{2x+1} - 7(2^x) + 6 = 0$

$2^1.2^{2x} - 7(2^x) + 6 = 0$

$2(2^x)^2 - 7(2^x) + 6 = 0$

Let p = 2^x

⇨ $2p^2 - 7p + 6 = 0$
$(2p - 3)(p - 2) = 0$

Either 2p -3 = 0

p = $\frac{3}{2}$

or

p – 2 =0

p = 2

when p= $\frac{3}{2}$ => $2^x = \frac{3}{2}$

$\log 2^x = \log\frac{3}{2}$

$x \log 2 = \log\frac{3}{2}$

$x = \frac{\log\frac{3}{2}}{\log s} = 0.585$

When p = 2

$2^x = 2 = 2^1$

x = 1

Hence x = 1 and x = 0.585 (3d.p)

Example 6

Show that

$\frac{3(2^{x+1})-4(2^{x-1})}{2^{x+1}-2^x} = 4$

Solution

$\frac{3(2^{x+1})-4(2^{x-1})}{2^{x+1}-2^x} = 4$

Handling terms on the LHS

$\frac{3(2^{x+1})-4(2^{x-1})}{2^{x+1}-2^x}$

$= \frac{3(2^x x2^1)-4(2^x.2^1)}{2^x.2^1-2^x}$

$= \frac{2^x(3\,.\,2^1-4\,.\,2^1)}{2^x(2^1-1)} = \frac{6-2}{1} = 4$

Example 7

Solve $x^{\frac{4}{3}} = 81$

$x^{\frac{4}{3}x\frac{3}{4}} = 81^{\frac{3}{4}}$

x = $\left(\sqrt[4]{81}\right)^2$= $3^3 = 27$

Solving equations with squares

Example 8

$\sqrt{2x+5} = x + 1$

Square both sides

$\left(\sqrt{2x+5}\right)^2 = (x+1)^2$

2x + 5 = x^2+ 2x + 1

x^2 = 4

x = ± 2

Testing/checking using -2

$\sqrt{2x+5} = x + 1$

$\sqrt{2\,x-2+5} = -2 + 1$

1 ≠ -1

Hence -2 is **not** a solution to the equation

Testing/checking using 2

$\sqrt{2x+5} = x+1$

$\sqrt{2\ x2+5} = 2+1$

3 = 3

Hence 2 is the solution to the equation

Example 9

Solve for x: $\sqrt{x+2} = 4$

Square both sides

$\left(\sqrt{x+2}\right)^2 = 4^2$

x + 2 = 16

x = 14

Finding square roots of terms containing rational and irrational quantities

When finding roots of terms expressed in the form a + $\sqrt{b}$, where a is a rational and b is an irrational quantity, we let the root to be in the form of $\pm(\sqrt{x_1} + \sqrt{x_2})$ where x_1 and x_2 are integers.

Example 8

Find the square root of 6 +2$\sqrt{5}$

Let $\pm(\sqrt{x_1} + \sqrt{x_2})$ be square root of 6 +2$\sqrt{5}$

⇨ $\pm(\sqrt{x_1} + \sqrt{x_2}) = \sqrt{6\ +2\sqrt{5}}$

Squaring both sides

$(\sqrt{x_1} + \sqrt{x_2})^2 = \left(\sqrt{6+\sqrt{5}}\right)^2$

$x_1 + x_2 + 2\sqrt{x_1 . x_2}$ =6 +2$\sqrt{5}$

Comparing terms on the two sides

$x_1 + x_2 = 6$

$x_1 = 6 - \ x_2$ ………………..(i)

$\mathrm{x_1 . x_2} = 5$……………………(ii)

Substituting eqn. (i) into eqn. (ii)

$(6 - \ x_2)x_2$ = 5

${x_1}^2 - 6x_2 + 5 = 0$

${x_1}^2 - x_2 - 5x_2 + 5 = 0$

$x_2(x_2 - 1) - 5(x_2 - 1) = 0$

$(x_2 - 1)((x_2 - 5) = 0$

Either: $x_2 - 1$ = 0 => $x_2 = 1$

Or $x_2 - 5$ = 0 => $x_2 = 5$

When$x_2 = 1$: $x_1 = 6 - 1$ = 5

When$x_2 = 5$: $x_1 = 6 - 5$ = 1

Hence the square root of 6 +2$\sqrt{5}$ is $\pm\left(1+\sqrt{5}\right)$

Example 10

Find the square root of 8 - 2$\sqrt{15}$

et $\pm(\sqrt{x_1} - \sqrt{x_2})$ be square root of 8 -2$\sqrt{15}$

$\pm(\sqrt{x_1} + \sqrt{x_2}) = \sqrt{8 - 2\sqrt{15}}$

Squaring both sides

$(\sqrt{x_1} - \sqrt{x_2})^2 = \left(\sqrt{8 - 2\sqrt{15}}\right)^2$

$x_1 + x_2 - 2\sqrt{x_1 . x_2} = 8\ - 2\sqrt{15}$

Comparing terms on the two sides

$x_1 + x_2 = 8$

$x_1 = 8 - \ x_2$ ………………..(i)

$x_1 . x_2 = 15$……………………(ii)

Substituting eqn. (i) into eqn. (ii)

$(8 - \ x_2)x_2$ = 15

${x_1}^2 - 8x_2 + 15 = 0$

${x_1}^2 - 3x_2 - 5x_2 + 15 = 0$

$x_2(x_2 - 3) - 5(x_2 - 3) = 0$

$(x_2 - 3)((x_2 - 5) = 0$

Either: $x_2 - 5 = 0$ => $x_2 = 5$

Or $x_2 - 3 = 0$ => $x_2 = 3$

When $x_2 = 5$: $x_1 = 8 - 5 = 3$

When $x_2 = 3$: $x_1 = 8 - 3 = 5$

Hence the square root of 8 -2$\sqrt{15}$ is $\pm(\sqrt{5} - \sqrt{3})$

Revision exercise 1

1. Simplify
   (i) $9a^2 \div 27a^{-4}$ $\left[\frac{2}{3}a^6\right]$
   (ii) $(6a^{-3}) \div (9a^{-4})^2$ $\left[\frac{2}{27}a^5\right]$
   (iii) $\frac{2a^{-3}b^2}{7c^{-4}d^2}$ $\left[\frac{2b^2c^4}{7a^3d^2}\right]$
   (iv) $(x^4yz^{-3})^2 \times \sqrt{x^{-5}y^2z} \div (xz)^{\frac{1}{2}}$ $[x^5yz^{-6}]$
   (v) $\sqrt[4]{y^3}x\sqrt{y^{\frac{1}{2}}}$ $\left[y^{\frac{5}{4}}\right]$
2. Evaluate
   (a) $(64)^{-\frac{3}{2}}$ [16]
   (b) $\left(\frac{8}{27}\right)^{-\frac{1}{3}}$ $\left[\frac{3}{2}\right]$
   (c) $\left(\frac{1}{25}\right)^{\frac{1}{2}}$ $\left[\frac{1}{5}\right]$
   (d) $\left(\frac{8}{27}\right)^{\frac{2}{3}}$ $\left[\frac{4}{9}\right]$
   (e) $\left(\frac{243}{512}\right)^{-\frac{2}{3}}$ [1.6445]
3. Solve the following equations
   (a) $98x^2 = 2$ [x= 0.1429]
   (b) $x^{-3} = 8$ $\left[x = \frac{1}{2}\right]$
   (c) $\frac{1}{32}x^3 = 8x^{-1}$[x = 4]
   (d) $\frac{9}{25}x = \frac{5}{3}x^{-2}$ $\left[x = \frac{5}{3}\right]$
   (e) $\frac{2}{14}x^{-2} + 14x = 0$ [x = -0.2169]
4. Solve for x
   (a) $3^{2x+1} + 3 = 10(3^x)$ [x = 1 or x = -1]
   (b) $2^{2x-1} + \frac{3}{2} = 2^{x+1}$[x =0, x =1.585]
   (c) $7^x = 3^{1-x}$ [x=0.3608]
   (d) $7x^{\frac{1}{2}}+2 = 0$ $\left[x = \frac{4}{49}\right]$
   (e) $5x^{\frac{2}{3}} = x^{-\frac{1}{3}}$ $\left[x = \frac{1}{5}\right]$
   (f) $4x^{-\frac{1}{3}} = 5x^{\frac{1}{6}}$ $\left[x = \frac{16}{25}\right]$
   (g) $6x^{\frac{2}{3}} - \frac{2}{3}x^{-\frac{1}{2}} = 0$ [x = 0.077]
   (h) $8x^{-2} = 343x^{\frac{1}{2}}$ [x=0.003562]
5. Show that
   (a) $\frac{(2^{2x} - 3.2^{2x-2})(3^x - 2.3^{x-2})}{3^{x-4}(4^{x+3} - 2^{2x})} = \frac{1}{4}$
   (b) $\frac{(1+a)^{\frac{1}{2}} - \frac{1}{3}a(1+a)^{-\frac{2}{3}}}{(1+a)^{\frac{2}{3}}} = \frac{3+2a}{3(1+a)^{\frac{4}{3}}}$
   (c) $(a - a^{-1})\left(a^{\frac{4}{3}} - a^{\frac{2}{3}}\right) = \frac{a^2 - a^{-2}}{a^{-\frac{1}{3}}}$
   (d) $\frac{a^{\frac{1}{2}}+ab}{ab - b^2} - \frac{\sqrt{a}}{\sqrt{a-b}} = \sqrt{\frac{a}{b}}$
6. Solve
   (a) $x^{\frac{1}{3}} - 3 = 28x^{-\frac{1}{3}}$ [x =-64, x = 343]
   (b) $2x^{\frac{1}{4}} = 9 - 4x^{-\frac{1}{4}}$[x =$\frac{1}{16}$, x = 256]
   (c) $x^3 + 8 = 9x^{\frac{3}{2}}$ [x = 1, x= 4]
   (d) $2x^{\frac{1}{3}} = \frac{81}{8}x^{-1}$ [x = 8.6967]
   (e) $49x^{-\frac{5}{6}} - \frac{8}{81}x^{\frac{7}{6}} = 0$ [x=22.2739]
   (f) $x^{\frac{2}{3}} - x^{\frac{1}{3}} - 2 = 0$[x=-1]
   (g) $x^{\frac{1}{2}} - 5x^{\frac{1}{4}} + 6 = 0$[x=-1]
   (h) $6x^{\frac{1}{3}} + 5 + x^{-\frac{1}{3}} = 0$ $\left[x = \frac{1}{2}, x = \frac{1}{3}\right]$
7. Solve for x
   (a) $\sqrt{x+2} - x = 0$[x=2]
   (b) $\sqrt{1+x} = 1 + \sqrt{1-x}$ $\left[x = \frac{\sqrt{3}}{2}\right]$
   (c) $(3-x)^{\frac{1}{2}} = (1+x)^{\frac{1}{2}} + (2-x)^{\frac{1}{2}}$ [x = -0.92665]
   (d) $\sqrt{x+6} = \sqrt{1-3x} - \sqrt{4-x}$ [-5]
8. Without using mathematical tables or calculators, find the value of
   $\frac{(\sqrt{5}+2)^2 - (\sqrt{5}-2)^2}{8\sqrt{5}}$ [1]
9. Find the square root of the following
   (a) 6 + 2$\sqrt{5}$ $[\pm(1 + \sqrt{5})]$
   (b) 18 - 2$\sqrt{12}$$[\pm(\sqrt{0.695} - \sqrt{17.303})]$
   (c) 18 - 2$\sqrt{2}$$[\pm(\sqrt{0.1118} - \sqrt{17.8882})]$

Examples exponential growth and decay that reduce to quadratics, using the laws of indices and interpret the solutions within context.

Exponential equations can sometimes be transformed into quadratic equations using the **laws of indices** and a suitable **variable substitution**. The solutions to the resulting quadratic equation must then be interpreted in the context of the original exponential problem, particularly ruling out non-physical results like negative amounts or impossible timeframes.

Example 11

A bacteria culture's growth is modeled by the equation $4^t - 20(2^t) + 64 = 0$, where t is the time in hours and the population is related to powers of 2**.** find the time(s) t when the population reaches a certain state.

Solution

Applying laws of indices

$4^t = 2^{2t} = (2^t)^2$

The equation becomes

$(2^t)^2 - 20(2^t) + 64 = 0$,

Let $x = 2^t$

Substituting x for 2^t

$x^2 + 20x + 64 = 0$

Solving quadratic equation

$(x – 4)(x – 16) = 0$

Either $x – 4 = 0$; $x = 4$

Or $(x – 16) = 0$; $x = 16$

Reversing substitution

For x = 4, $2^t = 4 = 2^2$; t =2

For x = 16, $2^t = 16 = 2^4$; t =4

Interpretation:

Both solutions, t = 2hours and t= 4hours are positive and physically possible. This means the bacteria population level described by the equation occurs at two different points in time, perhaps indicating the population rises, peaks, and falls due to resource limitations (though the initial model might be a simplified representation).

Example 12

A rural livestock population grows geometrically: $P_t = P_0 \cdot b^t$. Field data suggests each season’s herd equals the sum of the previous two seasons’ herds (a “Fibonacci-like” recurrence): $P_t = P_{t-1} + P_{t-2}$. Find the value of b

Derivation using indices:

Write all terms with the exponential form:

$P_0 \cdot b^t = P_0 \cdot b^{t-1} + P_0 \cdot b^{t-2}$

Divide through by $P_0 \cdot b^{t-2}$:

$b^2 = b + 1$

Rearranging

$b^2 - b - 1 = 0$

Solving quadratic equation

$b = \frac{1 \pm \sqrt{1+4}}{2}$

either b = 1.618 or -0.618

Since the base b is a growth multiplier per season, so b>0, we discard the negative value. A negative base is not meaningful for herd size, and b<1 would imply decay (contradicting the recurrence).

Hence the value of b = 1.618

Interpretation: The herd grows by about 61.8% each season.

Example 13

A radioactive substance's remaining mass (in grams) is described by the equation $9^x - 12(3^x) + 27 = 0$ is the number of half-lives passed. Find the number of half-lives for a specific remaining mass.

Solution

Applying laws of indices

$9^t = 3^{2t} = (3^t)^2$

The equation becomes

$(3^t)^2 - 12(2^t) + 27 = 0$,

Let $x = 3^t$

Substituting x for 3^t

$x^2 + 12x + 27 = 0$

Solving quadratic equation

$(x - 3)(x - 9) = 0$

Either $x - 3 = 0$; $x = 3$

Or $(x - 9) = 0$; $x = 9$

Reversing substitution

For x = 3: $3^t = 3 = 3^1$; t = 1

For x = 9: $3^t = 9 = 3^2$; t = 2

Interpretation:

The solutions are t = 1 and t = 2 half-lives. Both values are positive and meaningful in the context of time passing. The substance reaches the specific mass level after 1 half-life and again after 2 half-lives. This might happen if the "remaining mass" value in the equation is an intermediate value the substance passes through during its decay process.

Example 14

A community's clean water supply capacity S(t) is being upgraded so it scales approximately like $S(t)=200\ (1.05)^{2t}$ (two compounding improvements per year), while water demand grows as $D(t)=500\ (1.05)^t$. When will supply and demand balance?

Solution

The supply and demand will balance when S(t) = D(t)

$200\ (1.05)^{2t} = 500\ (1.05)^t$

Substituting $x = (1.05)^t$ the equation becomes

$200x^2 = 500x$

$200x^2 - 500x = 0$

x(200x – 500) = 0

x = 0 or x = 2.5

Discard x = 0 (not possible since x=$(1.05)^t$>0).

Reverse substation

Logarithms

A logarithm is an exponent, an index or power

The logarithm of a positive quantity p to a given base q is defined as the index or power to which the bases q must be raised to make it equal to P. i.e. $\log_q p = x \; means \; that \; q^x = p$ or x s the logarithm of p to base q

- x is the power (index, logarithm or exponent)
- q is the base
- p is the number (which must be positive)

Example 15

Find the values of x in the following

(a) $\log_2 8 = x$
Solution
8 = 2^3
∴ $\log_2 8 = 3$; x = 3

(b) $\log_x 25 = 2$
Solution
25 = 5^2
⇨ x^2 = 5^2

∴ $x = 5$

Example 16

Evaluate

(a) $\log_{27} 9\sqrt{3}$

Solution

Let $\log_{27} 9\sqrt{3}$ = x

$27^x = 9\sqrt{3}$

$3^{3x} = 3^2.3^{\frac{1}{2}} = 3^{\frac{5}{2}}$

Equating powers

3x =$\frac{5}{2}$

x = $\frac{5}{6}$

∴ $\log_{27} 9\sqrt{3}$ = $\frac{5}{6}$

(b) $\log_{\frac{1}{2}} \frac{1}{4}$

Solution

Let $\log_{\frac{1}{2}} \frac{1}{4}$ = x

$\left(\frac{1}{2}\right)^x = \frac{1}{4}$

$= \left(\frac{1}{2}\right)^2$

Equating powers x = 2

∴ $\log_{\frac{1}{2}} \frac{1}{4}$ = 2

Example 17

Given that $\log_3 x = 2\log_3 4 - \log_3 5 + \log_3 9$, find the value of x. (05marks)

$(1.05)^t$ =2.5

t = $\frac{\log 2.5}{\log 1.05}$ =18.78

Interpretation

After about 19 time units (e.g., years if t is in years), the upgraded supply catches up with demand.

$\log_3 x = \log_3 \frac{4^2 x 9}{5}$

$= \log_3 \frac{16x9}{5}$

$= \log_3 28.8$

Comparing both sides

x = 28.8

Example 18

Given that p = $\log_a (a^3 y^{-2})$ **and q =** $\log_a ay^2$**, find the value of p + q. (05 marks)**

Solution

P + Q = $\log_a (a^3 y^{-2}) + \log_a ay^2$

$= \log_a (a^3 y^{-2} x\ ay^2)$

$= \log_a (a^4)$

$= 4\log_a a$

= 4 x 1 = 4

Example 19

Evaluate $\frac{\log_6 216 + \log_2 64}{\log_3 243 - \log_{10} 0.1}$

Solution

$\frac{\text{Log}_6 216 + \log_2 64}{\log_3 243 - \log_{10} 0.1} = \frac{\log_6 6^3 + \log_2 2^6}{\log_3 3^5 - \log_{10} 10^{-1}}$

$= \frac{3\log_6 6 + 6\log_2 2}{5\log_3 3 - -1\log_{10} 10}$

$= \frac{3x1 + 6x1}{5x1 + 1x1}$

$= \frac{9}{6}$

$= 1.5$

Rules of logarithms

(a) (i) $\log_a a = 1$

Proof

Let $\log_a a = x$

$a^x = a^1$

x = 1

$\therefore \log_a a = 1$

(ii) $\log_a 1 = 0$

Proof

Let $\log_a 1 = x$

$a^x = a^0$

x = 0

$\therefore \log_a 1 = 0$

(b) The power rule

$\log_a P^q = \text{q}\log_a P$

Proof

Let $\log_a P = x$

$a^x = P$

Raising each to the power q

$a^{qx} = P^q$

⇨ $\log_a P = \log_a a^{qx} = qx$

$\therefore \log_a P^q = \text{q}\log_a P$

(c) The addition/multiplication rule

$\log_a pq = \log_a p + \log_a q$

Proof

Let $\log_a p = x$ and $\log_a q = y$

$p = a^x$ and $q = a^y$

$pq = a^x . a^y = a^{x+y}$

$\log_a pq = \log_a a^{x+y} = x + y$

$\therefore \log_a pq = \log_a p + \log_a q$

(d) The subtraction/division rule

$\log_a \left(\frac{p}{q}\right) = \log_a p - \log_a q$

Proof

Let $\log_a p = x$ and $\log_a q = y$

$p = a^x$ and $q = a^y$

$\frac{p}{q} = \frac{a^x}{a^y} = a^{x-y}$

$\log_a \left(\frac{p}{q}\right) = \log_a a^{x-y} = x - y$

$\therefore \log_a \left(\frac{p}{q}\right) = \log_a p - \log_a q$

(e) Change of base

$\log_a p = \frac{\log_q p}{\log_q a}$

Let $\log_a p = x$, then $a^x = p$

⇨ $\log_q a^x = \log_q p$

$x\log_q a = \log_q p$

$x = \frac{\log_q p}{\log_q a}$

$\therefore \log_a p = \frac{\log_q p}{\log_q a}$

Example 20

Evaluate

(a) $\log_2 8\sqrt{2}$

Solution

Either: let $\log_2 8\sqrt{2}$= x

⇨ $2^x = 8\sqrt{2}$

$= 2^3 . 2^{\frac{1}{2}}$

$= 2^{\frac{7}{2}}$

$x = \frac{7}{2}$

$\therefore \log_2 8\sqrt{2} = \frac{7}{2}$

Or $\log_2 8\sqrt{2} = \log_2 \left(2^3 . 2^{\frac{1}{2}}\right)$

$= \log_2 2^{\frac{7}{2}}$

$= \frac{7}{2}\log_2 2$

$= \frac{7}{2}$

(b) $\log_a \frac{1}{a}$

Solution

Let $\log_a \frac{1}{a}$ = x

$a^x = a^{-1}$

$x = -1$

$\therefore \log_a \frac{1}{a}$ = -1

Example 21

Express each of the following as a single logarithm

(a) Log 4 + log3

Solution

Log 4 + log3 = log (4 x 3)

= log 12

(b) Log 5 + log 18 – log3

Solution

Log 5 + log 18 – log3 = log$\left(\frac{5 \, x 18}{3}\right)$

= log30

Example 22

Show that $\log_a p = \frac{1}{\log_p a}$. Hence solve the equation $\log_5 x + 2\log_x 5 = 3$

Solution

Let $\log_a p = x$

⇨ $a^x = p$

Introducing log to base p on both sides

$\log_p a^x = \log_p p$

$x\log_p a = 1$

$x = \frac{1}{\log_p a}$

$\therefore \log_a p = \frac{1}{\log_p a}$

Then,

$\log_5 x + 2\log_x 5 = 3$

$\log_5 x + \frac{2}{\log_5 x} = 3$

Let y = $\log_5 x$

⇨ y + $\frac{2}{y} = 3$

$y^2 - 3y + 2 = 0$

(y – 1)(y – 2) = 0

Either y = 1 or y = 2

When y = 1: $\log_5 x = 1$; x = $5^1 = 5$

When y = 2: $\log_5 x = 2$; x = $5^2 = 25$

x = 5 and x = 25

Example 23

Solve $\log_x 5 + 4\log_5 x = 4$

Expressing terms on LHS to $\log_5$.

$\frac{\log_5 5}{\log_5 x} + 4\log_5 x = 4$

$\frac{1}{\log_5 x} + 4\log_5 x = 4$

Let $\log_5 x = y$

$\frac{1}{y} + 4y = 4$

$4y^2 - 4y + 1 = 0$

(2y – 1)(2y – 1) = 0

2y = 1

$y = \frac{1}{2}$

⇨ $\log_5 x = \frac{1}{2}$

$x = 5^{\frac{1}{2}} = \sqrt{5}$

Example 24

Show that $2\log 4 + \frac{1}{2}\log 25 - \log 20 = 2\log 2$.

Solution

Handling the left hand side

$2\log 4 + \frac{1}{2}\log 25 - \log 20$

$= 2\log 2^2 + \frac{1}{2}\log 5^2 - (\log 4 + \log 5)$

$= 2\log 2^2 + \frac{1}{2}\log 5^2 - \log 4 - \log 5$

$= 4\log 2 + \log 5 - 2\log 2 - \log 5$

= 2log 2

Example 25

(a) (i) Find $\log_9 27\sqrt{3}$ without using tables

Solution

Let $\log_9 27\sqrt{3}$ = x

$9^x = 27\sqrt{3}$

$(3^2)^x = 3^3.3^{\frac{1}{2}}$

$3^{2x} = 3^{\frac{7}{2}}$

Equating indices

$2x = \frac{7}{2}$

x =1.75

Or

Changing the base from 9 to 3

$\log_9 27\sqrt{3} = \frac{\log_3 27\sqrt{3}}{\log_3 9}$

$= \frac{\log_3 27 + \log_3 \sqrt{3}}{\log_3 9}$

$= \frac{\log_3 3^3 + \log_3 3^{\frac{1}{2}}}{\log_3 3^2}$

$= \frac{3+\frac{1}{2}}{2} = \frac{7}{4}$

$= 1.75$

(ii) Simplify $(\log_a b^2)(\log_b a^3)$

$$(\log_a b^2)(\log_b a^3) = (\log_a b^2)\frac{(\log_a a^3)}{\log_a b}$$

$$= (2\log_a b)\frac{(3\log_a a)}{\log_a b}$$

= 2 x 3 = 6

Or

$$(\log_a b^2)(\log_b a^3) = (2\log_a b)(3\log_b a)$$

$$= \left(\frac{2\log_{ba} b}{\log_b a}\right)(3\log_b a)$$

= 2 x 3 = 6

(b) Express $\log_{25} xy$ in terms of $\log_5 x$ *and* $\log_5 y$. Hence solve the simultaneous equations:

$\log_{25} xy = 4\frac{1}{2}$

$\frac{\log_5 x}{\log_5 y}$ = -10

Solution

(ii) By change of base rule

$$\log_{25} xy = \frac{\log_5 xy}{\log_5 25} = \frac{\log_5 x + \log_5 y}{\log_5 5^2}$$

$$= \frac{\log_5 x + \log_5 y}{2}$$

$$\therefore \log_{25} xy = \frac{\log_5 x + \log_5 y}{2}$$

Hence solving

$\log_{25} xy = 4\frac{1}{2}$

$\frac{\log_5 x + \log_5 y}{2} = \frac{9}{2}$

$\log_5 x + \log_5 y = 9$ ………………… (i)

$\frac{\log_5 x}{\log_5 y}$ = -10

$\log_5 x = -10\log_5 y$ …………….(ii)

Substituting eqn. (ii) into eqn. (i)

$-10\log_5 y + \log_5 y = 9$

$\log_5 y = -1$

y = $5^{-1} = \frac{1}{5}$

Substituting $\log_5 y$ into equation (ii)

$\log_5 x = 10$

$x = 5^{10}$

$\therefore\ x = 5^{10}$ and y = $\frac{1}{5}$

Example 26

(a) Given that $\log_b a = x$ show that b =$a^{\frac{1}{x}}$ and deduce $\log_b a = \frac{1}{\log_a b}$

Solution

$\log_b a = x$

$b^x = a$

$(b^x)^{\frac{1}{x}} = a^{\frac{1}{x}}$

$b = a^{\frac{1}{x}}$

Taking log to base a on both sides

$\log_a b = \log_a a^{\frac{1}{x}}$

$\log_a b = \frac{1}{x}\log_a a = \frac{1}{x}$

But x = $\log_b a$

$\therefore \log_a b = \frac{1}{\log_b a}$

(b) Find the value of x and y such that

(i) $\log_{10} x + \log_{10} y = 1.0$

$\log_{10} x - \log_{10} y = \log_{10} 2.5$

Solution

$\log_{10} x + \log_{10} y = 1.0$ ……………(i)

$\log_{10} x - \log_{10} y = \log_{10} 2.5$ …(ii)

Eqn. (i) + eqn. (ii)

2$\log_{10} x$ = $\log_{10} 10 + \log_{10} 2.5$

$\log_{10} x^2 = \log_{10} 25$

⇨ $x^2 = 25$

x = 5

Substituting x into eqn. (i)

$\log_{10} 5 + \log_{10} y = 1.0$

$\log_{10} y = \log_{10} 10 - \log_{10} 5$

$\log_{10} y = \log_{10} 10 \div 5 = \log_{10} 2$

y = 2

Hence x = 5 and y = 2

(ii) Simplify $2^x.3^y = 432$

Solution

$2^x.3^y = 432 = 2^4.3^3$

Comparing exponents

x = 4 and y = 3

(c) Simplify $\frac{1+\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt{3}}$

Solution

By rationalizing

$$\frac{1+\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt{3}} = \frac{(1+\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})}$$

$$= \frac{\sqrt{2}-\sqrt{3}+2-\sqrt{6}+\sqrt{6}-3}{2-3}$$

$$= \frac{\sqrt{2}-\sqrt{3}-1}{-1}$$

$$= 1+\sqrt{3}-\sqrt{2}$$

Example 27

Prove that $\log_6 x = \frac{\log_3 x}{1+\log_3 2}$. Given that $\log_3 2 = 0.631$, find without using tables or calculator $\log_6 4$ correct to 3significant figures

Solution

$$\log_6 x = \frac{\log_3 x}{\log_3 6} = \frac{\log_3 x}{\log_3 (2\,x 3)} = \frac{\log_3 x}{\log_3 3 + \log_3 2}$$

$$= \frac{\log_3 x}{1+\log_3 2}$$

Substituting for $\log_3 2 = 0.631$

$$\log_6 x = \frac{\log_3 2^2}{1+\log_3 2} = \frac{2\log_3 2}{1+\log_3 2} = \frac{2\,x\,0.631}{1+0.631}$$

$$= 0.774$$

Revision exercise 2

1. Evaluate
   (a) $\log_{\frac{1}{5}} 25\sqrt{5}$ $\left[-\frac{5}{2}\right]$
   (b) $\log_3 27$ [3]
2. Express the following as a single logarithm
   (i) Log15 - $\frac{1}{2}log9$ [log 5]
   (ii) 3log2 + 2log5 – log 20 [log 10]
3. Given that $\log_b a$ and $\log_c b = a$, show that $\log_c a = ac$
4. (a) solve the equation
   (i) $\log_a 4 + \log_4 a^2$
   $[a = 2 \text{ and } a = 4]$
   (ii) $\log_{14} x = \log_7 4x$ $\left[\frac{1}{196}\right]$
5. Without using tables or calculator show that $\frac{2log9 + log8 - \log 375}{\frac{1}{3}log6 - log5^{\frac{1}{3}}}$= 9
6. If$\log_2 x + \log_4 x + \log_{16} x = \frac{21}{16}$. Find the value of x [x = 1.6818]
7. Given$\log_a b = \log_d c$, show that $\log_c a = \log_d b$. Hence or otherwise solve the equation$\log_{9x} 64 = \log_x 4$. [x=3]
8. Solve the simultaneous equations
   $\log_{10}(y-x)$= 0
   $2\log_{10}(21+x)$ [(x,y) =(-5, -4) or (4,5)
9. Given that $\log_2 x + 2\log_4 y = 4$. Show that xy = 16. Solve simultaneous equations
   $10\log_{10}(x+y) = 1$
   $\log_2 x + 2\log_4 y = 4$. [(x, y) =(2,8)or (8,2)
10. (a) If $\log_b a = x$, show that b = $a^{\frac{1}{x}}$ and deduce that $\frac{1}{\log_a b}$.

(b) Solve

(i) $\log_x 4 + \log_4 x^2 = 3$ $[x = 2 \text{ or } 4]$

(ii) $2^{2x-1} + \frac{3}{2} = 2^{x+1}$[x=0 or 1.585]

11. Prove that $\log_8 x = \frac{2}{3}\log_4 x$. Hence without using tables or calculator, evaluate $\log_8 6$ correct to three significant figure, if $\log_4 3 = 0.7925$ [0.862]
    (e) $\sqrt{x+2} - x = 0$[x=2]
    (f) $\sqrt{1+x} = 1 + \sqrt{1-x}$ $\left[x = \frac{\sqrt{3}}{2}\right]$
    (g) $(3-x)^{\frac{1}{2}} = (1+x)^{\frac{1}{2}} + (2-x)^{\frac{1}{2}}$ [x = -0.92665]
    (h) $\sqrt{x+6}$ =$\sqrt{1-3x} - \sqrt{4-x}$ [-5]

10. Without using mathematical tables or calculators, find the value of $\frac{(\sqrt{5}+2)^2-(\sqrt{5}-2)^2}{8\sqrt{5}}$ [1]
11. Find the square root of the following

(d) $6 + 2\sqrt{5}$ $[\pm(1+\sqrt{5})]$
(e) $18 - 2\sqrt{12}$ $[\pm(\sqrt{0.695}-\sqrt{17.303})]$
(f) $18 - 2\sqrt{2}$ $[\pm(\sqrt{0.1118}-\sqrt{17.8882})]$

Thank You

Dr. Bbosa Science