Subsidiary Mathematics

SENIOR FIVE term 3

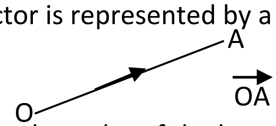
TOPIC 2/3: Vectors

Vectors

A vector is a quantity which has both magnitude and direction. Examples include, force, displacement, acceleration, momentum and velocity.

Representation of a vector

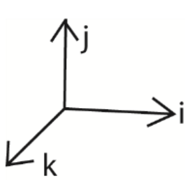
A vector is represented by a line with an arrow to indicate the direction of the vector.



where the order of the letters shows the direction

Vectors in dimensions

Vectors can be represented in three dimensions as I, j and k along the x, y and z- axes respectively



Resultant of vectors

When several vectors (V_1, V_2, V_3… V_n) are acting on a point object, the net vector R, is calculated as the vector sum

R = $V_1 + V_2 + V_3 + \ldots + Vn = \sum_{r=1}^{r=N} V_r$

Example 1

Find the resultant of the following vectors

(a) (2i + 3j + 3k) and (2i + 4j – 8k)

$$R = \begin{pmatrix}2\\3\\3\end{pmatrix} + \begin{pmatrix}2\\4\\-8\end{pmatrix} = \begin{pmatrix}2+2\\3+4\\3-8\end{pmatrix}\begin{pmatrix}4\\7\\-5\end{pmatrix} \; or\; 4i + 7j - 5k$$

(b) (7i – 4j + 3k), (5i -2j + 8k), (i –k)

$$R = \begin{pmatrix}7\\-4\\3\end{pmatrix} + \begin{pmatrix}5\\-2\\8\end{pmatrix} + \begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}7+5+1\\-4-2+0\\3+8-1\end{pmatrix} = \begin{pmatrix}13\\-6\\10\end{pmatrix} or\; 13i - 6j + 10k$$

Example 2

The resultant of (5i -2j), (7i + 4j), (ai + bj) and (-3i + 2j) is (5i + 5j). Find the values of a and b.

$$R = \binom{5}{-2} + \binom{7}{4} + \binom{a}{b} + \binom{-3}{2} = \binom{5}{5}$$

$$= \binom{9+a}{4+b} = \binom{5}{5}$$

9 + a = 5; a = -4 also 4 + b = 5; b = 1

Example 3

The resultant of the forces (3i + (a-c)j)N, ((2a + 3c)I + 5j)N and (4i, 6j)N acting on a particle is (10i + 12j)N. find

(i) Values of a and c

$$R = \binom{3}{a-c} + \binom{2a+3c}{5} + \binom{4}{6} = \binom{10}{12}$$

2a + 3c + 7 = 10

2a + 3c = 3 ……….. (i)

a –c +11 = 12

a – c = 1 …………. (ii)

(i) + 3(ii)

5a = 6; a = 1.2

from eqn. (ii)

c = 0.2

(ii) magnitude of (2a + 3c)I + 5j

R = (2a + 3c)I + 5j = (1.2 x 2 + 3 x 0.2)i + 5j = 3i + 5j

$$|R| = \sqrt{3^2 + 5^2} = 5.831N$$

Magnitude or modulus of a vector

This is the length of a vector

(i) Given R = xi + yj; $|R| = \sqrt{x^2 + y^2}$

(ii) Given R = xi + yj + zk; $|R| = \sqrt{x^2 + y^2 + z^2}$

Example 5

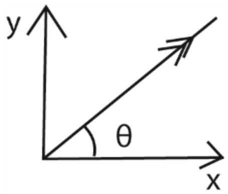
Find the magnitude of the following vectors

(a) 3i + 4j; $|R| = \sqrt{3^2 + 4^2}$ =5

(b) 3i + 2j – 6k; $|R| = \sqrt{3^2 + 2^2 + (-6)^2} = 7$

Direction of a vector

Consider R = xi + yj



$\theta = \tan^{-1}\left(\frac{x}{y}\right)$

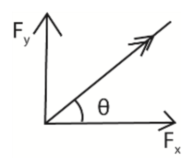
Example 6

Find the magnitude and direction of the resultant of each of the following

(a) (2i + 3j)N, (5t – 2j)N, (-3i, 3j)

$R = \begin{pmatrix}2\\3\end{pmatrix} + \begin{pmatrix}5\\-2\end{pmatrix} + \begin{pmatrix}-3\\3\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix}$

$|R| = \sqrt{4^2 + 4^2}$ = 5.6569N

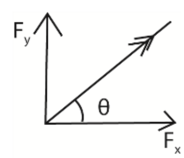


$\theta = \tan^{-1}\left(\frac{4}{4}\right)$= 45^0

(b) $\begin{pmatrix}2\\4\end{pmatrix}N, \begin{pmatrix}-6\\-5\end{pmatrix} Nand \begin{pmatrix}2\\1\end{pmatrix}N$

$R = \begin{pmatrix}2\\4\end{pmatrix} +, \begin{pmatrix}-6\\-5\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-2\\0\end{pmatrix}$

$|R| = \sqrt{(-2)^2 + 0^2}$ = 2N



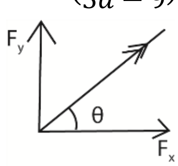
$\theta = \tan^{-1}\left(\frac{0}{-2}\right)$= 180^0

Example 7

Four forces of ai + (a-1)j, 3i + 2aj, 5i -6j, and –i -2j act on a particle. The resultant forces make an angle of 45^0 with horizontal. Find a. Hence determine the magnitude of the resultant force.

$R = \begin{pmatrix}a\\a-1\end{pmatrix} + \begin{pmatrix}3\\2a\end{pmatrix} + \begin{pmatrix}5\\-6\end{pmatrix} + \begin{pmatrix}-1\\-2\end{pmatrix}$

$= \begin{pmatrix}a+7\\3a-9\end{pmatrix}$



$\frac{a+7}{3a-9} = \tan^{-1}(45^0)$ = 1

a + 7 = 3a – 9, a = 8

$R = \begin{pmatrix}a+7\\3a-9\end{pmatrix} = \begin{pmatrix}8+7\\3\,x\,8-9\end{pmatrix} = \begin{pmatrix}15\\15\end{pmatrix}$

$|R| = \sqrt{15^2 + 15^2}$ = 21.21N

Unit vector

This a vector whose magnitude is unit (1)

Unit vector of r denoted by $\quad r = \frac{r}{|r|}$

Example 8

Find the unit vector of a = 6i -2j + 3k

Solution

$a = \frac{6i - 2j + 3k}{\sqrt{6^2 + (-2)^2 + 3^2}} = \frac{1}{7}(6i - 2j + 3k)$

Parallel vectors

If vectors a and b are parallel, then one of them is a scalar multiple of the other.

If a vector r of magnitude $|r|$ moves in direction xi + yj + zk then, $r = |r|\left(\frac{xi + yj + zk}{\sqrt{x^2+y^2+z^2}}\right)$

Example 9

Find the vector, V which has a magnitude of 15 units and is parallel to 16i + 12j

$V = 15 \text{ x } \frac{16i + 12j}{\sqrt{16^2+12^2}} = 15 \text{ x } \frac{16i + 12j}{20} = 12i + 9j$

Example 10

A body of velocity v and of magnitude 20m/s moves in the direction 6i + 8j. Find V.

$V = 20 \text{ x } \frac{6i + 8j}{\sqrt{6^2+8^2}} = 20 \text{ x } \frac{6i + 8j}{10} = 12i + 16j$

Example 11

A force of magnitude 12N acts on a body in the direction 2i + j + 2k. Find the force

$V = 12 \text{ x } \frac{2i + j + 2k}{\sqrt{2^2+1^2+2^2}} = 12x \frac{2i + j + 2k}{3} = 8i + 4j + 8k$

Example 12

The force A of magnitude 5N in the direction with unit vector $\frac{1}{5}(3i + 4j)$ and force B of magnitude 13N in the direction with unit vector $\frac{1}{13}(5i - 12j)$. Find the resultant forces of A and B.

$FA = \frac{1}{5}(3i + 4j)x5 = 3i + 4j$

$FB = \frac{1}{13}(5i - 12j)\ x\ 13 = 5i - 12j$

$F = \binom{3}{4} + \binom{5}{-12} = \binom{8}{-8}$

$|F| = \sqrt{8^2 + (-8)^2} = 11.3137N$

Example 13

A particle P moves through a displacement of 2m when acted o by two forces F_1 and F_2. Find the work done by the resultant force, if $F_1 = i - j$ and $F_2 = 10$ N and acts in the direction 4i + 3j

Solution

$F = \binom{1}{-1} + \binom{8}{6} = \binom{9}{5}$

$|F| = \sqrt{9^2 + 5^2} = 10.2956N$

$W = |F|\ x\ d = 10.2956 \text{ x } 2 = 20.5912J$

F_1 = i – j

$F_2 = 10\ x\ \frac{4i + 3j}{\sqrt{4^2+3^2}}$ = 8i + 6j

Revision exercise 1

(Answers are given in square brackets at the end of each question)

1. Find the resultant of each of the following forces
   (a) (6i + 2j)N, (-5i + j)N, (3i – 3j)N. [(4i)N]
   (b) (2i + 4j)N, (3i -5j), (6i +2j)N, (-7i – 7j)N. [(4i -6j)N]
   (c) (2i + 3j -7k)N, (2i + 5k)N, (3j + 4k)N. [(4i + 6j + 2k)N]
2. The resultant of forces (5i + 7j), (ai + bj) and (bi –aj)N is a force (11i + 5j)N. Find a and b. [a = 4, b = 2]
3. Find the magnitude and direction of the resultant of each of the following;
   (a) (-2i + 5j)N, (I + 2j)N. [7.07N at 98.1^0]
   (b) (6i + 2j)N, (4i – 3j)N. [10.05N at 354.3^0]
   (c) (3i + 2j), (-5i + j)N. [3.61N at 124^0]
4. A force of magnitude 50N acts on a body in the direction 24i + 7j. Find the force. [(48i + 14j)]
5. Two forces F_1 and F_2 have magnitude αN and βN and act in the direction i -2j and 4i + 3j respectively. Given that the resultant of F_1 and F_2 is (48i + 14j). Find the magnitude of αN and βN. [α = $8\sqrt{5}$N and β = 50N]
6. If a = 3i + 4j, b = 4i + 20j and c = 5i – 19j; find the
   (i) resultant of a and b [(7i + 24j)]
   (ii) resultant of a and c [(8i – 15j)]
   (iii) vector is parallel to a and has magnitude of 15 unit [(9i + 12j)]
   (iv) vector parallel to (a + b) and has a magnitude of 100 units [(28i + 96j]
7. If a = 2i + 5j, b = -7i + 7j and 14i. Find the;
   (i) resultant of a and b [(-5i + 12j)]
   (ii) resultant of a, b and c [(9i + 12j)]
   (iii) |b| [$7\sqrt{2}$]
   (iv) |a + b + c| [15units]
   (v) vector is parallel to a and has a magnitude of $5\sqrt{29}$ units. (10i + 25j)
   (vi) Vector is parallel to (a + b + c) and has magnitude 90 units. [(54i + 72j)]
8. If a = i -3j + 2k, b = 5i + 4j and c = 3i + j + 4k. Find the
   (i) resultant of a and b [(6i + j + 2k)]
   (ii) resultant of a, b and c.[(9i + 2j + 6k)
   (iii) |a| [$\sqrt{14}$]
   (iv) |a + b + c|[11units]
   (v) Vector parallel to (a + b + c) and has magnitude 5 units [$\frac{5}{11}(9i + 2j + 6k)$]
9. If a = 2i + 7j + 7k, b = 6i -3j + 2k and c = -4j -3k. find the
   (i) resultant a and b [8i +4jj + 9k]
   (ii) resultant a and c [2i + 3j + 4k

(iii) |b| [7units]
(iv) |a + b + c| [10 units]
(v) vector is parallel to |a + b + c| and has magnitude of 50 units [40i + 30k]

Scalar products or dot products

The dot product of two vectors p=$\begin{pmatrix} a \\ b \end{pmatrix}$ and q= $\begin{pmatrix} c \\ d \end{pmatrix}$ is given by **p.q =$\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} . \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{pmatrix}$= ac - bd**

The dot product of two vectors a and b inclined at angle θ is given by

$$\boldsymbol{a.b} = |\boldsymbol{a}||\boldsymbol{b}|\ \boldsymbol{cos\theta}$$

Note

If two vectors are perpendicular then the angle between them is 90^0 and

$$\boldsymbol{a.b} = |\boldsymbol{a}||\boldsymbol{b}|\ \boldsymbol{cos\ 90 = 0}$$

Example 14

The vector a = 3i + 2j and b = 4i – 5j

Determine

(a) **Magnitude of b**

$|b|$ = $\sqrt{4^2 + (-5)^2}$ **= 6.403**

(b) **A.b (05marks)**

(3 x 4)+ (2 x -5) = 12 – 10 = 2

Example 15

Given a = $\begin{pmatrix} 5 \\ -12 \end{pmatrix}$ **and b =** $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$**, find the;**

(a) **dot product of a and b. (02 marks)**

(5 x -3) + (-12x 4) = -93

(b) **angle between the vectors a and b. (03 marks)**

Solution

Let the angle be θ

$\sqrt{5^2 + (-12)^2}.\sqrt{(-3)^2 + 4^2}$**cosθ = -93**

13 **x 5 cosθ = 93**

$cos\theta = \frac{-93}{65}$

$\theta = 0$

Example 16

Point A, Band C have position vectors, 2j, 4i, and 2i – 2j respectively in the x – y plane.

(a) Find 2OA + 3OB – 4OC. (04 marks)

$$2\binom{0}{2} + 3\binom{4}{0} - 4\binom{2}{-2} = \binom{0+12-8}{4+0+8} = \binom{4}{12}$$

(b) Determine;

(i) AB and AC (04marks)

AB = AO + OB = $\binom{0}{-2} + \binom{4}{0} = \binom{4}{-2}$

AC = AO + OC = $\binom{0}{-2} + \binom{2}{-2} = \binom{2}{-4}$

(ii) AB + AC (03mark)

$$\binom{4}{-2} + \binom{2}{-4} = \binom{6}{-6}$$

(iii) angle BAC (05 marks)

Dot product of vector BA = $\binom{-4}{2}$**and AC =** $\binom{2}{-4}$ **= (-4 x 2) + (2 x -4) = -16**

Let the angle be θ

$|BA||AC|cos\theta = dot\ product$

$\sqrt{4^2+(2)^2}.\sqrt{(2)^2+(-4)^2}$**cosθ = -16**

$20cos\theta = -16$

$\theta = \cos^{-1}\left(\frac{16}{20}\right)$**= 143.1**^{0}

Example 17

If OA = $\binom{6}{5}$**, OB =** $\binom{9}{2}$ **and OC =**$\binom{7}{0}$

(a) find the vectors;

(i) BC

Solution

BC = BO + OC

=$\binom{-9}{-2} + \binom{7}{0} = \binom{-2}{-2}$

(ii) AB (06 marks)

AB = AO + OB

$=\begin{pmatrix}-6\\-5\end{pmatrix}+\begin{pmatrix}9\\2\end{pmatrix}=\begin{pmatrix}3\\-3\end{pmatrix}$

(b) **Show that vectors AB and BC are perpendicular. (03 marks)**

For perpendicular vectors

AB.BC =0

AB.BC = $\begin{pmatrix}3\\-3\end{pmatrix}.\begin{pmatrix}-2\\-2\end{pmatrix}$ **= (3 x -2)+(-3 x -2) = -6 + 6 = 0**

Hence AB and BC are perpendicular

(c) **Determine the magnitude of the vector 2BC – 3AB. (06 marks)**

2BC – 3AB = 2$\begin{pmatrix}-2\\-2\end{pmatrix}-3\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}-4\\-4\end{pmatrix}-\begin{pmatrix}9\\-6\end{pmatrix}=\begin{pmatrix}-13\\2\end{pmatrix}$

$|2BC-3AB|$ = $\sqrt{(13)^2+2^2}$ = $\sqrt{169+4}$ =$\sqrt{173}$ **= 13.15**

Example 18

If a = i – 2k and b = 3i – 3j + k, find

(i) a.b (ii) the angle between a and b

Solution

(i) a.b = $\begin{pmatrix}1\\0\\-2\end{pmatrix}.\begin{pmatrix}3\\-3\\1\end{pmatrix}$= 3 + 0 + -2 = 1

(ii) $\theta = \cos^{-1}\left(\frac{a.b}{|a||b|}\right) = \cos^{-1}\frac{1}{\sqrt{1^2+(2)^2}\sqrt{3^2+(-3)^2+1^2}}$ = 84.1^0

Example 19

If p = 2i – j + 3k and q = i + 4j + 3k; find the angle between p and q.

Solution

p.q = $\begin{pmatrix}2\\-1\\3\end{pmatrix}.\begin{pmatrix}1\\4\\3\end{pmatrix}$ = 2 + -4 + 9 = 7

$\theta = \cos^{-1}\left(\frac{p.q}{|p||q|}\right) = \cos^{-1}\frac{7}{\sqrt{2^2+(-1)^2+3^2}\sqrt{1^2+4^2+3^2}}$ = 68^0

Example 20

If the angle between two vectors a = xi + 2j and b = 3i + j is 45^0.Find the two possible values of constant x.

Solution

$\begin{pmatrix}x\\2\end{pmatrix}.\begin{pmatrix}3\\1\end{pmatrix}=\sqrt{x^2+2^2}.\sqrt{3^2+1^2}$

$x^2 + 3x – 4 = 0$

(x + 4)(x -1) = 0

x = -4 and x = 1

3x + 2 = $\sqrt{x^2 + 2^2}.\sqrt{10}.\frac{\sqrt{2}}{2}$

$(3x + 2)^2 = (x^2 + 4).\ 10 \times \frac{2}{4}$

Example 21

If p = 2αi + 7j – k and q = 3αi + αj + 3k.Find the value of the scalar α if the vectors are perpendicular

Solution

$$\begin{pmatrix} 2\alpha \\ 7 \\ -1 \end{pmatrix}.\begin{pmatrix} 3\alpha \\ \alpha \\ 3 \end{pmatrix} = 0$$

$6\alpha^2 + 7\alpha - 3 = 0$

$\alpha = \frac{1}{3}$ and $\alpha = \frac{3}{2}$

Example 22

The points P and Q have position vectors OP= –2i – 5j and OQ = i – 2j respectively. R is appoint such that OR = OP + λPQ.

(a) Find the:

(i) value of OP.OQ

-1 x 2 + -5 x -2 = 8

(ii) angle between the two vectors OP and OQ. (07 marks)

OP.OQ=$|OP||OQ|cos\theta$

8 = $\sqrt{(-2)^2 + (-5)^2}.\sqrt{(1)^2 + (-2)^2}cos\theta$

8 = $\sqrt{29}.\sqrt{5}cos\theta$

8 = $\sqrt{145}cos\theta$

cosθ = $\frac{8}{\sqrt{145}}$

$\theta = \cos^{-1}\frac{8}{\sqrt{145}}$ **= 48.37^0(2D)**

(b) Determine the

(i) vector PQ

PQ = PO + OQ

= $\binom{2}{5} + \binom{1}{-2}$

= $\begin{pmatrix}3\\3\end{pmatrix}$ **Or (3i + 3j)**

(ii) vector OR in terms of λ

OR = $\begin{pmatrix}-2\\-5\end{pmatrix} + \lambda\begin{pmatrix}3\\3\end{pmatrix} = \begin{pmatrix}-2+3\lambda\\-5+3\lambda\end{pmatrix}$

(iii) the value of λ for which OR is perpendicular to PQ. (08 marks)

When OR is perpendicular to PQ

Then OR.PQ = 0

$\begin{pmatrix}-2+3\lambda\\-5+3\lambda\end{pmatrix}.\begin{pmatrix}3\\3\end{pmatrix}$

-6 + 9λ- 15 + 9λ =

18λ = 21

λ = $\frac{21}{18} = \frac{7}{6}$

Revision exercise 2

(Answers are given in square brackets [ ] at the end of each question)

1. Find the scalar products for each of the following pairs of vectors.
   (i) a = 2i + j, b = i - 3j [-1]
   (ii) a = 3i, b = -2i + j [-6]
   (iii) a = 5i + j - 2k, b= 4i + 3j – 8k [39]
   (iv) 2i + 4j – 15k and -8i + 2j – k [7]
   (v) $\begin{pmatrix}2\\1\end{pmatrix}$ *and* $\begin{pmatrix}3\\-2\end{pmatrix}$ [4]
   (vi) $\begin{pmatrix}5\\-1\end{pmatrix}$ *and* $\begin{pmatrix}2\\4\end{pmatrix}$ [6]
   (vii) $\begin{pmatrix}0\\5\\-2\end{pmatrix}$ *and* $\begin{pmatrix}-3\\2\\1\end{pmatrix}$ [8]
   (vi) $\begin{pmatrix}5\\2\\7\end{pmatrix}$ *and* $\begin{pmatrix}1\\2\\-1\end{pmatrix}$ [2]
2. Find the angles between each of the following pairs of vectors
   (i) 3i + 4j and 5i – 12j [121^0]
   (ii) 3i and -2j [90^0]
   (iii) 2i + 3j – 6k and 2i + j + 2k [10^0]
   (iv) i + 2j – k and –I + 2j – k [48^0]
   (v) (vii) $\begin{pmatrix}0\\1\\-1\end{pmatrix}$ *and* $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ [120^0]
   (v) $\begin{pmatrix}3\\1\end{pmatrix}$ *and* $\begin{pmatrix}1\\-2\end{pmatrix}$ [82^0]
   (vi) $\begin{pmatrix}6\\-8\end{pmatrix}$ *and* $\begin{pmatrix}5\\4\end{pmatrix}$ [92^0]
   (viii) $\begin{pmatrix}2\\3\end{pmatrix}$ *and* $\begin{pmatrix}1\\-1\end{pmatrix}$ [73^0]
3. If a = αi + 2j – k and b = 5i –αj + k. Find the value of the scalar α if the vectors are perpendicular [$\frac{1}{3}$]
4. If a = 2i + αj and b = -α – k. Find the value of the scalar α if the vectors are perpendicular [0]
5. If a = 4i + 5j and b = qi – 8j. Find the value of scalar q if the vectors are perpendicular. [10]
6. If a = 6i – j and b = 2i + pk. Find the value of scalar p if the vectors are perpendicular [12]

7. Given $\begin{pmatrix} q \\ 2+q \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \\ 4-q \end{pmatrix}$ are perpendicular vectors. Find the value of q. [18]
8. If a = qi + 8j +(3q + 1)k and b = (q+1)I + (q-1)j – 2k. Find the value of the possible values of constant q if the vectors are perpendicular. [2 or -2]

Thank You

Dr. Bbosa Science