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Subsidiary Mathematics

SENIOR FIVE term 2

TOPIC 2/4: Series

Series

Introductions

Numbers arranged in a definite order a sequence. Each number in the sequence is derived from a particular rule.

The terms below are examples of sequences

- (a) 1, 3, 5, 7, 9 is a sequence of odd numbers
- (b) 2, 3, 5, 7, 11 ... is a sequence of prime number
- (c) 4, 16, 64 is a sequence formed by multiplying the preceding number by 4 to give the next number

Series are categorized into two:

- Arithmetic progression (A.P)
- Geometric progression

Arithmetic progression (A.P)

This is a series in which each term is obtained from the preceding one by addition or subtraction of a constant quantity.

The series 1 + 3 + 5 + 7 + 9 is an A.P

Note the following in an A.P

- (i) The first term of an A.P is denoted a. the first letter of the English alphabet
- (ii) There is a common difference d. in the progression, a = 1 and d = 2.
- (iii) Given the first term, a and the common difference, d
 - 1st term = a
 - 2nd term = a + d
 - 3rd term = a + 2d
 - nth term (U_n) = a + (n - 1)d
- (iv) The sum (S_n) of the first n terms of an A.P is given by,

$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

Or

$$S_n = \frac{n}{2}(a + L) \text{ where } L \text{ is the last term.}$$

Example 1

Find the 30th term of a series that has an nth term given by $\frac{1}{2}(32 - n)$

Solution

$$U_n = \frac{1}{2}(32 - n)$$

$$U_{30} = \frac{1}{2}(32 - 30) = 1$$

Example 2

The first term of an arithmetic progression (A.P) is 73 and the 9th term is 25. Determine the common difference

Solution

$$U_n = a + (n - 1)d$$

$$25 = 73 + (9 - 1)d$$

$$25 = 73 + 8d$$

$$d = -6$$

Example 3

The 3rd, 5th and 8th terms of A.P are $3n + 8$, $n + 34$, and $n^3 + 15$ respectively. Find the value of n and hence the common difference of the A.P

Solution

$$a + 2d = 3n + 8 \dots\dots\dots (i)$$

$$a + 4d = n + 24 \dots\dots\dots(ii)$$

$$a + 7d = n^3 + 15 \dots\dots\dots(iii)$$

$$\text{Eqn. (ii) - eqn. (i)}$$

$$2d = -2n + 16$$

$$d = -n + 8 \dots\dots\dots (iv)$$

$$\text{eqn. (iii) - eqn. (i)}$$

$$5d = n^3 - 3n + 7 \dots\dots\dots (v)$$

Substituting eqn. (iv) into eqn. (v)

$$a = 2$$

Substitute for a in (v)

$$36 + b = 33$$

$$b = -3$$

Substituting for a and b in eqn. (i)

$$3 \times 2 - 3 + c = 4$$

$$c = 1$$

Hence $a = 2$, $b = -3$ and $c = 1$

Example 4

The ninth term of an Arithmetic Progression (A.P) is greater than the fifth term by 6. The sum of the first twelve terms is 123. Find the

(a) common difference of the A.P

Solution

$$n^{\text{th}} \text{ term} = a + (n-1)d$$

$$9^{\text{th}} \text{ term} = a + 8d$$

$$5^{\text{th}} \text{ term} = a + 4d$$

$$\Rightarrow a + 8d = a + 4d + 6$$

$$4d = 6$$

$$d = 1.5$$

(b) first term of the A.P.

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$123 = \frac{12}{2}(2a + 11 \times 1.5)$$

$$6(2a + 16.5) = 123$$

$$12a = 24$$

$$a = 2$$

Example 5

The sum of the first 16 terms of an arithmetic progression (A. P) is 1088. The 16th term is twice the 8th term. Determine the value of the first term of the A.P. (05marks)

Solution

Let

The first term = a

And common difference = d

n^{th} term in A.P = $a + (n - 1)d$

$$8^{\text{th}} \text{ term} = a + 7d$$

$$16^{\text{th}} \text{ term} = a + 15d$$

$$\text{But } 2(a + 7d) = a + 15d$$

$$2a + 14d = a + 15d$$

$$\Rightarrow a = d$$

Sum (S_n) of the first n terms is given by

$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

$$\Rightarrow 1088 = \frac{16}{2}(2a + (16 - 1)d)$$

$$1088 = 8(2a + 15d)$$

$$\text{But } a = d$$

$$\Rightarrow 1088 = 8 \times 17a$$

$$a = 8$$

Hence the first term is 8

Example 6

The first term of A.P is 73 and the common difference is -6, find the number of terms that must be added to give a sum of 96

Solution

$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

$$\frac{n}{2}(2 \times 73 - 6(n - 1)) = 96$$

$$n(73 - 3(n - 1)) = 96$$

$$73n - 3n^2 + 3n = 96$$

$$3n^2 - 76n + 96 = 0$$

$$n = \frac{76 \pm \sqrt{76^2 - 4 \times 3 \times 96}}{2 \times 3}$$

$$n = \frac{76 \pm 68}{6}$$

$$n = 24$$

Hence the number of terms that must be added to give a sum of 96 are 24

Example 7

The tenth term of an arithmetic progression (A.P) is 29 and the fifteenth term is 44. Find the value of the first term and the common difference. Hence find the sum of the first 30 terms

Solution

$$U_n = a + (n - 1)d$$

$$U_{10} = a + (10 - 1)d$$

$$a + 9d = 29 \dots\dots\dots(i)$$

$$U_{15} = a + (15 - 1)d$$

$$a + 14d = 44 \dots\dots\dots(ii)$$

$$\text{Eqn (ii) - eqn. (i)}$$

$$5d = 15$$

$$d = 3$$

Substituting d in eqn. (i)

$$a + 9 \times 3 = 29$$

$$a = 2$$

Sum of the first 30 terms

$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

$$S_n = \frac{30}{2}(2 \times 2 + 3(30 - 1)) = 1365$$

Example 8

The 5th term of an arithmetic progression (A.P) is 12 and the sum of the first 5 terms is 80. Determine the first term and common difference.

Solution

$$U_n = a + (n - 1)d \text{ [} U_n = \text{nth term, } a = \text{first term and } d = \text{common difference}]$$

$$U_5 = a + (5 - 1)d$$

$$a + 4d = 12 \dots\dots\dots(i)$$

The sum of the first n terms,

$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

$$S_5 = \frac{5}{2}(2a + (5 - 1)d)$$

$$10a + 20d = 80 \times 2 = 160 \dots\dots\dots(ii)$$

$$\text{Eqn. (ii)} - 5\text{eqn. (i)}$$

$$5a = 100$$

$$a = 20$$

Substituting for a in eqn. (i)

$$20 + 4d = 12$$

$$d = -2$$

Hence the first term = 20 and the common difference = -2

Example 9

(a) Prove that $\sum_{r=1}^n r = \frac{n}{2}(n + 1)$

Solution

$$\begin{aligned} S_n &= 1 + 2 + 3 + \dots + n \\ + S_n &= n + (n - 1) + (n - 2) + \dots + 1 \\ \hline 2S_n &= (n+1) + (n+1) + (n+1) + \dots + (n+1) \\ 2S_n &= n(n + 1) \\ S_n &= \frac{n}{2}(n + 1) \end{aligned}$$

(b) Use your answer in (a) to deduce

(i) $\sum_{r=1}^n (3r - 1) = \frac{n}{2}(3n + 1)$

Note to deduce is to use the already existing result to work out other problems

$$\begin{aligned} S_n &= \frac{n}{2}[\text{last term} + \text{first term}] \\ \sum_{r=1}^n (3r - 1) &= \frac{n}{2}(3n - 1 + (3 - 1)) \\ &= \frac{n}{2}(3n - 1 + 2) \\ &= \frac{n}{2}(3n + 1) \end{aligned}$$

(ii) $\sum_{r=0}^n (r + 5) = \frac{1}{2}(n + 1)(n + 10)$

$$= \sum_{r=1}^n (r + 5) + \sum_{r=0} (r + 5)$$

$$= \frac{n}{2}((n + 1) + (1 + 5)) + 5$$

$$= \frac{1}{2}(n^2 + 11n) + 5$$

$$= \frac{1}{2}(n^2 + 11n + 10)$$

$$= \frac{1}{2}(n + 1)(n + 10)$$

Inserting geometric means

Like for A.Ps, the terms inserted between given two values of a G.P are known as geometric means.

If n terms are inserted, then the total number of terms will be n+ 2 with the two extreme values representing the first and last terms respectively

Example 10

(a) Insert two geometric means between 2 and 16

Solution

1st term a = 2

4th term, $ar^3 = 16$

$$2(r^3) = 18$$

$$r = 2$$

the second term, $ar = 2 \times 2 = 4$

the third term, $ar^2 = 2 \times 2^2 = 8$

(b) Insert three geometric means between 1 and 81

a = 1

the 5th term $ar^4 = 81$

$$1(r^4) = 81$$

$$r = 3$$

the second term, $ar = 1 \times 3 = 3$

the third term, $ar^2 = 1 \times 3^2 = 9$

the third term, $ar^3 = 1 \times 3^3 = 27$

Revision exercise 1

- Find the 5th and 8th terms of a series that has an nth term given by $(-1)^n(2n + 1)$
[-11, 17]
- The first term of an arithmetic progression (A.P) is $\frac{1}{2}$. The sixth term of the A.P is four times the fourth term. Find the common difference of the A.P $\left[\frac{-3}{14}\right]$
- The sum of p terms of an arithmetic progression is q and the sum of q terms is p; find the sum of p + q terms
- (a) The first four terms of an A.P are 5, 11, 17 and 23. Find the 30th term and the sum of the first 30 terms [179, 2760]
(b) the second term of an A.P is 7 and the 7th term is -8. Find the first term, common difference and the sum of the first 14 terms [10, -3, -133]
- (a) An A.P has the first term of 2 and common difference 5. Given that the sum of the first n terms of the progression is 119, calculate n [7]
(b) the sum of the first five terms of an A.P is $\frac{65}{2}$. Also, five times the 7th term is the same as six times the second term. Find the first term and the common difference
 $\left[a = 6, d = \frac{1}{4}\right]$
- The sum of the first n terms of the series is $n(n + 2)$. Find the first three terms. [3, 5, 7]
- In an A.P, the 1st term is 13 and 15th term is 11. Find the common difference and sum of the first 20 terms [7, 1590]
- (a) Show that $\ln 2^r$, $r = 1, 2, 3$, is an arithmetic progression
(b) find the sum of the first 10 terms of the progression [38.1231]
(c) Determine the least value of m for which the sum of the first 2m terms exceeds 883.7 [25]
- In an arithmetic progression $u_1 + u_2 + u_3 + u_4 = 15$ and $u_{16} = -3$. Find the greatest integer

N such that $U_N \geq 0$. Determine the sum of the first N terms of the progression. [$N = 14, S_{14} = 136.5$]

Geometric progression (G.P)

It is a series in which each term is obtained from the preceding one by multiplication or division by a constant quantity.

Observations

- The first term of G.P is also denoted, a
- The common ratio is r
- Given a and r
1st term = a
2nd term = ar
3rd term = ar^2
4th term = ar^3
nth term = ar^{n-1}

The sum (S_n) of the first n terms of G.P

The sum (S_n) of the first n terms of G.P is given by

$$S_n = \frac{a(r^n - 1)}{r - 1} \text{ for } r > 1 \text{ or}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} \text{ for } r < 1$$

Sum to infinity of a G.P

We have seen that the sum of n terms of a G.P

$$\text{for } r < 1 \text{ is } S_n = \frac{a(1 - r^n)}{1 - r}$$

Now for $-1 < r < 1$ i.e. $|r| < 1$, as $n \rightarrow \infty$, $r^n \rightarrow 0$

$$\text{Therefore } S_n = \frac{a(1 - 0)}{1 - r} = \frac{a}{1 - r}$$

Hence the sum of a GP to infinity for $|r| < 1$ converges to $S_\infty = \frac{a}{1 - r}$ and diverges for $r > 1$ and $r < -1$

The geometric mean (G.M)

Suppose three numbers a, b and c are consecutive terms of GP, then, the middle term is the geometric mean.

The common ratio, $r = \frac{b}{a} = \frac{c}{b}$

$$\Rightarrow b^2 = ac$$

$$b = \sqrt{ac}, \text{ the geometric mean}$$

Example 11

The sum to infinity of a Geometric Progression (GP) is $\frac{25}{4}$ and the first term is 5. Find the

(a) common ratio of the GP. (03marks)

$$S_{\infty} = \frac{a}{1-r}$$

$$\Rightarrow \frac{5}{1-r} = \frac{25}{4}$$

$$25(1-r) = 20$$

$$25r = 5$$

$$r = \frac{1}{5} = 0.2$$

(b) sum of the first ten terms of the GP. (02marks)

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\Rightarrow S_{10} = \frac{5(1-(0.2)^{10})}{1-0.2} = 6.25$$

Example 12

Find the sum of the first ten terms of the geometric progression (G.P); $8 + 4 + 2 + \dots$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$a = 8, r = \frac{4}{8} = \frac{1}{2}, n = 10$$

$$S_{10} = \frac{8(1-(0.5)^{10})}{1-(0.5)} = 16$$

Example 13

A GP has 3rd term 7 and 5th term 847. Find the possible values of the common ratio and the corresponding 4th terms

Solution

$$U_3 = ar^2 = 7$$

$$\Rightarrow ar^2 = 7 \dots\dots\dots(i)$$

$$U_5 = ar^4 = 847$$

$$\Rightarrow ar^4 = 847 \dots\dots\dots(ii)$$

Eqn. (ii) ÷ eqn. (i)

$$r^2 = 121$$

$$r = \pm 11$$

From eqn (i)

$$a = \frac{7}{121}$$

The 4th term, $U_4 = ar^3$

If $r = 11, U_4 = \frac{7}{121} (11)^3 = 77$

If $r = -11, U_4 = \frac{7}{121} (-11)^3 = -77$

Example 14

In a G.P the 2nd term is 15 and the 5th term is -405. Find the sum of the first 8 terms

Solution

$$U_2 = ar = 15$$

$$\Rightarrow ar = 15 \dots\dots\dots(i)$$

$$U_5 = ar^4 = -405$$

$$\Rightarrow ar^4 = -405$$

Eqn. (ii) ÷ eqn. (i)

$$r^3 = -27$$

$$r = -3$$

From eqn. (i), $a = -5$

Since $r < 1$

$$S_n = \frac{a(1-r^n)}{r-1} = \frac{-5(1-(-3)^8)}{1-(-3)} = \frac{32800}{4} = 8200 =$$

Example 15

In the geometric series $u_1 + u_2 + u_3 + \dots$

$$u_1 + u_3 = 26 \text{ and } u_3 + u_5 = 650.$$

Find the possible values of u_4

Solution

$$u_1 + u_3 = 26$$

$$a + ar^2 = 26$$

$$a(1 + r^2) = 26 \dots\dots\dots (i)$$

$$u_3 + u_5 = 650.$$

$$ar^2 + ar^4 = 650$$

$$ar^2(1 + r^2) = 650 \dots\dots\dots(ii)$$

$$\text{Eqn. (ii)} \div \text{eqn. (i)}$$

$$r^2 = 25$$

$$r = \pm 5$$

From eqn. (i)

$$a(1 + 25) = 26$$

$$a = 1$$

$$u_4 = ar^3$$

$$\text{If } r = 5; u_4 = a(5)^3 = 125$$

$$\text{If } r = -5; u_4 = a(-5)^3 = -125$$

Example 16

In a Geometric Progression (G.P), the difference between the fifth and the second term is 156. The difference between the seventh and the fourth is 1404 Find the possible values of the common ratio.

$$U_5 - U_2 = 156$$

$$ar^4 - ar = 156$$

$$ar(r^3 - 1) = 156 \dots\dots\dots (i)$$

$$U_7 - U_4 = 156$$

$$Ar^6 - ar^3 = 1404$$

$$ar^3(r^3 - 1) = 156 \dots\dots\dots (ii)$$

$$\text{Eqn. (ii)} \div \text{eqn. (i)}$$

$$\frac{ar^3(r^3-1)}{ar(r^3-1)} = \frac{1404}{156}$$

$$r^2 = 9$$

$$r = \pm 3$$

$$\therefore r = 3 \text{ and } r = -3$$

Example 17

(a) The first three terms of a Geometric progression (G.P) are 4, 8 and 16. Determine the sum of the first ten terms of the G.P. (04marks)

solution

$$a = 4, ar = 8$$

$$4r = 8$$

$$r = 2$$

$$S_n = \frac{a(1-r^n)}{r-1}$$

$$S_{10} = 4\left(\frac{2^{10}-1}{2-1}\right) = 4092$$

(b) An Arithmetic Progression (A.P) has a common difference of 3. A Geometric Progression (G.P) has a common ratio of 2. A sequence is formed by subtracting the term of the A.P from the corresponding terms of the G.P. The third term of the sequence is 4. The sixth term of the sequence is 79. Find the first term of the

(i) A.P (08 marks)

(ii) G.P (06 marks)

A.P

$$x, x+3, x+6, x+9, x+12, x+15, \dots\dots$$

G.P

$$y, 2y, 4y, 8y, 16y, 32y, \dots\dots$$

$$4y - (x + 6) = 4$$

$$4y - x = 10 \dots\dots\dots(i)$$

$$32y - (x + 15) = 79$$

$$32y - x = 94 \dots\dots\dots (ii)$$

$$\text{Eqn. (ii)} - \text{Eqn. (i)}$$

$$28y = 84, \Rightarrow y = 3$$

Substituting for y into eqn. (i)

$$12 - x = 10$$

$$x = 2$$

(i) A.P, $U_1 = 2$

(ii) G.P, $U_1 = 3$

Example 18

The sum of the first n terms of a geometric progression (G.P) is $\frac{4}{3}(4^n - 1)$. Find the n^{th} term as an integral power of 2

Solution

$$S_n = \frac{a(1 - r^n)}{r - 1}$$

Comparing with $S_n = \frac{4}{3}(4^n - 1)$

$$a = 4$$

$$r - 1 = 3, r = 4$$

The n^{th} term, $U_n = ar^{n-1}$

$$= 4 \times 4^{n-1} = 2^2 \times 2^{2(n-1)} = 2^{2+2n-2} = 2^{2n}$$

Example 19

Find three numbers in geometrical progression such that their sum is 26 and their product is 216

Solution

Let the numbers be $\frac{a}{r}$, a and ar

$$\text{Product} = \left(\frac{a}{r}\right)(a)(ar) = 216$$

$$a^3 = 216 = 6^3$$

$$a = 6$$

$$\therefore \text{the terms are } \frac{6}{r}, 6 \text{ and } 6r$$

$$\text{Sum of terms } \frac{6}{r} + 6 + 6r = 26$$

$$\Rightarrow 6r^2 - 26r + 6 = 0$$

$$3r^2 - 13r + 3 = 0$$

$$(r - 3)(3r - 1) = 0$$

$$\text{Either } r - 3 = 0; r = 3$$

$$\text{Or } 3r - 1 = 0; r = \frac{1}{3}$$

When $r = 3$

the terms are $\frac{6}{3} = 2, 6$ and $6 \times 3 = 18$

when $r = \frac{1}{3}$

the terms are $6 \div \frac{1}{3} = 18, 6$ and $6 \times \frac{1}{3} = 2$

Hence the terms in their order are 2, 6, 18 or 18, 6, 2

Inserting geometric means

Like for A.Ps, the terms inserted between given two values of a G.P are known as geometric means.

If n terms are inserted, then the total number of terms will be $n + 2$ with the two extreme values representing the first and last terms respectively

Example 20

(c) Insert two geometric means between 2 and 16

Solution

$$1^{\text{st}} \text{ term } a = 2$$

$$4^{\text{th}} \text{ term, } ar^3 = 16$$

$$2(r^3) = 18$$

$$r = 2$$

the second term, $ar = 2 \times 2 = 4$

the third term, $ar^2 = 2 \times 2^2 = 8$

(d) Insert three geometric means between 1 and 81

$$a = 1$$

$$\text{the } 5^{\text{th}} \text{ term } ar^4 = 81$$

$$1(r^4) = 81$$

$$r = 3$$

the second term, $ar = 1 \times 3 = 3$

the third term, $ar^2 = 1 \times 3^2 = 9$

the fourth term, $ar^3 = 1 \times 3^3 = 27$

Mixed terms of A.P and G.P

These are problems involving both A.Ps and G.Ps. when handling we make use of their respective properties.

Example 21

A geometric progression (G.P) and an arithmetic progression (A.P) have the same first term. The sum of their first, second and third terms are 6, 10.5 and 18 respectively. Calculate the sum of their 5th terms.

Solution

Terms	G.P	A.P	Sum
1st	a	a	$2a = 6$(i)
2nd	a + d	ar	$a + d + ar = 10.5$(ii)
3 rd	a + 2d	ar ²	$a + 2d + ar^2 = 18$..(iii)

From eqn. (i): $2a = 6$; $a = 3$

From eqn. (ii): $3 + d + 3r = 10.5$

$$d + 3r = 7.5 \text{ (iv)}$$

From eqn. (iii) $3 + 2d + 3r^2 = 18$

$$2d + 3r^2 = 15 \text{(v)}$$

Eqn. (v) – 2eqn. (iv)

$$3r^2 - 6r = 0$$

$$3r(r - 2) = 0$$

$$r - 2 = 0$$

$$r = 2$$

Substitute for r into eqn. (iv)

$$d + 6 = 7.5$$

$$d = 1.5$$

Sum of their fifth terms

$$= (a + 4d) + ar^4$$

$$= (3 + 4 \times 1.5) + 3 \times 2^4 = 57$$

Example 22

The 1st, 4th and 8th terms of A.P form a G.P. if the first term is 9, find the

- Common difference of the A.P
- Common ratio of the G.P
- Difference in sums of the first 6 terms of the progressions.

Solution

Given that a, a + 3d, a + 7d form a G.P

Substituting for a = 9, the terms are

9, 9 + 3d, 9 + 7d

$$\text{For a G.P, } r = \frac{9+3d}{9} = \frac{9+7d}{9+3d}$$

$$\Rightarrow (9 + 3d)^2 = 9(9 + 7d)$$

$$81 + 54d + 9d^2 = 81 + 63d$$

$$9d^2 - 9d = 0$$

$$9d(d - 1) = 0$$

$$\text{Either } d - 1 = 0; d = 1$$

$$\text{Or } d = 0$$

When d = 0 all terms of A.P are equal

Hence the common difference d = 1

Example 23

- The sum of the first m terms of a progression is $m(2m + 11)$
 - Show that the progression is an A.P
 - Determine the nth term of the progression

Solution

$$\text{Given } S_m = m(2m + 11)$$

$$\text{First term} = S_1 = 1(2 \times 1 + 11) = 13$$

$$S_2 = 2(2 \times 2 + 11) = 30$$

$$\text{Second term} = 30 - 13 = 17$$

$$S_3 = 3(2 \times 3 + 11) = 51$$

$$\text{Third term} = 51 - 20 = 21$$

The progression is 13, 17, 21, Hence A.P with the first term 13 and common difference, $d=4$

$$\begin{aligned} \text{(ii) } U_n &= S_n - S_{n-1} \\ &= n(2n + 11) - (n - 1)(2(n - 1) + 11) \\ &= 9 + 4n \end{aligned}$$

Example 24

(a) The first, second and last term of an A.P are a, b, c respectively. Prove that the sum of all terms is $\frac{(a+b)(b+c-2a)}{2(b-a)}$

Solution

If the 1st term is a and the second term is b ; the common difference, $d = (b - a)$

Last term (nth terms) $c = a + (n - 1)d$

$$c = a + (n - 1)(b - a)$$

$$\text{i.e. } n - 1 = \frac{c - a}{b - a} \Rightarrow n = \frac{b + c - 2a}{b - a}$$

$$\text{but } S_n = \frac{1}{2}n(a + L)$$

$$= \frac{1}{2} \left(\frac{b + c - 2a}{b - a} \right) (a + c)$$

$$= \frac{(a + b)(b + c - 2a)}{2(b - a)}$$

(b) The first, second and last terms of a GP are a and b . show that the sum of the first n terms is $\frac{a^n - b^n}{a^{n-2}(a - b)}$

Solution

$$\text{Common ratio} = \frac{b}{a}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a \left(1 - \left(\frac{b}{a} \right)^n \right)}{1 - \left(\frac{b}{a} \right)}$$

$$= \frac{a(a^n - b^n)a}{a^n(a - b)} = \frac{a^n - b^n}{a^{n-2}(a - b)}$$

Example 25

(a) Calculate the sum to infinity of the following terms

$$\text{(i) } 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

Solution

$$a = 1 \text{ and } r = \frac{1}{2}$$

$$S_\infty = \frac{a}{1 - r} = \frac{1}{1 - \frac{1}{2}} = 2$$

$$\text{(ii) } \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \dots$$

Solution

$$a = \frac{1}{5} \text{ and } r = \frac{1}{5}$$

$$S_\infty = \frac{a}{1 - r} = \frac{\frac{1}{5}}{1 - \frac{1}{5}} = \frac{1}{4}$$

(b) Work out the following

$$\text{(i) } \sum_{r=0}^{\infty} \left(\frac{1}{3} \right)^r$$

Solution

$$\sum_{r=0}^{\infty} \left(\frac{1}{3} \right)^r = 1 + \frac{1}{3} + \frac{1}{9} + \dots$$

$$a = 1 \text{ and } r = \frac{1}{3}$$

$$S_\infty = \frac{a}{1 - r} = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$

$$\text{(ii) } \sum_{r=2}^{\infty} \left(-\frac{1}{8} \right)^r$$

Solution

$$\sum_{r=2}^{\infty} \left(-\frac{1}{8} \right)^r = \frac{1}{64} - \frac{1}{512} + \frac{1}{4096} - \dots$$

$$A = \frac{1}{64} \text{ and } r = -\frac{1}{8}$$

$$S_\infty = \frac{a}{1 - r} = \frac{\frac{1}{64}}{1 + \frac{1}{8}} = \frac{1}{72}$$

$$\text{(iii) } \sum_{r=0}^{\infty} a^r$$

Solution

$$a = 1 \text{ and } r = a$$

$$S_\infty = \frac{a}{1 - r} = \frac{1}{1 - a}$$

$$\text{(iv) } \sum_{r=1}^{\infty} (3x)^{r+1}$$

Solution

$$\sum_{r=1}^{\infty} (3x)^{r+1} = 9x^2 + 27x^3 + 81x^4 + \dots$$

$$a = 9x^2 \text{ and } r = 3x$$

$$S_\infty = \frac{a}{1 - r} = \frac{9x^2}{1 - 3x}$$

Example 26

(a) Express the following as fractions using approach of sum of a G.P to infinity

(i) $0.\dot{4}$

Solution

$$\begin{aligned} 0.\dot{4} &= \frac{4}{10} + \frac{4}{100} + \frac{1}{1000} + \dots \\ &= \frac{4}{10} \left(1 + \frac{1}{10} + \frac{1}{100} + \dots \right) \\ &= \frac{4}{10} \left(\frac{1}{1 - \frac{1}{10}} \right) = \frac{4}{10} \times \frac{10}{9} = \frac{4}{9} \end{aligned}$$

(ii) $3.1\dot{2}\dot{7}$

Solution

$$\begin{aligned} 3.1\dot{2}\dot{7} &= 3 + \frac{1}{10} + \frac{27}{1000} + \frac{27}{10000} + \dots \\ &= 3 + \frac{1}{10} + \frac{27}{1000} \left(1 + \frac{1}{100} + \frac{1}{10000} + \dots \right) \\ &= \frac{31}{10} + \frac{27}{1000} \left(\frac{1}{1 - \frac{1}{100}} \right) = \frac{31}{10} + \frac{27}{1000} \left(\frac{100}{99} \right) \\ &= \frac{31}{10} + \frac{3}{110} = \frac{344}{110} = \frac{172}{55} \end{aligned}$$

Hence $3.1\dot{2}\dot{7} = \frac{172}{55}$

(b) The sum to infinity of a GP is 7 and the sum of the first two terms is $\frac{48}{7}$. Find the common ratio and the first term of the GP with positive common ratio

Solution

$$\begin{aligned} S_{\infty} &= 7 \\ \Rightarrow \frac{a}{1-r} &= 7 \\ a &= 7(1-r) \dots\dots\dots(i) \end{aligned}$$

But $S_2 = \frac{48}{7}$

$$a + ar = \frac{48}{7}$$

$$a(1+r) = \frac{48}{7} \dots\dots\dots(ii)$$

substituting eqn. (i) into eqn. (ii)

$$7(1-r)(1+r) = \frac{48}{7}$$

$$49(1-r^2) = 48$$

$$49r^2 = 1$$

$$r^2 = \frac{1}{49}$$

$$r = \pm \frac{1}{7}$$

Considering appositive ratio

From eqn. (i)

$$a = 7\left(1 - \frac{1}{7}\right) = 6$$

Revision exercise 2

- The common ration of a GP is -5 and the sum of the first seven terms of the progression is 449. Find the first three terms. $\left[\frac{1}{29}, \frac{-5}{29}, \frac{25}{29}\right]$
- In the geometrical series $\sum_{r=1}^n u_r$, $u_5 - u_2 = 156$ and $u_7 - u_4 = 1404$. Find the possible values of the common ratio and corresponding values of u_1 [
 $r = 3, a = 156; r = -3, a = \frac{13}{7}$
- The sum of the second and third terms of a G.P is 9. It the seventh term is eight times the fourth term, find the
(a) The first term and the common ratiom $[a = \frac{3}{2} \text{ and } r = 2]$
(b) The sum of the fourth and first term $[36]$
- Find the sum of ten terms of geometrical series 2, -4, 8 $[-682]$
- The second and the third terms of a G.P progression are 24 and $12(b + 1)$ respectively. Find b if the sum of the first three terms of the progression is 76 $\left[\frac{1}{2} \text{ or } 2\right]$
- The sum of the 2nd and 3rd terms of a G.P is 12. The sum of the 3rd and 4th terms is -36. Find the 1st term and common ratio $[a = 2, r = -3]$
- What is the smallest number of terms of GP 5, 10, 20 that can give a sum greater than 500, 000 $[n = 17]$

8. The first, fourth and eighth terms of Arithmetic progression (A.P) form a geometric progression. If the first term is 9, find
- The common difference of A.P [1]
 - The common ratio of the G.P $\left[\frac{4}{3}\right]$
 - The difference in the sums of the first 6 terms of the progressions [55.7049]
9. The second, third and ninth terms of an A.P form a G.P. find the common ratio of the G.P [6]
10. (a) The sum of the first 10 terms of an AP is 120. The sum of the next 8 terms is 240. Find the sum of the next 6 terms [264]
 (b) the arithmetic mean of the a and b is three times their geometric mean. Show that $\frac{a}{b} = 7 \pm 12\sqrt{2}$
11. The first three terms of a geometric series are 1, p, and q. Given also that 10, q and p are the first three terms of an arithmetic series. Show that $2p^2 - p - 10 = 0$
 Hence find the possible values of p and q [p = -2 and q = 4 or p = $\frac{5}{2}$ and q = $\frac{25}{4}$]

Application of A.Ps and G.Ps to interest rates

If a sum of money P is invested at a simple interest rate of r% per annum, the amount received after n years is given by $A = P + I$ where $I = \frac{P \times r \times n}{100}$

By substitution we have

$$A = P + \frac{P \times r \times n}{100} = P \left(1 + \frac{nr}{100}\right)$$

The interest for one year is $\frac{Pr}{100}$, for 2 years is $\frac{2Pr}{100}$, for n year = $\frac{nPr}{100}$. Therefore the various amounts of interest after one, two, three, etc. years form an AP

On the other hand, if the principal P is invested at compound interest rate of r% per annum, the interest being added annually, the amount after

one year is $\left(1 + \frac{r}{100}\right)$, after two years is $P \left(1 + \frac{r}{100}\right)^2$, after 3 years is

$$P \left(1 + \frac{r}{100}\right)^3 \text{ and after n years } P \left(1 + \frac{r}{100}\right)^n$$

Hence the amounts after one, two, three, etc. years for a GP.

Note: if with compound interest is added half annually as much as when added yearly, but it is added twice as much. Hence amount A =

$$P \left(1 + \frac{r}{100}\right)^{2n}$$

Now suppose that instead of adding the interest annually, it is the principal, P which is added annually,

$$\text{Amount after 1}^{\text{st}} \text{ year} = P \left(1 + \frac{r}{100}\right)^1$$

$$\text{Amount after 2}^{\text{nd}} \text{ year} = P \left(1 + \frac{r}{100}\right)^2$$

$$\text{Amount after 3}^{\text{rd}} \text{ year} = P \left(1 + \frac{r}{100}\right)^3$$

$$\text{Amount after n}^{\text{th}} \text{ year} = P \left(1 + \frac{r}{100}\right)^n$$

Total amount after n years

$$= P \left(1 + \frac{r}{100}\right)^1 + P \left(1 + \frac{r}{100}\right)^2 + \dots + P \left(1 + \frac{r}{100}\right)^n$$

$$= P \left[\left(1 + \frac{r}{100}\right)^1 + \left(1 + \frac{r}{100}\right)^2 + \dots + \left(1 + \frac{r}{100}\right)^n \right]$$

This is a G.P with:

$$\text{first term} = \frac{100+r}{100} = (100+r)\%$$

$$\text{And common ratio} = \frac{100+r}{100} = (100+r)\%$$

$$S_n = P(100+r)\% \left[\frac{(100+r)\%-1}{(100+r)\%-1} \right]$$

Example 23

- (a) Find the amount at the end of ten years when 500000 shillings is invested at 5% compound interest
- the interest being added annually

Solution

$$A = P \left(1 + \frac{r}{100}\right)^n$$
$$= 500000 \left(1 + \frac{5}{100}\right)^{10}$$
$$A = 814,447.3134$$

(ii) the interest being added twice a year

Solution

$$A = P \left(1 + \frac{r}{100}\right)^{2n}$$
$$= 500000 \left(1 + \frac{5}{100}\right)^{20}$$
$$A = 1,326,648.853$$

(b) Find the amount at the end of ten years when 500000 shillings is invested at 5% simple interest

$$A = \frac{nPr}{100} = \frac{10 \times 500000 \times 5}{100} = 750,000$$

1. John opened an account in the bank and deposited 200,000 shillings every month for ten months without withdrawing. Find how much money he accumulated after 10 months if the bank offered 10% compound interest per month. [3,506,233.412]

Peter deposited sh. 100,000 at the beginning of every year for 5 years; find

Thank You

Dr. Bbosa Science